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Aspects of Brill-Noether geometry in Moduli theory of Algebraic and Tropical curves

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Introduction

The Brill-Noether theory for smooth curves is the study of *special* line bundles, i.e. of the line bundles L on a curve C such that $H^1(C, L) \neq 0$. By Riemann-Roch this means that the H^0 cohomology or space of holomorphic sections is larger than expected, hence that the divisor corresponding to the line bundle moves in a larger linear system. This theory is a broad and classical area of algebraic geometry, which dates back to Riemann and his work on Abelian functions [R876]. However, the study has been deeply and extensively carried out by Brill and Noether in the XIX century ([BN873]). Through the years many aspects of the theory of special linear series have been investigated, so that each of them has inspired a separate research area. Among these areas, for instance, we recall the classical study of the Brill-Noether varieties

$$W_d^r(C) = \{L \in \mathrm{Pic}^d C : h^0(L) > r\},$$

and their dimension. This study focused on the properties of linear series on a curve C, hence suggesting the possible projective models of C. The original motivation to study these varieties was to classify differences among curves. To this aim the structure of the Brill-Noether varieties becomes interesting only when we look at special linear series.

In this setting there are other aspects coming up, such as the projective normality of curves, which is the property for a canonically embedded curve of having a complete linear series cut by the hypersurfaces of the ambient space in any degree. We notice that the canonical bundle is special.

Moreover it is worth mentioning Abel maps (we will recall the definition later), which take values in the Brill-Noether varieties $W_d^0(X)$. If the degree is $g-1$ we have that when X is smooth the variety $W_{g-1}^0(X)$ is a divisor inside $\mathrm{Pic}^{g-1}X$, the so called Theta divisor of the Jacobian variety of X. From this point of view we can include the study of the Theta divisor in the Brill-Noether theory, and this naturally leads to approach the Torelli theorem (for details on all of these subjects see [ACGH]).

The Brill-Noether theory for smooth curves has been widely developed, but when we think of families of curves we have to consider singular curves as well. However, for singular curves not much is known yet. We point out some recent developments in the case of *binary* curves in [C5].

In this thesis we present some generalizations of Brill-Noether problems for distinguished classes of curves: singular and tropical curves. The study of singular curves and their properties is still very active and open: many of the classical theorems proved for smooth curves hold no longer for singular ones. We will work on curves with at most planar singularities. Of course the easiest to handle are nodal curves, and in particular stable curves, that allow us to work with their moduli.

Tropical geometry is a recent branch of mathematics which relates algebro-geometric objects to purely combinatorial ones in such a way that ideally one should be able to obtain results in algebraic geometry after studying their combinatorial tropical counterpart. A tropical curve C of genus g in the sense of [BMV09] is a marked graph Γ endowed with some extra data (see Definitions 2.2.1 and 2.2.3). Given their graph nature, we can interpret tropical curves as degenerations of smooth curves, but we can as well find a different and deeper relation between stable curves and tropical ones given by duality. Indeed a tropical curve is associated to a graph which can be thought of as the dual graph of a stable curve (see subsection 2.5.3). So we can actually view singular and tropical curves as being part of the class of *degenerate* curves, i.e. curves arising as limits of smooth curves through a degenerating process.

The motivations for my work are, on the one hand, my interest in the Brill-Noether theory itself, since there is a lot of beautiful classical geometry involved. Moreover many proofs of general facts concerning moduli of curves have been obtained in history by looking at the properties of suitable degenerate curves, as singular and tropical curves are. Hence in this perspective I hope that my results can help interpret in an easier way classical problems.

The problems

Chapter 1 is devoted to singular curves, and we tackle the following problems:

1. Given a smooth projective curve C of genus g and a natural number $d \geq 1$, we can consider the product C^d and define the *Abel map of degree* d

$$\alpha^d_C : C^d \longrightarrow \mathrm{Pic}^d C, \qquad (p_1, \ldots, p_d) \mapsto \mathcal{O}_C(\sum_{i=1}^{d} p_i);$$

it is a regular map, and in degree 1 it is injective when $g \geq 1$. In the smooth case the image $\mathrm{Im}\alpha^d_C$ of the Abel map coincides with the Brill-Noether variety $W^0_d(C)$. Of course it is interesting to approach the problem of extending Abel maps to singular curves in such a way that they have a geometric meaning.

If X is a singular curve we can still define Abel maps: we consider the decomposition of X in irreducible components, $X = C_1 \cup \ldots \cup C_\gamma$, and set $\dot{X} := X \setminus X^{\text{sing}}$, where X^{sing} is the set of singular points of X, and $\dot{C}_i = C_i \cap \dot{X}$. Then, let $\underline{d} = (d_1, \ldots, d_\gamma)$ be a multidegree with $d_i \geq 0$ for any i, and set

$$\dot{X}^{\underline{d}} := \dot{C}_1^{d_1} \times \cdots \times \dot{C}_\gamma^{d_\gamma};$$

$\dot{X}^{\underline{d}}$ is a smooth irreducible variety of dimension $d = |\underline{d}|$, open and dense in $X^{\underline{d}} := C_1^{d_1} \times \cdots \times C_\gamma^{d_\gamma}$. We set

$$\alpha_X^{\underline{d}} : \dot{X}^{\underline{d}} \longrightarrow \text{Pic}^{\underline{d}} X, \qquad (p_1, \ldots, p_d) \mapsto \mathcal{O}_X(\sum_{i=1}^{d} p_i),$$

and we call it the *Abel map of multidegree* $\underline{d}$; it is a regular map. Abel maps for integral curves have been studied by Altman and Kleiman in [AK80], and later on in [EGK00], [EGK02], [EK05]. We notice that the completion of Abel maps for integral curves was a major step to prove *autoduality* of the compactified Jacobian ([EGK02]). This is an important property connected with the study of the fibers of the Hitchin fibration for $GL(n)$ ([AIK76], [N10]).

For reducible curves, the problem of completing the Abel maps is open with a few exceptions as we shall explain. As it is well known, the non separatedness of the Picard functor, together with combinatorial hurdles, make the case of reducible curves much more complex. The first step in this direction was taken by Caporaso and Esteves in [CE06], where they construct Abel maps of degree 1 for stable curves. However, they do not describe explicitly the closure of the image of the completion of the map. It is interesting to notice that they consider stable curves as limits of smooth ones, approaching this way the study of Abel maps for families of curves. In this setting, the completion of α_X^1 can be viewed as a specialization to the singular fiber of the Abel maps of the smooth fibers.

Further improvements have been achieved for Gorenstein curves by Caporaso, Coelho and Esteves in [CCE08] using torsion free sheaves, and by Coelho and Pacini in [Co07] and [CP09], where, respectively, they construct Abel maps of degree 2 for curves with two components and two nodes, and in any degree for curves of compact type. So in all other cases this problem remains open.

On the other hand the situation is better understood in case $d = g - 1$ in [C2]: if X is a nodal connected curve of genus g, denote by $A_{\underline{d}}(X)$ the closure of $\text{Im}\alpha_X^{\underline{d}}$ inside $\text{Pic}^{\underline{d}} X$. Let

$$W_{\underline{d}}(X) := \{L \in \text{Pic}^{\underline{d}} X : h^0(L) > 0\};$$

in Theorem 3.1.2. the author proves that if $\underline{d}$ is a stable multidegree such that $|\underline{d}| = g - 1$, then

$$A_{\underline{d}}(X) = W_{\underline{d}}(X),$$

and hence that the Brill-Noether variety $W_{\underline{d}}(X)$ is irreducible. Let $\overline{P_X^{g-1}}$ be the compactified Jacobian in degree $g - 1$; it has a polarization given by the Theta divisor $\Theta(X)$, and the pair $(\overline{P_X^{g-1}}, \Theta(X))$ is a semiabelic stable pair as in [A02]. It turns out that the varieties $A_{\underline{d}}(X_S) = W_{\underline{d}}(X_S)$, where X_S is a partial normalization of X at a set S of nodes, are the sets which give a stratification of $\Theta(X)$ (see Theorem 4.2.6. in [C2]).

In the first section of Chapter 1 we generalize this stratification in lower degree and give a characterization of the closure of the image of the Abel map of multidegree $\underline{d}$ for some classes of nodal curves, inside the compactified Picard variety $\overline{P_X^d}$ constructed in

[C1]. We recall that in this construction every point of $\overline{P_X^d}$ corresponds to a pair $(\widehat{X}_S, \widehat{M}_S)$ where $\widehat{X}_S$ is the blow up of X at a set S of nodes of X, and $\widehat{M}_S$ is a *balanced* line bundle (see below) of multidegree $\underline{d}$ on $\widehat{X}_S$ up to equivalence. So our question can be posed in the following way: which points of $\overline{P_X^d}$ are limits of effective Weil divisors on X?

We will study the following cases: irreducible curves on the one hand, and two types of reducible curves, namely curves of compact type and binary curves. Curves of compact type have the advantage and the special property that the generalized Jacobian is compact. Binary curves are nodal curves made of two smooth rational components meeting at $g+1$ points. They form a remarkable class of reducible curves since they present the basic problems as all reducible curves, yet simpler combinatorics. Indeed, they have been used in the past as test cases for results later generalized to all stable curves, see for instance [C5],[Br99].

In order to answer our question, let X_S be a partial normalization of a nodal curve X at a set S of nodes. We define the set

$$W^+_{\underline{d}_S}(X_S) = \{L \in \mathrm{Pic}^{\underline{d}_S} X_S : h^0(Z, L|_Z) > 0 \text{ for all subcurves } Z \subseteq X_S\},$$

and consider the union of the $W^+_{\underline{d}_S}(X_S)$ when S varies among the subsets of X^{sing} and $\underline{d}_S$ is the restriction to X_S of a balanced multidegree $\widehat{\underline{d}}_S$ on the partial blow up $\widehat{X}_S$. Similarly to [C2][Theorem 4.2.6], we define

$$\widetilde{W}_d(X) := \coprod_{\substack{\emptyset \subset S \subset X^{\mathrm{sing}} \\ \widehat{\underline{d}}_S \in B_d^{\geq 0}(\widehat{X}_S)}} W^+_{\underline{d}_S}(X_S),$$

where $B_d^{\geq 0}(\widehat{X}_S)$ is the set of strictly balanced multidegrees $\widehat{\underline{d}}_S \geq \underline{0}$ on $\widehat{X}_S$ such that $|\widehat{\underline{d}}_S| = d$, and $\underline{d}_S = \widehat{\underline{d}}_S|_{X_S}$.

In paragraph 1.2.1 we study directly the closure inside $\overline{P_X^d}$ of $A_d(X)$ for irreducible curves, and we prove that $\overline{A_d(X)} = \overline{W_d(X)}$ giving a description of it in terms of the Brill-Noether varieties $W^0_{d-\delta_S}(X_S)$ where X_S is the normalization of X at a set of nodes S, and $\delta_S = \sharp S$.

In paragraph 1.2.2 we turn our attention to reducible curves: we describe the structure of the varieties $W_{\underline{d}}(X)$ for curves of compact type, which is quite natural, and in the last part we develop the study of $A_{\underline{d}}(X)$ and its closure inside $\overline{P_X^d}$ for binary curves. We characterize it in terms of the varieties $W_{\underline{d}_S}(X_S)$. If X is a binary curve of genus g and $1 \leq d \leq g-1$, we prove that the closure inside $\overline{P_X^d}$ of the union of the varieties $A_{\underline{d}}(X)$ as $\underline{d}$ varies among balanced multidegrees on X, is exactly $\widetilde{W}_d(X)$. In other words, we define

$$\overline{A_d(X)} := \overline{\bigcup_{\underline{d} \in B_d^{\geq 0}(X)} A_{\underline{d}}(X)} \subset \overline{P_X^d},$$

then the main theorem states that

$$\widetilde{W}_d(X) = \overline{A_d(X)} \subset \overline{P_X^d}. \tag{1}$$

Finally we study the simpler case when $d = 1$ giving a characterization of the closure of the image of the Abel map for all the stable curves such that the set $B_1^{\geq 0}(X)$ of strictly balanced multidegrees $\underline{d} \geq \underline{0}$ is nonempty, i.e. the so called *d-general* curves.

2. Let C be a smooth curve of genus g over an algebraically closed field k. The canonical bundle ω_C induces an embedding of C in $\mathbb{P}^{g-1}$ if and only if C is not hyperelliptic; we indicate the power $\omega_C^{\otimes n}$ by ω_C^n for any $n \in \mathbb{N}$. One says that C is *projectively normal* if the maps

$$H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(k)) \to H^0(C, \omega_C^k) \tag{2}$$

are surjective for every $k \geq 1$. In other words, C is projectively normal if and only if the hypersurfaces of degree k in $\mathbb{P}^{g-1}$ cut a complete linear series on C for any k. If $k = 1$ and the map (2) is surjective, we say that C is *linearly normal*, which means that the curve is embedded via a complete linear series. If ω_C is ample, then an equivalent formulation states that C is projectively normal if the maps

$$\mathrm{Sym}^k H^0(C, \omega_C) \to H^0(C, \omega_C^k) \tag{3}$$

are surjective for every $k \geq 1$, because the surjectivity of all these maps when ω_C is ample implies the very ampleness of ω_C.

If C is a smooth, non-hyperelliptic curve, Castelnuovo and Noether proved that its canonical model is projectively normal (see [ACGH]). For curves, though, the problem becomes harder: in the case of integral curves, in [KM09] the authors generalize Castelnuovo's approach proving that linear normality is equivalent to projective normality. For reducible curves yet not much is known: properties of the canonical map for Gorenstein curves, i.e. the map induced by the dualizing sheaf, are investigated in [CFHR99], whereas in [F04] the author gives a sufficient condition for line bundles on non-reduced curves to be *normally generated* (see 1.4.9). The projective normality of reducible curves is studied in [S91]; more in general, since the problem of studying projective normality reduces to the study of multiplication maps, we refer to [B01] and [F04] for these items.

In the second section of chapter 1 we investigate the projective normality of reducible curves restricting the problem to suitable subcurves. The first step is to study the *quadratic normality*, i.e. the surjectivity of the maps in (2) for $k = 2$. Let X be a connected, reduced and Gorenstein projective curve of genus g with ω_X very ample. Assume that X has planar singularities at the points lying on at least two irreducible components. Our main result about quadratic normality is the following theorem.

Theorem 1. *Let X be a curve as above, and set $X = A \cup B$ with A, B connected subcurves being smooth at $D := A \cap B$. If $A \neq \emptyset$ and the map*

$$\mu_{\omega_A, \omega_X|_A} : H^0(A, \omega_A) \otimes H^0(A, \omega_X|_A) \to H^0(X, \omega_A \otimes \omega_X|_A)$$

is surjective, then X is quadratically normal.

We also study certain multiplication maps in order to establish sufficient conditions that imply the surjectivity of the map in (3) for some k (*k-normal generation*) assuming to know the surjectivity for $(k-1)$ (see Proposition 1.4.23). Moreover at the end of the section we carry out a proof of the projective normality for binary curves following the approach suggested by Castelnuovo-Noether in [ACGH], and see some applications of our results. This part is based on a joint work with Edoardo Ballico (see [BB10]).

3. In the third section of chapter 1 we study some properties of semistable curves that are related to Brill-Noether theory: in subsection 1.3.1 we prove Martens' theorem and Mumford's theorem for irreducible nodal curves generalizing the approach described in [ACGH]. Then, in subsection 1.3.2 we turn our attention to the possible projective models of semistable *$k-$gonal* curves. A nodal curve is said to be *$k-$gonal* if it admits a regular smoothing such that the general fiber is a smooth curves having a $\mathfrak{g}^1_k$ (i.e. a pencil of degree k).

We study nodal curves with two components (which are often called *vine curves*) investigating about sufficient and necessary conditions in order for them to be $k-$gonal. In other words we list the properties that a vine curve must have in order to be k-gonal, and vice-versa. This study is carried out more extensively for trigonal curves, and the techniques we use refer to [Br99], [C2], [C6], [EM02] and to [HM82] for the specific use of admissible covers. We also introduce the concept of *weakly $k-$gonal* curves, defined as the curves possessing a $\mathfrak{g}^1_k$, and in the case of weakly trigonal curves we investigate, as Caporaso does for hyperelliptic curves in [C6], if they are trigonal. As one can expect the answer is negative.

Chapter 2 is devoted to tropical curves, and more in general to tropical moduli. The classical Torelli map $\mathrm{t}_g : \mathcal{M}_g \to \mathcal{A}_g$ is the modular map from the moduli space $\mathcal{M}_g$ of smooth curves of genus g to the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g, sending a curve C into its Jacobian variety $\mathrm{Jac}(C)$, naturally endowed with the principal polarization given by the class of the theta divisor Θ_C. The Torelli map has been widely studied as it allows to relate the study of curves to the study of linear (although higher-dimensional) objects, i.e. abelian varieties. Among the many known results on the Torelli map t_g, we mention: the injectivity of the map t_g (proved by Torelli in [T13]) and the many different solutions to the so-called Schottky problem, i.e. the problem of characterizing the image of t_g (see the nice survey of Arbarello in the appendix of [M99]).

The aim of this chapter is to define and study a tropical analogous of the Torelli map and is based on a joint work with Margarida Melo and Filippo Viviani (see [BMV09]). In the paper [MZ07], Mikhalkin and Zharkov study abstract tropical curves and tropical abelian varieties. They construct the Jacobian $\mathrm{Jac}(C)$ and observe that the naive generalization of the Torelli theorem, namely that a curve C is determined by its Jacobian $\mathrm{Jac}(C)$, is false in this tropical setting. However, they speculate that this naive generalization should be replaced by the statement that the tropical Torelli map $t_g^{\rm tr}: M_g^{\rm tr} \to A_g^{\rm tr}$ has tropical degree one, once it has been properly defined!

In [CV1], Caporaso and Viviani determine when two tropical curves have the same Jacobians. They use this to prove that the tropical Torelli map is indeed of tropical degree one, assuming the existence of the moduli spaces $M_g^{\rm tr}$ and $A_g^{\rm tr}$ as well as the existence of the tropical Torelli map $t_g^{\rm tr}: M_g^{\rm tr} \to A_g^{\rm tr}$, subject to some natural properties. Indeed, a construction of the moduli spaces $M_g^{\rm tr}$ and $A_g^{\rm tr}$ for every g remained open so far, at least to our knowledge. However, the moduli space of n-pointed rational tropical curves $M_{0,n}^{\rm tr}$ was constructed by different authors (see [SS2], [Mi4], [GKM09], [KM09]). The aim of chapter 2 is to define the moduli spaces $M_g^{\rm tr}$ and $A_g^{\rm tr}$, the tropical Torelli map $\mathrm{t}_g: M_g^{\rm tr} \to A_g^{\rm tr}$ and to investigate an analogue of the Torelli theorem and of the Schottky problem.

With that in mind, we introduce slight generalizations in the definition of tropical curves and tropical principally polarized abelian varieties. A tropical curve C of genus g in the sense of [BMV09] is given by a marked graph (Γ, w, l) where (Γ, l) is a metric graph and $w: V(\Gamma) \to \mathbb{Z}_{\geq 0}$ is a weight function defined on the set $V(\Gamma)$ of vertices of Γ, such that $g = b_1(\Gamma) + |w|$, where $|w| := \sum_{v \in V(\Gamma)} w(v)$ is the total weight of the graph, and the marked graph (Γ, w) satisfies a stability condition (see Definitions 2.2.1 and 2.2.3). A (principally polarized) *tropical abelian variety* A of dimension g is a real torus $\mathbb{R}^g/\Lambda$ as before, together with a flat semi-metric coming from a positive semi-definite quadratic form Q with rational null-space (see Definition 2.3.1). To every tropical curve $C = (\Gamma, w, l)$ of genus g, it is associated a tropical abelian variety of dimension g, called the *Jacobian* of C and denoted by $\mathrm{Jac}(C)$, which is given by the real torus $(H_1(\Gamma, \mathbb{R}) \oplus \mathbb{R}^{|w|})/(H_1(\Gamma, \mathbb{Z}) \oplus \mathbb{Z}^{|w|})$, together with the positive semi-definite quadratic form $Q_{(\Gamma,l)}$ which vanishes on $\mathbb{R}^{|w|}$ and is given on $H_1(\Gamma, \mathbb{R})$ by $Q_{(\Gamma,l)}(\sum_{e \in E(\Gamma)} n_e \cdot e) = \sum_{e \in E(\Gamma)} n_e^2 \cdot l(e)$. The advantage of such a generalization in the definition of tropical curves and tropical abelian varieties is that the moduli spaces we will construct are closed under specializations (see subsection 2.2.1 for more details).

The construction of the moduli spaces of tropical curves and tropical abelian varieties is performed within the category of what we call *stacky fans* (see section 2.1.1). A stacky fan is, roughly speaking, a topological space given by a collection of quotients of rational polyhedral cones, called cells of the stacky fan, whose closures are glued together along their boundaries via integral linear maps (see definition 2.1.1).

The moduli space $M_g^{\rm tr}$ of tropical curves of genus g is a stacky fan with cells $C(\Gamma, w) = \mathbb{R}_{>0}^{|E(\Gamma|)}/\mathrm{Aut}(\Gamma, w)$, where (Γ, w) varies among stable marked graphs of genus g, consisting of all the tropical curves whose underlying marked graph is equal to (Γ, w) (see definition 2.2.5). The closures of two cells $\overline{C(\Gamma, w)}$ and $\overline{C(\Gamma', w')}$ are glued together along the faces that correspond to common specializations of (Γ, w) and (Γ', w') (see Theorem 2.2.8). Therefore, in $M_g^{\rm tr}$, the closure of a cell $C(\Gamma, w)$ will be equal to a disjoint union of lower dimensional cells $C(\Gamma', w')$ corresponding to different specializations of (Γ, w).

We describe the maximal cells and the codimension one cells of $M_g^{\rm tr}$ and we prove that

$M_g^{\rm tr}$ is pure dimensional and connected through codimension one (see Proposition 2.2.9). Moreover the topology with which $M_g^{\rm tr}$ is endowed is shown in [C8] to be Hausdorff.

The moduli space $A_g^{\rm tr}$ of tropical abelian varieties of dimension g is first constructed as a topological space by forming the quotient $\Omega_g^{\rm rt}/\,{\rm GL}_g(\mathbb{Z})$, where $\Omega_g^{\rm rt}$ is the cone of positive semi-definite quadratic forms in $\mathbb{R}^g$ with rational null space and the action of ${\rm GL}_g(\mathbb{Z})$ is via the usual arithmetic equivalence (see definition 2.3.5). In order to put a structure of stacky fan on $A_g^{\rm tr}$, one has to specify a ${\rm GL}_g(\mathbb{Z})$-admissible decomposition Σ of $\Omega_g^{\rm rt}$ (see definition 2.3.6), i.e. a fan decomposition of $\Omega_g^{\rm rt}$ into (infinitely many) rational polyhedral cones that are stable under the action of ${\rm GL}_g(\mathbb{Z})$ and such that there are finitely many equivalence classes of cones modulo ${\rm GL}_g(\mathbb{Z})$. Given such a ${\rm GL}_g(\mathbb{Z})$-admissible decomposition Σ of $\Omega_g^{\rm rt}$, we endow $A_g^{\rm tr}$ with the structure of a stacky fan, denoted by $A_g^{\rm tr,\Sigma}$, in such a way that the cells of $A_g^{\rm tr,\Sigma}$ are exactly the ${\rm GL}_g(\mathbb{Z})$-equivalence classes of cones in Σ quotiented out by their stabilizer subgroups (see Theorem 2.3.7).

Among all the known ${\rm GL}_g(\mathbb{Z})$-admissible decompositions of $\Omega_g^{\rm rt}$, one will play a special role in this paper, namely the (second) Voronoi decomposition which we denote by V. The cones of V are formed by those elements $Q \in \Omega_g^{\rm rt}$ that have the same Dirichlet-Voronoi polytope ${\rm Vor}(Q)$ (see definition 2.3.9). We denote the corresponding stacky fan by $A_g^{\rm tr,V}$ (see definition 2.3.11). We describe the maximal cells and the codimension one cells of $A_g^{\rm tr,V}$ and we prove that $A_g^{\rm tr,V}$ is pure-dimensional and connected through codimension one (see Proposition 2.3.12). $A_g^{\rm tr,V}$ admits an important stacky subfan, denoted by $A_g^{\rm zon}$, formed by all the cells of $A_g^{\rm tr,V}$ whose associated Dirichlet-Voronoi polytope is a zonotope. We show that ${\rm GL}_g(\mathbb{Z})$-equivalence classes of zonotopal Dirichlet-Voronoi polytopes (and hence the cells of $A_g^{\rm zon}$) are in bijection with simple matroids of rank at most g (see Theorem 2.3.16).

After having defined $M_g^{\rm tr}$ and $A_g^{\rm tr,V}$, we show that the tropical Torelli map

$$\begin{aligned} t_g^{\rm tr} : M_g^{\rm tr} &\to A_g^{\rm tr,V} \\ C &\mapsto {\rm Jac}(C), \end{aligned}$$

is a map of stacky fans (see Theorem 2.4.5).

We then prove a Schottky-type and a Torelli-type theorem for $t_g^{\rm tr}$. The Schottky-type theorem says that $t_g^{\rm tr}$ is a full map whose image is equal to the stacky subfan $A_g^{\rm gr,cogr} \subset A_g^{\rm zon}$, whose cells correspond to cographic simple matroids of rank at most g (see Theorem 2.4.10). The Torelli-type theorem says that $t_g^{\rm tr}$ is of degree one onto its image (see Theorem 2.4.15). Moreover, extending the results of Caporaso and Viviani [CV1] to our generalized tropical curves (i.e. admitting also weights), we determine when two tropical curves have the same Jacobian (see Theorem 2.4.14).

Finally, we define the stacky subfan $M_g^{\rm tr,pl} \subset M_g^{\rm tr}$ consisting of planar tropical curves (see definition 2.5.7) and the stacky subfan $A_g^{\rm gr} \subset A_g^{\rm zon}$ whose cells correspond to graphic simple matroids of rank at most g (see definition 2.5.1). We show that $A_g^{\rm gr}$ is also equal to the closure inside $A_g^{\rm tr,V}$ of the so-called principal cone $\sigma^0_{\rm prin}$ (see Proposition 2.5.4). We prove that $t_g^{\rm tr}(C) \in A_g^{\rm gr}$ if and only if C is a planar tropical curve and that $t_g^{\rm tr}(M_g^{\rm tr,pl}) =$

$A_g^{\mathrm{gr,cogr}} := A_g^{\mathrm{cogr}} \cap A_g^{\mathrm{gr}}$ (see Theorem 2.5.12).

As an application of our tropical results, we study a problem raised by Namikawa in [N80] concerning the extension $\overline{\mathrm{t}}_g$ of the (classical) Torelli map from the Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ of $\mathcal{M}_g$ to the (second) Voronoi toroidal compactification $\overline{\mathcal{A}_g}^V$ of $\mathcal{A}_g$ (see subsection 2.5.3 for more details). More precisely, in Corollary 2.5.13, we provide a characterization of the stable curves whose dual graph is planar in terms of their image via the compactified Torelli map $\overline{\mathrm{t}}_g$, thus answering affirmatively to [N80, Problem (9.31)(i)]. The relation between our tropical moduli spaces M_g^{tr} (resp. $A_g^{\mathrm{tr,V}}$) and the compactified moduli spaces $\overline{\mathcal{M}_g}$ (resp. $\overline{\mathcal{A}_g}^V$) is that there is a natural bijective correspondence between the cells of the former and the strata of the latter; moreover these bijections are compatible with the Torelli maps t_g^{tr} and $\overline{\mathrm{t}}_g$. This allows us to apply our results about t_g^{tr} to the study of $\overline{\mathrm{t}}_g$, providing thus the necessary tools to solve Namikawa's problem.

Chapter 1

Aspects of Brill-Noether theory for singular curves

1.1 Notation

Let us recall some basic facts about the construction in [C1] that we will use in what follows. We work over an algebraically closed field k. Throughout the paper a *curve* will be a reduced projective variety of pure dimension 1 over k. Moreover, we will deal with nodal curves, although some statements are more general. Let then X be a nodal curve, and let $X^\nu \stackrel{\nu}{\longrightarrow} X$ be its normalization; if $X^\nu = \sqcup_{i=1}^{\gamma} C_i^\nu$ is the decomposition of X^ν into smooth components of genus g_i for every $i = 1, \dots, \gamma$, then the arithmetic genus of X is $g = \sum_{i=1}^{\gamma} g_i + \delta - \gamma + 1$. If Z is a subcurve of X of genus g_Z and $Z^c = \overline{X \setminus Z}$, we will denote by $\delta_Z = \sharp Z \cap Z^c$ and if ω_X is the dualizing sheaf of X, we set $\deg_Z \omega_X = \deg \omega_X|_Z = 2g_Z - 2 + \delta_Z$.

A curve X of genus $g \geq 2$ is said to be *stable* if it is connected and if every component $E \cong \mathbb{P}^1$ is such that $\delta_E \geq 3$, which is equivalent to saying that the curve has finite automorphism group. By a *quasistable* curve we mean a connected curve X such that every subcurve $E \cong \mathbb{P}^1$ has $\delta_E \geq 2$ and the ones with $\delta_E = 2$, i.e. the exceptional components, don't intersect. If S is a set of nodes of a stable curve X, throughout the paper we will denote by X_S the normalization of X at the nodes in S, and by $\widehat{X}_S$ the quasistable curve obtained by "blowing up" X at S. In what follows we will often call $\widehat{X}_S$ a partial blow up of X. Obviously X_S is the complement in $\widehat{X}_S$ of all the exceptional components.

In [C1] Caporaso constructs a compactification $\overline{P}_{d,g} \to \overline{M}_g$ of the universal Picard variety, such that the fiber over a smooth curve X of genus $g \geq 2$ is its Picard variety $\mathrm{Pic}^d X$, whereas if X is a stable curve in $\overline{M}_g$, then the fiber over it is $\overline{P_X^d}$, a connected and projective scheme, which has a meaningful description in terms of line bundles on the partial blowups of X.

Indeed, let X be a quasistable curve of genus g and $L \in \mathrm{Pic}^d X$; we denote the multidegree of L by

$$\underline{d} = (d_1, \dots, d_\gamma),$$

where, if $X = \bigcup_{i=1}^{\gamma} C_i$ is the decomposition of X in irreducible components, we have $d_i = \deg L|_{C_i}$ and $d = |\underline{d}|$. We say that $\underline{d}$ is *balanced* if for any connected subcurve Z of X we have that

$$d\frac{w_Z}{2g-2} - \frac{\delta_Z}{2} \le d_i \le d\frac{w_Z}{2g-2} + \frac{\delta_Z}{2}, \tag{1.1}$$

where $w_Z = \deg_Z \omega_X$, and for any exceptional component E of X we have $L|_E = \mathcal{O}_E(1)$.

We say $\underline{d}$ is *strictly balanced* if strict inequalities hold in (1.1) for every $Z \subsetneq X$ such that $Z \cap Z^c \not\subset X_{\text{exc}}$, where X_{exc} is the subcurve of the exceptional components of X (see [C7]). We will denote by $\overline{B_d(X)}$ the set of balanced multidegrees on X, and by $B_d(X)$ its subset of strictly balanced ones.

We are going to introduce the scheme $\overline{P_X^d}$ by looking at its stratification; so let X be a stable curve of genus $g \ge 2$, then, for any d, $\overline{P_X^d}$ is a connected, reduced scheme of pure dimension g, such that

$$\overline{P_X^d} = \coprod_{\substack{\emptyset \subset S \subset X^{\text{sing}} \\ \underline{d} \in B_d(\widehat{X}_S)}} P_S^{\underline{d}}, \tag{1.2}$$

where $P_S^{\underline{d}} \cong \text{Pic}^{\underline{d}_S} X_S$, $X_S \subset \widehat{X}_S$ as above, and $\underline{d}_S = \underline{d}|_{X_S}$. In particular, the points in $\overline{P_X^d}$ are in one-to-one correspondence with equivalence classes of strictly balanced line bundles. Any such class is determined by S and by $M \in \text{Pic} X_S$. Hence a point of $\overline{P_X^d}$ can be denoted by $[M, S]$, where if $\widehat{M_S}$ is a class of line bundles in $B_d(\widehat{X}_S)$, then $M := \widehat{M_S}|_{X_S}$, and, by construction, when restricted to every exceptional component of $\widehat{X}_S$, $\widehat{M_S}$ is equal to $\mathcal{O}(1)$.

A node n of X is said to be *separating* if $X \setminus \{n\}$ is not connected; we denote by X^{sing} the set of nodes of X, and by X_{sep} the subset of separating nodes.

If X is a nodal connected curve of genus g, we will denote by $A_{\underline{d}}(X)$ the closure of $\text{Im}\alpha_X^{\underline{d}}$ inside $\text{Pic}^{\underline{d}} X$. Moreover we define the Brill-Noether variety:

$$W_{\underline{d}}(X) := \{L \in \text{Pic}^{\underline{d}} X : h^0(L) > 0\}.$$

Let $\nu_S : X_S \to X$ be the normalization of X at the nodes in S. It induces the pullback map $\nu_S^* : \text{Pic}^{\underline{d}} X \to \text{Pic}^{\underline{d}} X_S$; if $M \in \text{Pic}^{\underline{d}} X_S$, we denote by $F_M(X)$ the fiber of ν_S^* over M, and by $W_M(X)$ the intersection $F_M(X) \cap W_{\underline{d}}(X)$.

1.2 A compactification of the image of the Abel Map

Let $\alpha_X^{\underline{d}}$ be the Abel map of multidegree $\underline{d}$ of a stable curve X of genus $g \ge 2$. We want to describe the closure of $\text{Im}\alpha_X^{\underline{d}}$ inside the compactified Jacobian $\overline{P_X^d}$ constructed in [C1]. We start by examining the case of irreducible curves.

1.2.1 Irreducible curves

Let X be an irreducible nodal curve of genus g and, for $d \geq 1$, consider the Brill-Noether variety $W_d(X)$. As a subvariety of $\mathrm{Pic}^d X$, we are interested in studying its closure $\overline{W_d(X)}$ in the compactified Picard Variety $\overline{P_X^d}$, using the description given in [C1]. It will turn out that $\overline{W_d(X)}$ is strongly related to the image of the Abel map, that we are going to define. Let $\dot{X} := X \setminus X^{\mathrm{sing}}$ be the smooth locus of X; since X is irreducible, we have that $\dot{X}^d$ is a smooth irreducible variety of dimension d, open and dense in X^d. Now, for $d \geq 1$, let

$$\begin{array}{cccc} \alpha_X^d : & \dot{X}^d & \longrightarrow & \mathrm{Pic}^d X \\ & (p_1, \ldots, p_d) & \mapsto & \mathcal{O}_X(\sum\limits_{i=1}^{d} p_i); \end{array}$$

we call α_X^d the *Abel map of degree* d. It is a regular map, and obviously $\alpha_X^d(\dot{X}^d) \subset W_d(X)$. We denote by $A_d(X)$ the closure of $\alpha_X^d(\dot{X}^d)$ in $\mathrm{Pic}^d X$; of course $A_d(X) \subset W_d(X)$. Let us now introduce the following set

$$\widetilde{W}_d(X) := \{[M, S] \in \overline{P_X^d} \text{ s.t. } h^0(\widehat{X}_S, \widehat{M}_S) > 0\},$$

where $S \subset X^{\mathrm{sing}}$ with $\delta_S := \sharp S$, $\widehat{X}_S = X_S \cup \cup_{i=1}^{\delta_S} E_i$ is the blow up of X at the nodes of S, and, as we introduced in the previous section, $\widehat{M}_S$ is a class of line bundles in $B_d(\widehat{X}_S)$ such that its resctrictions to the components of $\widehat{X}_S$ are

$$\widehat{M}_S|_{X_S} =: M, \quad \widehat{M}_S|_{E_i} = \mathcal{O}(1) \text{ for any } i = 1, \ldots \delta_S.$$

Let us observe that since $h^0(\widehat{X}_S, \widehat{M}_S) = h^0(X_S, M)$ (see [C2][Lemma 4.2.5]), we have:

$$\widetilde{W}_d(X) = \{[M, S] \in \overline{P_X^d} \text{ s.t. } h^0(X_S, M) > 0\},$$

which is in turn equivalent to:

$$\widetilde{W}_d(X) \cong \bigsqcup_{S \subset X^{\mathrm{sing}}} W_{d-\delta_S}(X_S).$$

Theorem 1.2.1. *Let X be an irreducible curve of genus $g \geq 1$ with δ nodes. Then for any $d \geq 1$ we have:*

(i) $A_d(X) = W_d(X)$*, hence* $W_d(X)$ *is irreducible and* $\dim W_d(X) = \min\{d, g\}$*,*

(ii) $\overline{A_d(X)} = \overline{W_d(X)} = \widetilde{W}_d(X) \subset \overline{P_X^d}$.

Proof. We start by assuming that X has only one node n, and its normalization is $\nu : X_n \to X$, with $\nu^{-1}(n) = \{p, q\}$. Let us consider the regular dominant map

$$\begin{array}{cccc} \rho : & W_d(X) & \to & W_d(X_n) \\ & L & \mapsto & \nu^*(L); \end{array}$$

for any $M \in \mathrm{Im}\rho$ we denote by $W_M(X) = \rho^{-1}(M)$, the fiber of ρ. We recall that $W_M(X) \subset F_M(X)$, where $F_M(X) \cong k^*$ is the fiber of the pullback map $\nu^* : \mathrm{Pic}^d X \to \mathrm{Pic}^d X_n$. The cardinality of the fibers $W_M(X)$ is at least 0, so, since $\dim W_d(X_n) = d$, it follows that $\dim W_d(X) \leq d$; moreover $A_d(X)$ is irreducible of dimension d, hence we have that $A_d(X)$ is an irreducible component of $W_d(X)$. We want to prove that for any $M \in \mathrm{Im}\rho$, $W_M(X) \subset \overline{A_d(X)}$, so that $A_d(X) \subset W_d(X) \subset \overline{A_d(X)}$ implies that $W_d(X) = A_d(X)$ and $\overline{W_d(X)} = \overline{A_d(X)}$. We are now going to analyze all the possible cases.

(1) $M \in \mathrm{Im}\rho$ with $h^0(X_n, M) = 1$ and

$$h^0(X_n, M(-p)) = h^0(X_n, M(-q)) = h^0(X_n, M) - 1.$$

Then by [C2, Lemma 2.2.3], $W_M(X) = \{L_M\}$ with $L_M \in \mathrm{Im}\alpha_X^d$.

(2) $M \in \mathrm{Im}\rho$ with $h^0(X_n, M) \geq 2$ and

$$h^0(X_n, M(-p)) = h^0(X_n, M(-q)) = h^0(X_n, M) - 1.$$

We are going to show that there exist two points in $\overline{F_M(X)} \subset \overline{P_X^d}$ which are contained in $\overline{A_d(X)}$. Indeed,

$$\overline{F_M(X)} \setminus F_M(X) = \{[M(-p), n], [M(-q), n]\}.$$

Let us take $[M(-p), n]$; by [C2, Lemmas 2.2.3, 2.2.4] there exists $L \in \mathrm{Pic}^{d-1}X$ such that $\nu^*(L) = M(-p)$ and $L \in \mathrm{Im}\alpha_X^{d-1}$. Let now $p_t \in \dot{X}$ be a moving point specializing to the node, i.e.such that $p_t \overset{t\to 0}{\longrightarrow} n$. Of course $L(p_t) \in \mathrm{Im}\alpha_X^d$, and $L(p_t) \to [M(-p), n]$ as $t \to 0$. Then $[M(-p), n] \in \overline{A_d(X)}$. The same holds for $[M(-q), n]$, so we have that

$$\overline{F_M(X)} \setminus F_M(X) \subset \overline{A_d(X)}.$$

(3) $M \in \mathrm{Im}\rho$ with $h^0(X_n, M) = 1$ and

$$h^0(X_n, M(-p)) = h^0(X_n, M(-q)) = h^0(X_n, M).$$

Again we want to prove that $\overline{F_M(X)} \setminus F_M(X) \subset \overline{A_d(X)}$; so let M' be a line bundle on X_n not supported on either p or q such that $M = M'(hp + kq)$; then M' is as in (1) and $\deg M' = d'$ with $d' = d - (h + k)$. Let us consider $[M(-p), n] \in \overline{F_M(X)}$, then $M(-p) = M'(h'p + kq)$, where $h' = h - 1$. We choose a moving point p_t on X_n specializing to p as t goes to 0, and a moving point q_t on X_n such that q_t specializes to q. Now fix t, and take the line bundle $M_t'' := M'(h'p_t + kq_t)$ on X_n; by case (1), there exists $L_t'' \in \mathrm{Im}\alpha_X^{d-1}$ such that $\nu^*(L_t'') = M_t''$. We consider now one moving point $p_u \in \dot{X}$, such that $\nu^*(p_u)$ on X_n specializes to p when $u \to 0$. As well as we saw in case (2), $L_t''(p_u) \in \mathrm{Im}\alpha_X^d$ specializes to $[M_t'', n]$ as $u \to 0$. Hence $[M_t'', n] \in \overline{A_d(X)}$. Now let $t \to 0$: we see that, by construction, $[M_t'', n] \to [M(-p), n]$, hence $[M(-p), n] \in \overline{A_d(X)}$. Using the same argument, we get that $[M(-q), n] \in \overline{A_d(X)}$ as well.

(4) $M \in \mathrm{Im}\rho$ with $h^0(X_n, M) \geq 2$ and either p or q as base point. Choose, say, p as base point, i.e. $h^0(X_n, M(-p)) = h^0(X_n, M) = h^0(X_n, M(-q)) + 1$. Then there exists

$M' \in \mathrm{Pic}^{d'} X_n$, with $M = M'(hp)$, $d' = d - h$, and M' not supported on either p or q up to move the support away. We notice that $M(-p) = M'(h'p)$ with $h' = h - 1$, so, as before, we perform a double specialization to show that $[M(-p), n] \in \overline{A_d(X)}$. Concerning $[M(-q), n]$, we have that $M(-q) = M'(hp - q) =: M''(hp)$ for a suitable $M'' \in \mathrm{Pic}^{d'-1} X_n$. Moreover, since p is a base point of $M''(hp)$, $h^0(X_n, M'') \geq 1$. We take again a moving point p_t on X_n specializing to p, and a p_u on X such that $\nu^*(p_u)$ specializes to p on X_n. We fix t and denote $M''_t := M''(hp_t)$, then by [C2, Lemmas 2.2.3,2.2.4] there exists L''_t contained in $\mathrm{Im}\alpha_X^{d'-1}$ such that $\nu^*(L''_t) = M''_t$. We take $L''_t(p_u)$; letting $u \to 0$ we get that $L''_t(p_u) \to [M''_t, n] \in \overline{A_d(X)}$. Now we let $t \to 0$, and obtain $[M''_t, n] \to [M(-q), n]$, whence $[M(-q), n] \in \overline{A_d(X)}$.

(5) $M \in \mathrm{Im}\rho$ with $h^0(X_n, M) \geq 2$ and $h^0(X_n, M(-p)) = h^0(X_n, M(-q)) = h^0(X_n, M)$. Then there exists $M' \in \mathrm{Pic}^{d'} X_n$, with $M = M'(hp + kq)$, $d' = d - (h + k)$, and M' not supported on either p or q up to move the support away. As well as above, we consider $[M(-p), n]$ and $[M(-q), n]$ to show that they are contained in $\overline{A_d(X)}$. We proceed as in case (3) performing a double specialization, and recalling that $h^0(X_n, M') \geq 2$ by assumption.

Let $U \subset W_d(X_n)$ be the following set:

$$U := \{M \in W_d(X_n) \text{ s.t. } h^0(X_n, M) = 1, h^0(X_n, M(-p)) = h^0(X_n, M(-q)) = 0\};$$

this is of course an open set in $W_d(X_n)$, and it contains all the line bundles M studied in case (1). In particular for any $M \in U$, we have that $\overline{A_d(X)}$ intersects $\overline{F_M(X)}$ in only one point L_M, where $W_M(X) = \{L_M\}$. In order to verify this assertion, by (1) we just have to check that $[M(-p), n]$ and $[M(-q), n]$ are not contained in $\overline{A_d(X)}$, but this is obvious, since $h^0(X_n, M(-p)) = 0$, hence on the blow up $\hat{X}_n$ of X at n, $h^0(\hat{X}_n, \widehat{M(-p)}) = 0$. From the study of all the possibilities above, from (2) to (5), we get that for any $M \in \mathrm{Im}\rho$ which is not in U, $\overline{A_d(X)}$ contains at least two points of $\overline{F_M(X)}$, but since the generic M has $\sharp(\overline{F_M(X)} \cap \overline{A_d(X)}) = 1$, we have that for $M \in \mathrm{Im}\rho \setminus U$, the whole $\overline{F_M(X)}$ must be contained in $\overline{A_d(X)}$, hence for any $M \in \mathrm{Im}\rho$ we have that $\overline{W_M(X)} \subset \overline{A_d(X)}$.

So we have shown that $W_d(X) = A_d(X)$, with subsequent equality of their closures. In order to show that $\overline{W_d(X)} = \widetilde{W}_d(X)$, we argue like this: direction $\subset$ is obvious, since $\widetilde{W}_d(X)$ is a closed set in $\overline{P}_X^d$ containing $W_d(X)$. On the other hand, the analysis made above suggests that any $[N, n] \in \widetilde{W}_d(X)$ is also an element of $\overline{A_d(X)}$. Indeed if N has p and/or q as base points, we argue as in (3),(4),(5); if otherwise N does not contain p nor q in its support, by (2) we get that there exists $L(p_t) \in \mathrm{Im}\alpha_X^d$, such that $L(p_t)$ specializes to $[N, n]$ as $t \to 0$.

If the number of nodes δ is ≥ 2, we proceed by induction on δ. Indeed, let X be a nodal irreducible curve having δ nodes. We blow up X at one node n, so that $\hat{X}_n$ is the blown up curve, and X_n is the strict transform, and we have the normalization map $\nu : X_n \to X$ such that $\nu^{-1}(n) = \{p, q\}$. So again we look at the dominant morphism $\rho : W_d(X) \to W_d(X_n)$, and we prove that the fibers $W_M(X) \subset \overline{A_d(X)}$ for any $M \in \mathrm{Im}\rho$. As inductive hypothesis we assume that $W_d(X_n) = A_d(X_n)$ is irreducible of dimension d. This is the only point where we used the smoothness of X_n in the previous case when $\delta = 1$; hence reapplying the argument above, which is based on [C2, Lemmas 2.2.3,2.2.4], we get the conclusions for every δ and for every $d \geq 1$. $\square$

Remark 1.2.2. We observe that when $d \geq g$, with g the genus of X, it doesn't make sense referring to $W_d(X)$, since it is equal to $\mathrm{Pic}^d X$. On the other hand, when $d = 1$ we have that by [C2, Lemma 2.2.3], $W_1(X) = \mathrm{Im}\alpha^1_X = A_1(X)$, and when $d = g-1$ we get that the Theta divisor is irreducible in $\mathrm{Pic}^{g-1}X$.

Remark 1.2.3. From the equality $A_d(X) = W_d(X)$ for any d, we deduce an important fact; we use the previous notation, where X has δ nodes and X_n is the normalization at a node n. Let $L \in W_d(X)$ be such that $M = \nu^* L$ has $W_M(X) = F_M(X)$. Then $k^* = W_M(X)$, and we can denote its elements in the following way:

$$W_M(X) = \{L^c, c \in k^*\}.$$

By 1.2.1 we have that for any $c \in k^*$ there exists a family $L^c_t \in \mathrm{Im}\alpha^d_X$ such that $L^c_t \to L^c$. In particular, we will have that $L^c_t = \tilde{L}^c_t(hp^c_t + kq^c_t)$ for suitable $h, k, p^c_t, q^c_t \in \dot{X}$ such that $\nu^*(p^c_t)$ specializes to p on X_n, $\nu^*(q^c_t)$ specializes to q, and $\tilde{L}^c_t$ specializes to some effective line bundle on X not supported on n. Hence we can assume $\tilde{L}^c_t = \tilde{L}^c$ not depending on t; so, for any $c \in k^*$, we have $\tilde{L}^c(hp^c_t + kq^c_t) \to L^c$. If $\tilde{L}^c$ is such that no other effective line bundle is in its fiber, we have that $\tilde{L}^c = \tilde{L}$, and $\tilde{L}(hp^c_t + kq^c_t) \to L^c$, so in this case the limit depends only upon the choice of the moving points p^c_t and q^c_t. Equivalently, if $c \neq c'$ in k^*, there exist moving points p^c_t, q^c_t and $p^{c'}_t, q^{c'}_t$ such that $\tilde{L}(hp^c_t + kq^c_t) \to L^c$ and $\tilde{L}(hp^{c'}_t + kq^{c'}_t) \to L^{c'}$.

1.2.2 Reducible curves

Very little is known about Abel maps of reducible curves, even if recently a lot of effort has been put into studying the class of stable curves, see for example [C2], [C5], [C6], [Co07],[CP09]. We are going to study the relation among the varieties $W_{\underline{d}}(X)$, $A_{\underline{d}}(X)$ and their closures in $\overline{P^d_X}$. Let X be a reducible curve with components $C_1, \ldots, C_\gamma$; for any $\underline{d} = (d_1, \ldots, d_\gamma) \in \mathbb{Z}^\gamma$ with $|\underline{d}| = d$, we can consider the Brill-Noether variety $W_{\underline{d}}(X)$ that we defined in the introduction of the paper. Obviously if $d_i < 0$ for every $i = 1, \ldots, \gamma$, we get that $W_{\underline{d}}(X) = \emptyset$. On the other hand, if we assume $\underline{d} \geq \underline{0}$, i.e. $d_i \geq 0$ for every i, we can define the *Abel map of multidegree* $\underline{d}$. Set $\dot{X} := X \setminus X^{\text{sing}}$, and $\dot{C}_i = C_i \cap \dot{X}$; we define

$$\dot{X}^{\underline{d}} := \dot{C}_1^{d_1} \times \ldots \times \dot{C}_\gamma^{d_\gamma} \subset X^{\underline{d}} := C_1^{d_1} \times \ldots \times C_\gamma^{d_\gamma},$$

and

$$\begin{array}{cccc} \alpha^{\underline{d}}_X : & \dot{X}^{\underline{d}} & \longrightarrow & \mathrm{Pic}^{\underline{d}} X \\ & (p_1, \ldots, p_d) & \mapsto & \mathcal{O}_X(\sum_{i=1}^d p_i). \end{array}$$

As in the irreducible case, we denote by $A_{\underline{d}}(X)$ the closure of the set $\mathrm{Im}\alpha_X^{\underline{d}} \subset \mathrm{Pic}^{\underline{d}}X$. We are now going to introduce a set which will be crucial hereafter.

$$W_{\underline{d}}^{+}(X) := \{L \in \mathrm{Pic}^{\underline{d}}X \text{ s.t. } h^0(Z, L|_Z) > 0 \text{ for any subcurve } Z \subseteq X\}. \tag{1.3}$$

This definition suggests the following

Lemma 1.2.4. *Let $\underline{d} \geq \underline{0}$ be a multidegree on a reducible curve X. Then*

$$A_{\underline{d}}(X) \subset W_{\underline{d}}^{+}(X).$$

Proof. The proof is straightforward: the line bundles in $\mathrm{Im}\alpha_X^{\underline{d}}$ are of the form $\mathcal{O}_X(\sum_{i=1}^{d} p_i)$, hence their restriction to any subcurve of X has nonzero sections. Then by upper semicontinuity of the dimension of the H^0 this is still true for their limits in $A_{\underline{d}}(X)$. □

We start by studying the simplest case, i.e. when X is a curve of compact type.

Curves of compact type

When X is a curve of compact type, for any multidegree $\underline{d}$ we have that $\mathrm{Pic}^{\underline{d}}X$ is complete, hence so is $W_{\underline{d}}(X)$. However we are interested in the relation between $A_{\underline{d}}(X)$ and $W_{\underline{d}}(X)$. We start by assuming that X has two smooth components C_1, C_2 meeting at one node n, hence its normalization is the disconnected curve

$$C_1 \sqcup C_2 \xrightarrow{\nu} X,$$

with $\nu^{-1}(n) = \{p, q\}$. This induces the pullback map

$$\mathrm{Pic}^{(d_1,d_2)}X \xrightarrow{\nu^*} \mathrm{Pic}^{d_1}C_1 \times \mathrm{Pic}^{d_2}C_2,$$

which is an isomorphism, and given $L \in W_{\underline{d}}(X)$, we denote $(L_1, L_2) := \nu^*(L)$. We define the sets:

$$\begin{aligned} W_{\underline{d}}^{+}(X) :=& \quad \{L \in W_{\underline{d}}(X) \text{ s.t. } h^0(C_1, L_1) > 0, h^0(C_2, L_2) > 0\}, \\ W_{\underline{d}}^{+-}(X) :=& \quad \{L \in W_{\underline{d}}(X) \text{ s.t. } h^0(C_1, L_1) > 0, h^0(C_2, L_2) = 0\}, \\ W_{\underline{d}}^{-+}(X) :=& \quad \{L \in W_{\underline{d}}(X) \text{ s.t. } h^0(C_1, L_1) = 0, h^0(C_2, L_2) > 0\}; \end{aligned} \tag{1.4}$$

of course we have that $W_{\underline{d}}(X) = W_{\underline{d}}^{+}(X) \sqcup W_{\underline{d}}^{+-}(X) \sqcup W_{\underline{d}}^{-+}(X)$ set-theoretically.

Proposition 1.2.5. *Let X be a curve of compact type of genus g with two smooth components C_1, C_2 of genus resp. g_1, g_2. Let $\underline{d} \geq \underline{0}$ be a multidegree with $|\underline{d}| = d$ such that $1 \leq d \leq g-1$. We have:*

(i) if $d_1 \leq g_1 - 1$ and $d_2 \leq g_2 - 1$, then $W_{\underline{d}}(X)$ is connected and has 3 irreducible components, of dimensions $d, d_1 + g_2 - 1, d_2 + g_1 - 1$,

(ii) if $d_1 \geq g_1$ and $d_2 \leq g_2 - 1$ (up to swapping the indices), $W_{\underline{d}}(X)$ is connected and has 2 irreducible components.

Proof. In order to prove (i) we assume that $d_1 \leq g_1 - 1$ and $d_2 \leq g_2 - 1$. We consider the pullback map

$$\begin{array}{rccc} \nu^*: & \mathrm{Pic}^{\underline{d}} X & \xrightarrow{\cong} & \mathrm{Pic}^{d_1} C_1 \times \mathrm{Pic}^{d_2} C_2 \\ & L & \mapsto & (L_1, L_2); \end{array}$$

then by [C2, 2.1.1] using that $\delta = 1$,

$$W_{\underline{d}}^{+}(X) = (\nu^*)^{-1}(W_{d_1}(C_1) \times W_{d_2}(C_2)). \tag{1.5}$$

Now since C_1, C_2 are smooth curves, we have that $W_{d_i}(C_i)$ is irreducible of dimension d_i for $i = 1, 2$. Then $W_{\underline{d}}^{+}(X)$ is a closed irreducible set containing $A_{\underline{d}}(X)$. Since the fibers of ν^* have cardinality one, $\dim W_{\underline{d}}^{+}(X) = d$. By definition we know that $\mathrm{Im}\alpha_X^{\underline{d}} = (\nu^*)^{-1}(\mathrm{Im}\alpha_{C_1}^{d_1} \times \mathrm{Im}\alpha_{C_2}^{d_2})$, hence $\dim \mathrm{Im}\alpha_X^{\underline{d}} = d$, then $A_{\underline{d}}(X) = W_{\underline{d}}^{+}(X)$ and they both have dimension d.

The other two components of $W_{\underline{d}}(X)$ are the following ones: consider $L \in W_{\underline{d}}^{+-}(X)$; we have that $h^0(C_2, L_2) = 0$, and since L has nonzero sections, we have $h^0(C_1, L_1(-p)) > 0$. As in [C3] we define the set

$$\Lambda_p := \{L_1 \in \mathrm{Pic}^{d_1} C_1 \text{ s.t. } h^0(C_1, L_1(-p)) > 0\}, \tag{1.6}$$

and consider the isomorphism

$$\begin{array}{rccc} \phi_p: & \mathrm{Pic}^{d_1-1} C_1 & \longrightarrow & \mathrm{Pic}^{d_1} C_1 \\ & M & \mapsto & M(p). \end{array} \tag{1.7}$$

It is easy to see that $\Lambda_p = \phi_p(W_{d_1-1}(C_1))$, hence Λ_p is closed and irreducible of dimension $d_1 - 1$. Now consider the set

$$\overline{W_{\underline{d}}^{+-}}(X) := (\nu^*)^{-1}(\Lambda_p \times \mathrm{Pic}^{d_2} C_2);$$

it contains

$$W_{\underline{d}}^{+-}(X) = (\nu^*)^{-1}(\Lambda_p \times (\mathrm{Pic}^{d_2} C_2 \setminus W_{d_2} C_2))$$

as an open set, and $\dim \overline{W_{\underline{d}}^{+-}}(X) = d_1 + g_2 - 1$.

The last irreducible component of $W_{\underline{d}}(X)$ is the one containing the L's such that $h^0(C_1, L_1) = 0$ and $h^0(C_2, L_2) \neq 0$. Arguing as before, we define the set $\Lambda_q \subset \mathrm{Pic}^{d_2} C_2$, and the isomorphism $\phi_q : \mathrm{Pic}^{d_2-1} C_2 \to \mathrm{Pic}^{d_2} C_2$ sending $N \in \mathrm{Pic}^{d_2-1} C_2$ to $N(q)$. Hence $\Lambda_q = \phi_q(W_{d_2-1}(C_2))$, and the set

$$\overline{W_{\underline{d}}^{-+}}(X) := (\nu^*)^{-1}(\mathrm{Pic}^{d_1} C_1 \times \Lambda_q)$$

is the closure of $W_{\underline{d}}^{-+}(X)$, with $\dim \overline{W_{\underline{d}}^{-+}}(X) = d_2 + g_1 - 1$. Hence we have that

$$W_{\underline{d}}(X) = A_{\underline{d}}(X) \cup \overline{W_{\underline{d}}^{+-}}(X) \cup \overline{W_{\underline{d}}^{-+}}(X),$$

and their intersection is $(\nu^*)^{-1}(\Lambda_p \times \Lambda_q)$, having dimension $d_1 - 1 + d_2 - 1 = d - 2$. This implies that $W_{\underline{d}}(X)$ is connected.

Part (ii) comes from part (i), once we have noticed that if $d_1 \geq g_1$ and $d_2 \leq g_2 - 1$, then $h^0(C_1, L_1) > 0$, so $W_{\underline{d}}^{-+}(X) = \emptyset$. Hence

$$W_{\underline{d}}(X) = W_{\underline{d}}^{+}(X) \cup \overline{W}_{\underline{d}}^{+-}(X),$$

and their intersection is $(\nu^*)^{-1}(\Lambda_p)$, having dimension $d_1 - 1$. We notice that in this case by (1.5), $\dim W_{\underline{d}}^{+}(X) = g_1 + d_2$, which can be less than d. We prove that even in this case it holds that $W_{\underline{d}}^{+}(X) = A_{\underline{d}}(X)$. Indeed, inclusion ($\supset$) is obvious, and concerning ($\subset$), let us take a line bundle $L \in W_{\underline{d}}^{+}(X)$. Then we look at its pullback $M = \nu^*(L)$. Let $M = (\mathcal{O}_{C_1}(D_1 + \lambda p), \mathcal{O}_{C_2}(D_2 + \mu q))$ for some suitable divisors D_1 and D_2; we choose moving points p_t on $C_1 \cap X$ and q_t on $C_2 \cap X$, specializing resp. to p and q. We consider on $C_1 \sqcup C_2$ the line bundle:

$$M_t := (\mathcal{O}_{C_1}(D_1 + \lambda p_t), \mathcal{O}_{C_2}(D_2 + \mu q_t)),$$

and push it down to X, getting the (unique) line bundle $L_t \in \operatorname{Im}\alpha_X^{\underline{d}}$ such that $\nu^*(L_t) = M_t$. Then if we let t tend to 0, we get that L_t specializes to L, and hence that $L \in A_{\underline{d}}(X)$. So we conclude that $W_{\underline{d}}^{+}(X) = A_{\underline{d}}(X)$. It follows that $\dim W_{\underline{d}}(X) = \max\{g_1 + d_2, d_1 + g_2 - 1\}$. If vice-versa $d_2 \geq g_2$ and $d_1 \leq g_1 - 1$, we have that $W_{\underline{d}}^{+-}(X) = \emptyset$, $W_{\underline{d}}(X) = A_{\underline{d}}(X) \cup \overline{W}_{\underline{d}}^{-+}(X)$, $A_{\underline{d}}(X) \cap \overline{W}_{\underline{d}}^{-+}(X) = (\nu^*)^{-1}(\Lambda_q)$, and $\dim W_{\underline{d}}(X) = \max\{d_1 + g_2, d_2 + g_1 - 1\}$. □

Remark 1.2.6. We just observe that the case $d = g-1$ is carried out in [C2], but we obtain it as a by-product in 1.2.5(ii); since there are no strictly balanced multidegrees summing to $g-1$ on a curve of compact type, we get that $W_{\underline{d}}(X)$ is not irreducible.

In the sequel we will try to generalize our study to any curve of compact type, so take X as the union of irreducible smooth curves $C_1, \ldots, C_\gamma$, with g_i the genus of C_i and g the genus of X. Notice that since X is of compact type, we have that $\sharp(C_i \cap C_j) = 1$ for $i \neq j$, and this implies that the total number of nodes $\delta \leq \gamma - 1$; we denote by n_{ij} the intersection point $C_i \cap C_j$. Let $\underline{d} \geq \underline{0}$ be a multidegree on X, with $|\underline{d}| = d$, $1 \leq d \leq g-1$. Let

$$\nu : \bigsqcup_{i=1}^{\gamma} C_i \to X$$

be the total normalization map, ν^* the pullback as before, and denote by $(L_1, \ldots, L_\gamma)$ the pullback to $\bigsqcup_{i=1}^{\gamma} C_i$ of any $L \in \operatorname{Pic}^{\underline{d}} X$. If n_{ij} is a node, its branches on C_i, C_j will be called respectively p_j^i, p_i^j, distinguishing the curve they belong to by the position of indices.

Lemma 1.2.7. *Let X be a connected curve of compact type as above and $\underline{d} \geq \underline{0}$. Then $W_{\underline{d}}^{+}(X) = A_{\underline{d}}(X)$, is a (closed) irreducible component of $W_{\underline{d}}(X)$.*

Proof. The proof is straightforward: we see that, as we pointed out in the case $\gamma = 2$,

$$W_{\underline{d}}^{+}(X) = (\nu^*)^{-1}(W_{d_1}C_1 \times \cdots \times W_{d_\gamma}C_\gamma), \tag{1.8}$$

indeed X has a number of nodes $\delta = \gamma - 1$, so we apply [C2, 2.1.1] and obtain the equality. Since C_i is smooth for every i, by (1.8) $W_{\underline{d}}^{+}(X)$ turns out to be a closed irreducible set of dimension $d_1+\cdots+d_\gamma=d$, and it contains $A_{\underline{d}}(X)$. To see the inverse inclusion we argue as in 1.2.5(ii), proving that for any $L\in W^+_{\underline{d}}(X)$ there exists $L_t\in \mathrm{Im}\alpha^{\underline{d}}_X$ such that, if we let t tend to 0, we get that L_t specializes to L. So we have $W^+_{\underline{d}}(X)=A_{\underline{d}}(X)$ as we wanted. $\square$

What we are going to do now is to study the remaining irreducible components of $W_{\underline{d}}(X)$. To do this we need to introduce some notation: let $\delta_i=\sharp(C_i\cap\overline{X\setminus C_i})$ for $i=1,\dots,\gamma$, and let $\underline{I}$ be a $1\times\gamma$ vector where the j-th component is $I_j=+$ or $I_j=-$. Then we can define the set:

$$W^{\underline{I}}_{\underline{d}}(X):=\left\{L\in W_{\underline{d}}(X)\text{ s.t. } h^0(C_j,L_j)=0 \text{ if } I_j=-, \text{ and } h^0(C_j,L_j)>0 \text{ if } I_j=+\right\}.$$

Notice that if $I_j=+$ for every j, i.e. $\underline{I}=(+,\dots,+)$, we get $W^+_{\underline{d}}(X)$. Let us fix some vector $\underline{I}\neq(+,\dots,+)$; set

$$I^+:=\{j\in\{1,\dots,\gamma\},\, I_j=+\},$$

and

$$I^-:=\{h\in\{1,\dots,\gamma\},\, I_h=-\}.$$

We denote by p^j_h the branch on C_j of the point of intersection $C_j\cap C_h$, for $j\in I^+$, and some $h\in I^-$, if it exists. Moreover, we fix $j\in I^+$ and consider the disconnected curve $\overline{X\setminus C_j}=X^j_1\sqcup\cdots\sqcup X^j_{k_j}$. We observe that C_j has only one point of intersection with each X^j_l, for $l=1,\dots,k_j$. We denote the branches of this point on C_j and X^l_j resp. by p^j_l and p^l_j. If $(L_1,\dots,L_\gamma)$ are the restrictions of a line bundle L on X to each irreducible component of X, we denote by $L_{X^l_j}$ the restriction of L to the connected component X^l_j. Set:

$$\mathcal{L}_j:=\left\{l\in\{1,\dots,k_j\},\;\; h^0(L_{X^l_j})=0\right\},$$

and let

$$\Lambda_j:=\{L_j\in W_{d_j}(C_j),\, h^0(L_j(-\sum_{l\in\mathcal{L}_j}p^j_l))>0\}.$$

Now, still for $j\in I^+$, consider the set:

$$\widetilde{\Sigma}_j:=\left(\prod_{h\in I^-}(\mathrm{Pic}^{d_h}C_h\setminus W_{d_h}(C_h))\times\Lambda_j\times\prod_{l\in I^+,l\neq j}W_{d_l}(C_l)\right)\subset\prod_{i=1}^{\gamma}\mathrm{Pic}^{d_i}C_i. \tag{1.9}$$

and denote by Σ_j the set obtained from $\widetilde{\Sigma}_j$ by reordering the factors in such a way that the final order in Σ_j corresponds to the order of the components of $\underline{I}$, so for example, Λ_j will be the factor in the position of j in $\underline{I}$. We will denote by

$$\Sigma_{\underline{I}}=\bigcup_{j\in I^+}\Sigma_j. \tag{1.10}$$

We observe that Λ_j is irreducible (see 1.2.5), and its dimension depends on the cardinality λ_j of $\mathcal{L}_j$. Indeed $\dim(\Lambda_j)=d_j-\lambda_j$ and $0\leq\lambda_j\leq k_j$. It follows that Σ_j is irreducible for every $j\in I^+$.

Lemma 1.2.8. *We have that* $(\nu^*)^{-1}(\Sigma_{\underline{I}}) = W_{\underline{d}}^{\underline{I}}(X)$.

Proof. Inclusion ($\subset$) is easy by definition of $\Sigma_{\underline{I}}$, since an element of $(\nu^*)^{-1}(\Sigma_{\underline{I}})$ must have at least a nonzero section. On the other hand, given a line bundle $L \in W_{\underline{d}}^{\underline{I}}(X)$, we want to prove that $\nu^*(L) = (L_1, \dots, L_\gamma)$ belongs to $\Sigma_{\underline{I}}$. If for every $i \in I^+$, $h^0(L_{X_i^l}) \neq 0$ for every $l = 1, \dots, k_i$, then $\mathcal{L}_i = \emptyset$ and $\Lambda_i = W_{d_i}(C_i)$ for every i, hence in this case

$$\Sigma = \prod_{h \in I^-} (\mathrm{Pic}^{d_h} C_h \setminus W_{d_h}(C_h)) \times \prod_{l \in I^+} W_{d_l}(C_l)$$

up to reordering the factors in the left hand side, and therefore $\nu^*(L) \in \Sigma$. Now, assume that there exists $i \in I^+$ such that $\mathcal{L}_i \neq \emptyset$. Without loss of generality we can assume that $|\mathcal{L}_i| = 1$. Then in order to glue the sections and get a line bundle on X, it must be $h^0(L_i(-p_l^i)) > 0$, hence $L_i \in \Lambda_i$, and therefore

$$(L_1, \dots, L_\gamma) \in \Sigma_j \subset \Sigma_{\underline{I}}.$$

□

Even if we can't say precisely which is the dimension of the components of $W_{\underline{d}}^{\underline{I}}(X)$, we can count how many they are. By (1.10) we see that for any fixed $\underline{I}$, the number of irreducible components of $W_{\underline{d}}^{\underline{I}}(X)$ is $|I^+|$. Hence we can say that the number of irreducible components of $W_{\underline{d}}(X)$ is

$$\mathcal{N} := \left(1 + \sum_{\underline{I} \neq (+,\dots,+)} |I^+|\right). \tag{1.11}$$

Remark 1.2.9. We notice that depending on $\underline{d}$, some $\underline{I}$'s won't appear in (1.11); indeed, if there exists some $k \in \{1, \dots, \gamma\}$ such that $d_k \geq g_k$, then the component I_k of $\underline{I}$ must be $+$, so we will have a small number of $\underline{I}$'s, and hence a small number of irreducible components in $W_{\underline{d}}(X)$. Moreover, if $d_k \geq g_k$ for every k, we get that the only irreducible component of $W_{\underline{d}}(X)$ is $W_{\underline{d}}^{+}(X)$.

Binary curves

A binary curve of genus g is a nodal curve made of two smooth rational components intersecting at $g+1$ points. We are going to recall some properties that we will use throughout this paragraph. If X is a binary curve of genus $g \geq -1$, a multidegree $\underline{d} = (d_1, d_2)$ such that $|\underline{d}| = d$, is balanced on X if

$$m(d,g) := \frac{d-g-1}{2} \leq d_i \leq \frac{d+g+1}{2} =: M(d,g) \tag{1.12}$$

We say that $\underline{d}$ is strictly balanced if strict inequality holds. If $\widehat{X}_S$ is a quasistable curve obtained from a binary curve X by blowing up the nodes in S, then we call $E_1, \dots, E_{\sharp S}$ the exceptional components, so that if X_S is the partial normalization of X at the nodes in S, we have that $\widehat{X}_S = X_S \cup \cup_{i=1}^{\sharp S} E_i$.

Definition 1.2.10. A multidegree $\widehat{\underline{d}} = (d_1, \ldots, d_{2+\sharp S})$ on $\widehat{X}_S$ with $|\widehat{\underline{d}}| = d$, is balanced if the following hold:
(1) $d_i = 1$ for any $i = 3, \ldots, \sharp S$, i.e. $\widehat{\underline{d}}|_{E_i} = 1, \forall i$.
(2) $\widehat{\underline{d}}|_{X_S}$ is balanced on X_S.
$\widehat{\underline{d}}$ is strictly balanced if its restriction to X_S is strictly balanced on X_S.

Remark 1.2.11. Let X be a binary curve of genus g, and let X_n be the normalization of X at the node n, such that $\nu : X_n \to X$ is the associated map. Let $\underline{d} \geq \underline{0}$ be a balanced multidegree on X such that $|\underline{d}| = d \leq g-1$, then it is still balanced on X_n. Indeed, let us suppose by contradiction that

$$d_1 < m(d, g-1);$$

then it should be

$$d_1 < \frac{d-g}{2} \leq \frac{g-1-g}{2},$$

but then we would have that $d_1 < 0$, which cannot happen.

Lemma 1.2.12. *Let X be a quasistable curve, and $L \in \operatorname{Pic}^d X$ a balanced line bundle such that $\underline{\deg} L =: \underline{d}$ with $d \leq g-1$ and $h^0(L) \geq 1$. Then there exists a non exceptional irreducible component C of X such that for general $p \in C$*

$$h^0(L(p)) = h^0(L).$$

Proof. We fix a smooth point p on X. We know that $h^0(L(p)) \geq h^0(L)$. We suppose that $h^0(L(p)) = h^0(L)+1$; by Riemann-Roch this is equivalent to saying that $h^0(\omega_X \otimes L^{-1}(-p)) = h^0(\omega_X \otimes L^{-1})$. This holds if and only if p is a base point of $\omega_X L^{-1}$. But now we notice that, again by Riemann-Roch theorem,

$$h^0(\omega_X \otimes L^{-1}) = h^0(L) + 2g - 2 - d - g + 1 \geq h^0(L) \geq 1.$$

Therefore $\omega_X \otimes L^{-1}$ has some non vanishing section on X. If $E \subset X$ is an exceptional component, then $\deg_E \omega_X = 0$ and $\deg_E L = 1$, hence $\deg_E \omega_X \otimes L^{-1} = -1$, hence every section of $\omega_X \otimes L^{-1}$ vanishes on E. This implies that there must be a non exceptional component C of X such that the restriction to C of $H^0(\omega_X \otimes L^{-1})$ is non zero. Hence the general point $p \in C$ is not a base point of $\omega_X \otimes L^{-1}$. So we get our conclusions. ☐

Remark 1.2.13. We recall that if X is a nodal curve and $\underline{d} = \underline{g-1}$ is *stably balanced* as in section 1.3.1 in [C2], then $W_{\underline{d}}(X) = A_{\underline{d}}(X)$. It's very easy to see that if X is a binary curve and $\underline{d} = \underline{g-1} \geq \underline{0}$ is balanced, then $\underline{d}$ is strictly balanced and hence stably balanced. This implies that if X is a binary curve of genus g and $\underline{d} = \underline{g-1} \geq \underline{0}$ balanced, then $W_{\underline{d}}(X) = A_{\underline{d}}(X)$.

Lemma 1.2.14. *Let $\underline{d} \in \overline{B_d(X)}$ be such that $W_{\underline{d}}(X) \neq \emptyset$ where X is binary of genus g and $d \leq g-1$. Then $\underline{d} \geq \underline{0}$ and $\underline{d} \in B_d(X)$.*

Proof. By [C5][Proposition 12] if $d_i < 0$ and $d \leq g$ we have $W_{\underline{d}}(X) = \emptyset$, hence $\underline{d} \geq \underline{0}$. Now we have

$$m(d,g) = \frac{d-g-1}{2} \leq \frac{g-1-g-1}{2} = -1.$$

Therefore, if $\underline{d} \geq \underline{0}$, $d_i \neq m(d,g)$ for $i = 1,2$. Hence $\underline{d}$ is strictly balanced on X. ☐

We notice that by lemma 1.2.14, for a binary curve we have $W^+_{\underline{d}}(X) = W_{\underline{d}}(X)$.

Proposition 1.2.15. *Let $X = C_1 \cup C_2$ be a binary curve of genus g, L a line bundle on X of degree $\underline{d}$ balanced, with $0 < |\underline{d}| \leq g-1$, and $h^0(X,L) > 0$. Then there exists a family $L_t \in \mathrm{Im}\alpha^d_X$ such that $L_t \to L$ when $t \to 0$.*

Proof. Let L be a line bundle as in the hypothesis; we will use induction on the degree.

If $d = g-1$ by [C2] (see remark 1.2.13) we have that $A_{\underline{d}}(X) = W_{\underline{d}}(X)$.

Now let $d < g-1$; by lemma 1.2.12 we have that there exists a component of X, say C_1, such that for the general $p \in C_1$ we have that $h^0(L(p)) = h^0(L)$. By lemma 1.2.14 $L(p)$ has balanced multidegree on X. Hence we can apply induction and get that there exists a family $L'_t \in \mathrm{Im}\alpha^{d+1}_X$ such that $L'_t \to L(p)$. Like before we denote this family via

$$\mathcal{O}_X(a^1_t + \cdots + a^{d+1}_t) \to L(p). \tag{1.13}$$

We notice that p is a base point of $L(p)$. Let $\nu : X_n \to X$ be the normalization of X at a node n, as in remark 1.2.11. Then we can pullback (1.13) to X_n and get

$$\mathcal{O}_{X_n}(a^1_t + \cdots + a^{d+1}_t) \to L'(p), \tag{1.14}$$

where with abuse of notation we call the points on X and X_n in the same way, and $L' = \nu^*(L)$.

Now we divide the proof in two cases:

Case 1 : we assume that $h^0(L'(p)) = h^0(L')$.

We need to use a second induction on the number of nodes. The inductive statement is: if $\widetilde{L}$ and $\widetilde{L}(p)$ are balanced line bundles on Y binary curve with δ nodes, with $\underline{\deg}\widetilde{L} = \underline{d} \geq \underline{0}$ with $h^0(\widetilde{L}) > 0$, $h^0(\widetilde{L}(p)) = h^0(\widetilde{L})$ and there exists $O_Y(a^1_t + \cdots + a^{d+1}_t) \to \widetilde{L}(p)$, then $a^i_t \to p$ for some i.

The base of induction is obvious on a curve with no nodes, i.e. a smooth one. So we suppose that the statement above is true for X_n: in particular we know that $h^0(L'(p)) = h^0(L')$; then by induction it holds that in (1.14) there exists a^i_t such that

$$a^i_t \to p \text{ for some } i. \tag{1.15}$$

Up to reordering the points we can assume that $i = d+1$. Now, by applying (1.15) to (1.13) we get that

$$\mathcal{O}_X(a^1_t + \cdots + a^d_t) \to L,$$

and hence the conclusions in case 1.

Case 2 : we assume that $h^0(L'(p)) = h^0(L') + 1$. Then we have $h^0(L') = h^0(L)$. We have two possibilities: by applying Lemma 2.2.3 (2) and Lemma 2.2.4 (2) in [C2], either n is a base point of L, or $W_{L'}(X) = \{L\}$. In the first case we have that it must be true regardless of the choice of n, i.e. every node n of X must be a base point of L, which is impossible since the nodes are $g+1$ whereas the degree of L is $d < g-1$.

On the other hand, if $W_{L'}(X) = \{L\}$ we need a new inductive argument on the number of nodes. In this case the inductive statement is: let Y is a binary curve of genus g, $M \in \mathrm{Pic}^{\underline{d}}Y$ such that $\underline{d}$ is balanced and $d \leq g-1$ with $h^0(Y, M) > 0$. Then there exists $M_t \in \mathrm{Im}\alpha^{\underline{d}}_Y$ such that $M_t \to M$ when $t \to 0$.

The base of induction is given by a binary curve of genus 2, i.e. with 3 nodes, so that $d = 1$, and since $d = g-1$, by [C2] we have $W_{\underline{d}}(X) = A_{\underline{d}}(X)$, hence the conclusion holds.

We assume the inductive statement for X_n, so we get that there exists $L'_t \in \mathrm{Im}\alpha^{\underline{d}}_{X_n}$ such that $L'_t \to L'$. Since $L'_t \in \mathrm{Im}\alpha^{\underline{d}}_{X_n}$, for every t there exists $L_t \in \mathrm{Im}\alpha^{\underline{d}}_X$ such that $\nu^*(L_t) = L'_t$. By the fact that $W_{L'}(X) = \{L\}$, we conclude that $L_t \to L$.

□

Corollary 1.2.16. *Let X be a binary curve of genus g, and let $\underline{d} \geq \underline{0}$ be a balanced multidegree on X. Then $W_{\underline{d}}(X) = A_{\underline{d}}(X)$. In particular $W_{\underline{d}}(X)$ is irreducible of dimension d.*

Proof. The first assertion is implied by proposition 1.2.15. And of course this implies that $W_{\underline{d}}(X)$ is irreducible. By [C5][proposition 25] we have that the dimension of $W_{\underline{d}}(X) = d$. □

So far we have studied the closure of $\mathrm{Im}\alpha^{\underline{d}}_X$ inside $\mathrm{Pic}^{\underline{d}}X$ when X is a binary curve. The next step is to study its closure inside the compactified Picard variety $\overline{P^d_X}$.

Let $B_d(X)$ be the set of strictly balanced line bundles of multidegree $\underline{d}$ on X, with $|\underline{d}| = d$, and denote by $\overline{B_d(X)}$ the set of balanced multidegrees. The stratification of $\overline{P^d_X}$ as in [C7, Fact 2.2] is the following

$$\overline{P^d_X} = \coprod_{\substack{\emptyset \subset S \subset X^{\mathrm{sing}} \\ \underline{d} \in B_d(\widehat{X}_S)}} P^{\underline{d}}_S. \tag{1.16}$$

For any set S of nodes of X, if C_1, C_2 are the smooth components of X, $X_S = C_1 \cup C_2$, with $\delta_S = \sharp(C_1 \cap C_2) = \delta - \sharp S$, so that the total normalization is

$$C_1 \sqcup C_2 \xrightarrow{\nu_S} X_S,$$

and given $L \in W_{\underline{d}_S}(X_S)$, we denote $(L_1, L_2) := \nu_S^*(L)$. The stratification in (1.16) motivates the definition of

$$\widetilde{W}_d(X) := \bigsqcup_{\substack{\emptyset \subset S \subseteq X^{\mathrm{sing}} \\ \widehat{\underline{d}}_S \in B^{\geq 0}_d(\widehat{X}_S)}} W_{\underline{d}_S}(X_S), \tag{1.17}$$

where $B_d^{\geq 0}(\widehat{X}_S)$ is the set of strictly balanced multidegrees $\underline{e} \geq \underline{0}$ on $\widehat{X}_S$ such that $|\underline{e}| = d$, $\widehat{X}_S$ is the partial blow up of X at the nodes contained in S, X_S is the strict transform of X, $\widehat{\underline{d}}_S$ is a balanced multidegree on $\widehat{X}_S$ such that $|\widehat{\underline{d}}_S| = d$, whereas

$$\underline{d}_S = \widehat{\underline{d}}_S|_{X_S}$$

and $|\underline{d}_S| = d - \sharp S$.

We notice that, if $\underline{d} \in B_d^{\geq 0}(X)$, denoting by $\overline{A_{\underline{d}}(X)}$ the closure of $A_{\underline{d}}(X)$ in $\overline{P_X^d}$, similarly to lemma 1.2.4 we have the inclusion

$$\overline{A_{\underline{d}}(X)} \subset \widetilde{W}_d(X). \tag{1.18}$$

Definition 1.2.17. We denote by

$$\overline{A_d(X)} := \overline{\bigcup_{\underline{d} \in B_d^{\geq 0}(X)} A_{\underline{d}}(X)} \subset \overline{P_X^d}.$$

Theorem 1.2.18. *Let $X = C_1 \cup C_2$ be a binary curve of genus $g \geq 2$ with $\delta \geq 2$ nodes and smooth components. Take $1 \leq d \leq g-1$. Then*

$$\widetilde{W}_d(X) = \overline{A_d(X)} \subset \overline{P_X^d}.$$

Proof. Let us observe that, since $\sharp B_d^{\geq 0}(X) < \infty$, we have that

$$\overline{\bigcup_{\underline{d} \in B_d^{\geq 0}(X)} A_{\underline{d}}(X)} = \bigcup_{\underline{d} \in B_d^{\geq 0}(X)} \overline{A_{\underline{d}}(X)}.$$

For any $\underline{d} \in B_d^{\geq 0}(X)$, by (1.18) we get that inclusion ($\supset$) holds. Let us now prove inclusion ($\subset$). By (1.17) it is sufficient to show that for any $\emptyset \subset S \subset X^{\text{sing}}$,

$$W_{\underline{d}_S}(X_S) \subset \overline{A_{\underline{d}}(X)}$$

for a certain $\underline{d} \in B_d^{\geq 0}(X)$.

First of all we notice that by (1.17), we can equivalently write

$$\widetilde{W}_d(X) = \left\{[M, S] \in \overline{P_X^d} \text{ s.t. } M \in W_{\underline{d}_S}(X_S) \text{ with } \underline{d}_S \geq \underline{0}\right\}.$$

Let us assume that $\sharp S = 1$, with $S = \{n\}$; take $M \in W_{\underline{d}_S}(X_S)$ with $\underline{d}_S = \widehat{\underline{d}}_S|_{X_S}$ and $\widehat{\underline{d}}_S \in \overline{B_d^{\geq 0}(\widehat{X}_S)}$, and consider $[M, S] = [M, n]$. Thanks to the stratification of $\overline{P_X^d}$ there exists $\underline{d} = (d_1, d_2) \in B_d^{\geq 0}(X)$ such that either $\underline{d}_S = (d_1 - 1, d_2)$ or $\underline{d}_S = (d_1, d_2 - 1)$. We assume, with no loss of generality, that $\underline{d}_S = (d_1 - 1, d_2)$. Now by proposition 1.2.15 we know that $M \in A_{\underline{d}_S}(X_S)$, i.e. there exists a family $M_t \in \operatorname{Im}\alpha_{X_S}^{\underline{d}_S}$ such that M_t specializes to M on X_S as $t \mapsto 0$. By [C2, Lemmas 2.2.3,2.2.4] we have that for any t there exists $L_t \in \operatorname{Im}\alpha_X^{\underline{d}_S}$ such that the pullback of L_t to X_S is M_t. Let us fix t and take a moving point p_u on $C_1 \cap \dot{X}$ such that p_u specializes to n as $u \mapsto 0$. We see that by construction $\underline{\deg} L_t(p_u) = \underline{d}$ and $L_t(p_u) \in \operatorname{Im}\alpha_X^{\underline{d}}$; moreover

$$L_t(p_u) \stackrel{u \to 0}{\longrightarrow} [M_t, n] \in \overline{A_{\underline{d}}(X)}.$$

Now we let t tend to 0, so we obtain that $[M_t, n] \mapsto [M, n]$, and $[M, n] \in \overline{A_{\underline{d}}(X)}$.

We proceed by induction on $\sharp S$; we have just proved that when $\sharp S = 1$, then for any $\widehat{\underline{d}}_S \in \overline{B_d^{\geq 0}(\widehat{X}_S)}$ there exists $\underline{d} \in B_d^{\geq 0}(X)$ such that $W_{\underline{d}_S}(X_S) \subset \overline{A_{\underline{d}}(X)}$. Let us suppose that $S \subset X^{\text{sing}}$ is such that for any $\widehat{\underline{d}}_S \in \overline{B_d^{\geq 0}(\widehat{X}_S)}$ there exists $\underline{d} \in B_d^{\geq 0}(X)$ such that $W_{\underline{d}_S}(X_S) \subset \overline{A_{\underline{d}}(X)}$, or equivalently, $[M, S] \in \overline{A_{\underline{d}}(X)}$ for every $M \in W_{\underline{d}_S}(X_S)$.

We want to prove that for $T \subset X^{\text{sing}}$, with $T = S \cup n$ for any node n of X_S, then taking $\widehat{\underline{d}}_T \in \overline{B_d^{\geq 0}(\widehat{X}_T)}$ there exists $\underline{d} \in B_d^{\geq 0}(X)$ such that $[M_T, T] \in \overline{A_{\underline{d}}(X)}$ with $M_T \in W_{\underline{d}_T}(X_T)$.

We take an element $M_T \in W_{\underline{d}_T}(X_T)$, and consider $[M_T, T]$. By 1.2.15 we know that $M_T \in A_{\underline{d}_T}(X_T)$, hence there exists a family $M_T^t \in \text{Im}\alpha_{X_T}^{d_T}$ such that M_T^t specializes to M_T on X_T as $t \mapsto 0$. Let $\underline{d}_S = (d_1^S, d_2^S)$ be a multidegree on X_S such that $|\underline{d}_T| = |\underline{d}_S| - 1$ and $\widehat{\underline{d}}_S \in \overline{B_d^{\geq 0}(\widehat{X}_S)}$; it exists because of the stratification of $\overline{P_X^d}$. Let us assume that, say, $\underline{d}_T = (d_1^S - 1, d_2^S)$. Again [C2, Lemmas 2.2.3,2.2.4] imply that for any t there exists $M_S^t \in \text{Im}\alpha_{X_S}^{d_T}$ such that the pullback of M_S^t to X_T is M_T^t. We fix t and take $p_u \in C_1 \cap \dot{X}$ specializing to n on X_S; then we have that $\underline{\deg} M_S^t(p_u) = \underline{d}_S$, hence by inductive hypothesis $[M_S^t(p_u), S] \in \overline{A_{\underline{d}}(X)}$ for a certain $\underline{d} \in B_d^{\geq 0}(X)$. Then we have that

$$[M_S^t(p_u), S] \xrightarrow{u \to 0} [M_T^t, T] \in \overline{A_{\underline{d}}(X)},$$

and again letting t tend to 0 we obtain that

$$[M_T^t, T] \xrightarrow{t \to 0} [M_T, T] \in \overline{A_{\underline{d}}(X)},$$

as we wanted.

It follows that

$$\bigsqcup_{\substack{\emptyset \subset S \subseteq X^{\text{sing}} \\ \widehat{\underline{d}}_S \in B_d^{\geq 0}(\widehat{X}_S)}} W_{\underline{d}_S}(X_S) = \bigcup_{\underline{d} \in B_d^{\geq 0}(X)} \overline{A_{\underline{d}}(X)},$$

hence we get the conclusions. □

We are now going to investigate about the closure inside $\overline{P_X^d}$ of the set $A_{\underline{d}}(X)$ when $\underline{d}$ is a strictly balanced multidegree on X binary curve. Before, we need to recall some definitions introduced in [C1].

Definition 1.2.19. Let X and $\widehat{X}$ be two Deligne-Mumford semistable curves; we say that $\widehat{X}$ *dominates* X if they have the same stable model and if there exists a surjective morphism of $\widehat{X}$ onto X such that every component of $\widehat{X}$ is either contracted to a point or mapped birationally onto its image.

Definition 1.2.20. Let $\underline{d} \in \overline{B_d(X)}$ and $\widehat{\underline{d}} \in \overline{B_d(\widehat{X})}$. We say that $\widehat{\underline{d}}$ is a *refinement* of $\underline{d}$, and we denote it by

$$\widehat{\underline{d}} \preceq \underline{d},$$

if and only if $\widehat{X}$ dominates X via a map φ and for every subcurve Y of X there exists a subcurve $\widehat{Y}$ of $\widehat{X}$ such that φ maps $\widehat{Y}$ to Y and $|\underline{d}_Y| = |\widehat{\underline{d}}_{\widehat{Y}}|$.

We are now able to state:

Proposition 1.2.21. *Let X be a binary curve of genus $g \geq 2$ and $\underline{d} \geq \underline{0}$ a strictly balanced multidegree on X. Then*

$$\overline{A_{\underline{d}}(X)} = \bigsqcup_{\substack{\emptyset \subset S \subseteq X^{\text{sing}} \\ \widehat{\underline{d}}_S \in B_{\underline{d}}^{\geq 0}(\widehat{X}_S) \\ \widehat{\underline{d}}_S \preceq \underline{d}}} W_{\underline{d}_S}(X_S) \subset \widetilde{W}_d(X).$$

Proof. Inclusion ($\subset$) is obvious by an argument analogous to lemma 1.2.4. The proof of ($\supset$) is actually the same as in 1.2.18, i.e. we take an element $[M, S] \in \overline{P_X^d}$ such that $M \in W_{\underline{d}_S}(X_S)$ and $\widehat{\underline{d}}_S \preceq \underline{d}$ and we use the same argument as in 1.2.18, considering that $\widehat{\underline{d}}_S \preceq \underline{d}$, hence $W_{\underline{d}_S}(X_S) \subset \overline{A_{\underline{d}}(X)}$. □

Degree 1

We are now going to investigate what happens when the degree $d = 1$. Let X be a nodal connected curve of genus $g \geq 2$, let $C_1, \ldots, C_\gamma$ be its irreducible components, and set $g_i = g(C_i)$, and $\delta_i = \sharp(C_i \cap \overline{X \setminus C_i})$.

Lemma 1.2.22. *Let X be a semistable curve of genus $g \geq 2$ as above and $\underline{d}$ be a balanced multidegree on X such that $|\underline{d}| = 1$ and $\underline{d} \ngeq \underline{0}$. Then $W_{\underline{d}}(X) = \emptyset$.*

Proof. Let us suppose that C_i is an irreducible component of X such that $d_i < 0$. By the balancing condition we know that:

$$\frac{2g_i - 2 + \delta_i}{2g - 2} - \frac{\delta_i}{2} \leq d_i \leq \frac{2g_i - 2 + \delta_i}{2g - 2} + \frac{\delta_i}{2}. \tag{1.19}$$

Assume that there exists $L \in W_{\underline{d}}(X)$, and denote by $n_1, \ldots, n_{\delta_i}$ the nodes of $C_i \cap \overline{X \setminus C_i}$. Moreover, denote for simplicity $Z_i = \overline{X \setminus C_i}$ and let $q_1, \ldots, q_{\delta_i}$ be the branches of $n_1, \ldots, n_{\delta_i}$ on Z_i. If $Y \subset X$ is a subcurve of X, we denote by $L_Y := L_{|Y}$. Since by assumption $h^0(C_i, L_{C_i}) = 0$, we have that

$$h^0(X, L) = h^0(Z_i, L_{Z_i}(-q_1 - \cdots - q_{\delta_i})) > 0.$$

Hence we must have that $\deg L_{Z_i} \geq \delta_i$, and recalling that $\deg L_{C_i} = d_i = 1 - \deg L_{Z_i}$, it follows that

$$d_i \leq 1 - \delta_i. \tag{1.20}$$

Therefore we have to verify that

$$\frac{2g_i - 2 + \delta_i}{2g - 2} - \frac{\delta_i}{2} \leq 1 - \delta_i;$$

this holds if and only if

$$\delta_i \leq 2 - \frac{2g_i}{g}.$$

So we have two possibilities:

(i) either $g_i = 0$ and $\delta_i = 1$ or $\delta_i = 2$,

(ii) or $g_i \neq 0$ and $\delta_i = 1$.

In case (i), $C_i \cong \mathbb{P}^1$, so let us suppose that $\delta_i = 2$; hence C_i is an exceptional component of X, and by the balancing condition it must be $d_i = 1$. But by (1.20), we see that $d_i \leq -1$, and we get a contradiction.

Suppose now that $g_i \geq 0$ and $\delta_i = 1$. Then, by (1.33) we have that

$$d_i \geq \frac{-1}{2} + \frac{2g_i - 1}{2g - 2},$$

hence $d_i \geq 0$, which is again a contradiction in both cases (i) and (ii). Therefore $W_{\underline{d}}(X) = \emptyset$. □

By lemma 1.2.22 we have that the only possibility that $W_{\underline{d}}(X) \neq \emptyset$ is when $\underline{d} = (1, 0, \ldots, 0)$, up to swapping some indices. In particular, when $\underline{d} = (1, 0, \ldots, 0)$ and $L \in W_{\underline{d}}(X)$ we have that by [C6, Lemma 4.2.3] either $h^0(L) \leq 1$ or C_1 is a separating line, $h^0(L) = 2$ and $L_{\overline{X \setminus C_1}} = \mathcal{O}_{\overline{X \setminus C_1}}$. We have the following

Theorem 1.2.23. *Let X be a connected nodal curve; let $\underline{d} = (1, 0, \ldots, 0)$ be a multidegree on X. Then $A_{\underline{d}}(X) = W^+_{\underline{d}}(X)$.*

Proof. By lemma 1.2.4 we only need to prove that $A_{\underline{d}}(X) \supset W^+_{\underline{d}}(X)$. As usual we call $C_1, \ldots, C_\gamma$ the irreducible components of X, then up to reordering we have that $\underline{d}|_{C_1} = 1$. By [C6][Lemma 4.2.3] if $L \in W^+_{\underline{d}}(X)$ we must have $h^0(L) = 1$ unless C_1 is a separating line of X, we will discuss this case later, so suppose C_1 is not a separating line of X.

Then L has one nonzero section s on X. If it vanishes on a smooth point r of X, we have that $L = \mathcal{O}_X(r)$, so $L \in \mathrm{Im}\alpha^{\underline{d}}_X$. Otherwise, if there exists a node $n \in C_1 \cap C_1^c$ such that $s(n) = 0$, we normalize X at n; we denote $X' \xrightarrow{\nu} X$ the normalization at n. Now we have two possibilities:

(i) X' is connected, i.e. n is nonseparating. Let L' be the pullback of L to X'. Let us denote by C_2 the component of X such that $\{n\} = C_1 \cap C_2$ and by p, q the branches of n on X' with $p \in C_1$. With abuse of notation we call again s the pullback of s to L'. Then $s(p) = s(q) = 0$, but L' has degree one, so s doesn't vanish on other points of C_1, and in particular if $\{n_1, \ldots, n_l\}$ are the other nodes of $\overline{X' \setminus C_1} \cap C_1$, $s(n_i) \neq 0$. Notice that $l \geq 1$. Since the pullback of L to the total normalization of X is $(\mathcal{O}_{C_1}(p), \mathcal{O}_{C_2}, \ldots, \mathcal{O}_{C_\gamma})$, then s restricted to $X' \setminus C_1$ must be a constant, hence by what we just said a nonzero constant. In particular $s(q) \neq 0$, which is a contradiction. Therefore s cannot vanish on a nonseparating node of X.

(ii) X' is not connected. Then $X' = C_1 \sqcup Z_1$ and Z_1 is connected. In particular n is a separating node of X. The pullback of L to X' is $M = (\mathcal{O}_{C_1}(p), \mathcal{O}_{Z_1})$, where p is as in (i). Now let us consider a moving point $p_t \in C_1 \setminus p$, such that $p_t \mapsto p$. Let $M_t = (\mathcal{O}_{C_1}(p_t), \mathcal{O}_{Z_1}) \in \mathrm{Pic}X'$. Then the line bundle $\mathcal{O}_X(p_t)$ on X pulls back to M_t (abusing notation). As $p_t \mapsto p$,

$M_t \mapsto M$ and $L_t \mapsto \widetilde{L}$ such that $h^0(\widetilde{L}) \geq 1$. Since $h^0(X', M) = 1$ we have that L is the unique line bundle on X pulling back to M and such that $h^0(L) \geq 1$. Hence $L = \widetilde{L}$. Hence $L \in A_{\underline{d}}(X)$.

If C_1 is a separating line, since $L \in W^+_{\underline{d}}(X)$ we have $h^0(L) = 2$. Then we can choose $r \in C_1 \setminus (C_1 \cap C_1^c)$ such that $L = \mathcal{O}_X(r)$. So $L \in \mathrm{Im}\alpha^{\underline{d}}_X$. □

In what follows we are going to give a characterization of the closure of $A_{\underline{d}}(X)$ in $\overline{P^d_X}$ for stable curves in degree 1. Let then X be a stable curve. Let as usual $X = C_1 \cup \cdots \cup C_\gamma$ be the decomposition of X into irreducible components. If X_S is a partial normalization of X at a set S of nodes, we consider the decomposition of X_S in connected components:

$$X_S = X_1^S \sqcup \cdots \sqcup X^S_{r_S}.$$

We denote by

$$\widetilde{\nu}_S : \bigsqcup_{i=1}^{\gamma} C_i^S \longrightarrow X_S,$$

the partial normalization of X_S at all the nodes in the set

$$\bigcup_{i=1}^{\gamma} \left(C_i^S \cap (X_S \setminus C_i^S)\right).$$

We recall that by [C1], for any stable curve of genus $g \geq 2$ and any d, we have a decomposition

$$\overline{P^d_X} = \coprod_{\substack{\emptyset \subset S \subset X^{\mathrm{sing}} \\ \underline{d} \in B_d(\widehat{X}_S)}} P^{\underline{d}}_S.$$

We define

$$\widetilde{W_d}(X) := \bigsqcup_{\substack{\emptyset \subset S \subset X^{\mathrm{sing}} \\ \widehat{\underline{d}}_S \in B_d^{\geq 0}(\widehat{X}_S)}} W^+_{\underline{d}_S}(X_S),$$

where again $B_d^{\geq 0}(\widehat{X}_S)$ is the set of strictly balanced multidegrees $\widehat{\underline{d}}_S \geq \underline{0}$ on $\widehat{X}_S$ such that $|\widehat{\underline{d}}_S| = d$, and $\underline{d}_S = \widehat{\underline{d}}_S|_{X_S}$ with $|\underline{d}_S| = d - \sharp S$. We will also use the notation

$$\underline{d}_S = (d_1^S, \ldots, d_\gamma^S)$$

for the components of the multidegree $\underline{d}_S$ on X_S. Let

$$\overline{A_d(X)} := \overline{\bigcup_{\underline{d} \in B_d^{\geq 0}(X)} A_{\underline{d}}(X)} \subset \overline{P^d_X}.$$

Remark 1.2.24. When the degree $d = 1$, the elements of $B_1^{\geq 0}(X)$ are of the form $(d_1, \ldots, d_\gamma)$ with $d_i = 1$ for one suitable $i \in \{1, \ldots, \gamma\}$ and $d_j = 0$ for $j \neq i$. Thus, when we look at the set $B_1^{\geq 0}(\widehat{X}_S)$, if it is nonempty it must be $\sharp S = 1$.

We have the following result:

Theorem 1.2.25. *Let X be a stable curve of genus $g \geq 2$ with $B_1^{\geq 0}(X) \neq \emptyset$. Then*

$$\widetilde{W_1}(X) = \overline{A_1(X)} \subset \overline{P_X^1}.$$

Proof. Inclusion ($\supset$) holds by lemma 1.2.4. Now we prove inclusion ($\subset$). By hypothesis the set $\overline{A_1(X)}$ is nonempty. We want to prove that for any $\emptyset \subset S \subset X^{\mathrm{sing}}$ such that $\widehat{\underline{d}}_S \in B_1^{\geq 0}(\widehat{X}_S)$,

$$W^+_{\underline{d}_S}(X_S) \subset \overline{A_{\underline{d}}(X)}$$

for a certain $\underline{d} \in B_1^{\geq 0}(X)$. We can equivalently write

$$\widetilde{W_1}(X) = \left\{ [M,S] \in \overline{P_X^d} \text{ s.t. } M \in W^+_{\underline{d}_S}(X_S) \text{ with } \underline{d}_S \geq \underline{0} \right\}.$$

By remark 1.2.24 we can assume $\sharp S = 1$, with $S = \{n\}$, so hereafter we will write n instead of S. We take $M \in W^+_{\underline{d}_n}(X_n)$ with $\widehat{\underline{d}}_n \in \overline{B_1^{\geq 0}(\widehat{X}_n)}$, and consider $[M,n]$. Then by the stratification of $\overline{P_X^1}$ and remark 1.2.24 there exists $\underline{d} = (d_1, \dots, d_\gamma) \in B_{\underline{d}}^{\geq 0}(X)$ such that $d_i - 1 = d_i^n = 0$ for one i. We assume, with no loss of generality, that $d_1 - 1 = d_1^n = 0$; now $M \in \mathrm{Pic}_{\underline{d}_n} X_n$ is such that $M = (\mathcal{O}_{X_1^n}, \dots, \mathcal{O}_{X_{r_n}^n})$, where r_n is the number of connected components of X_n and it is 1 or 2 whether n is separating or not. Then, there exists $L \in \mathrm{Im}\alpha_X^{\underline{d}_n}$ such that the pullback of L to X_n is M, and of course $L = \mathcal{O}_X$. Let us take a moving point p_u on $C_1 \cap \dot{X}$ such that p_u specializes to n as $u \mapsto 0$. We see that by construction $\underline{\deg} L(p_u) = \underline{d}$ and $L(p_u) \in \mathrm{Im}\alpha_X^{\underline{d}}$; moreover

$$L(p_u) \xrightarrow{u \to 0} [M,n] \in \overline{A_{\underline{d}}(X)}.$$

□

Remark 1.2.26. In [CE06][Proposition 3.15], the authors characterize the locus Σ_g^1 in $\overline{M}_g$ of the curves such that $B_1(X)$ is empty, i.e. the so called 1-*general* curves. They prove that if $g \geq 2$ and g is odd, then the set Σ_g^1 is empty. Hence when g is odd, if $B_1^{\geq 0}(X) \subset B_1(X)$ is nonempty we are always in the case of theorem 1.2.25. If, otherwise, $B_1^{\geq 0}(X) = \emptyset$, then both the sets $\widetilde{W_1}(X)$ and $\overline{A_1(X)}$ are empty.

1.3 Notes on Brill-Noether theory of nodal curves

1.3.1 Martens' theorem and Mumford's theorem

In this subsection we are going to prove some classical theorems of the Brill-Noether theory for irreducible curves; they will be probably well known to the experts, but we couldn't find an exhaustive reference. For example some already extended results in this context are the very classical Riemann's Theorem and Clifford's Theorem. Some more results are presented in [BF02]. Before starting we recall from [C2, Proposition 5.2.1.], that if X is an irreducible nodal curve of genus $g \geq 3$ with δ nodes, it is hyperelliptic, i.e. $X \in \overline{H_g}$ if and only if it has a $\mathfrak{g}_2^1$.

Lemma 1.3.1. *Let X be a not hyperelliptic irreducible curve of genus $g \geq 4$, with $\delta \geq 2$ nodes. Then there exists $n \in X_{sing}$ such that the normalization of X at n is not hyperelliptic.*

Proof. Let $\{n_1, \dots, n_\delta\}$ be the set of nodes of X. Suppose by contradiction that for any $i = 1, \dots, \delta$ the partial normalization $Y_i \xrightarrow{\nu_i} X$ of X at n_i is a hyperelliptic curve of genus $g_{Y_i} = g - 1$. Let then Z be the normalization of X and of Y_i for any $i = 1, \dots, \delta$, such that, if $Z \xrightarrow{\nu} X$ and $Z \xrightarrow{\nu'_i} Y_i$ are the normalizations, we have that $\nu_i \circ \nu'_i = \nu$ for any i. We fix $i = 1, \dots, \delta$ and look at $Z \xrightarrow{\nu'_i} Y_i$; since Y_i is hyperelliptic, then so is Z, and by [C2, Proposition 5.2.1] we have that the hyperelliptic series $H_Z = O_Z(p_i + q_i)$ for $i = 1, \dots, \delta$, where $\nu^{-1}(n_i) = \{p_i, q_i\}$. But then if we look at $Z \xrightarrow{\nu} X$, again by [C2, Proposition 5.2.1] we get that there exists a line bundle $H_X \in \mathrm{Pic}^2 X$ such that $h^0(X, H_X) = 2$, i.e. X is hyperelliptic, which is a contradiction. □

Theorem 1.3.2 (Martens for irreducible curves). *Let X be an irreducible nodal curve of genus $g \geq 3$ with δ nodes; let $2 \leq d \leq g - 1$, $d \geq 2r$, $r > 0$. If X is not hyperelliptic, then $\dim W^r_d(X) \leq d - 2r - 1$.*

Proof. The proof is by induction on the number of nodes δ. So let $\delta = 1$; let $Z \xrightarrow{\nu} X$ be the normalization of X at its node n, hence Z is a smooth curve of genus $g_Z = g - 1$. Now we have two alternatives: either Z is hyperelliptic, or it is not. We distinguish the two cases:

Case I. If Z is not hyperelliptic, when $d \leq g - 2 = g_Z - 1$ we can apply Martens' Theorem for smooth curves and we get that $\dim W^r_d(Z) \leq d - 2r - 1$. Now let us consider the morphism

$$\begin{array}{rcl} \rho : W^r_d(X) & \longrightarrow & W^r_d(Z) \\ L & \mapsto & \nu^*(L); \end{array}$$

by classical Martens' theorem we also have that $\dim W^{r+1}_d(Z) \leq d - 2r - 2$, so we just have to look at the fibers of ρ. If $W^r_d(Z)$ doesn't have the two points of $\nu^{-1}(n) = \{q_1, q_2\}$ as fixed base points, then the fibers of ρ over $W^r_d(Z) \backslash W^{r+1}_d(Z)$ have dimension 0 by [C5, Lemma 9], and over $W^{r+1}_d(Z)$ they have dimension at most 1. Hence

$$\dim W^r_d(X) \leq d - 2r - 1.$$

If every line bundle in $W^r_d(Z)$ has $\{q_1, q_2\}$ as base points, then we have an injection

$$\begin{array}{rcl} W^r_d(Z) & \hookrightarrow & W^r_{d-2}(Z) \\ L & \mapsto & L(-q_1 - q_2). \end{array}$$

Therefore we have that $\dim W^r_d(Z) \leq (d - 2) - 2r - 1 = d - 2r - 3$, and hence $\dim W^r_d(Z) \leq d - 2r - 2$. When $d = g - 1 = g_Z$, we use the Serre duality to get

$$\dim W^r_{g-1}(Z) = \dim W^{r-1}_{g-3}(Z) \leq (g - 1) - 2r - 1$$

and

$$\dim W^{r+1}_{g-1}(Z) = \dim W^r_{g-3}(Z) \leq (g-1) - 2r - 2,$$

therefore we conclude arguing as before.

Case II. If Z is a hyperelliptic curve, we again have to distinguish two cases. Let H_Z the hyperelliptic series on Z, and let $\{q_1, q_2\} = \nu^{-1}(n)$.

(i) $H_Z = O_Z(q_1 + q_2)$; in this case, by [C2, Proposition 5.2.1] we get that X must be hyperelliptic, a contradiction.

(ii) Let $H_Z = O_Z(a + b)$, where a, b are points on Z such that $a + b \nsim q_1 + q_2$. We look at the dimension of $Im(\rho) \subseteq W^r_d(Z)$: in particular we want to show that $\dim Im(\rho) \leq d - 2r - 2$. We describe $Im(\rho)$ as follows:

$$Im(\rho) = \{M \in W^r_d(Z)\, s.t.\, W^r_M(X) \neq \emptyset\},$$

where $W^r_M(X)$ is the fiber of $W^r_d(X)$ over M via ρ. More precisely, we have that $Im(\rho) = V_0 \sqcup V_1$, where

$$V_0 = \{M \in W^r_d(Z)\, s.t.\, \dim W^r_M(X) = 0\},$$

and

$$V_1 = \{M \in W^r_d(Z)\, s.t.\, \dim W^r_M(X) = 1\}.$$

Now we ask what is the dimension of V_0 and V_1. Let us consider V_0; let $M \in W^r_d(Z)$. By [C2, Lemma 5.1.3.] we have that $\dim W^r_M(X) = 0$ if and only if $h^0(Z, M) = r + 1$ and $h^0(Z, M - q_1 - q_2) = h^0(Z, M - q_1) = h^0(Z, M - q_2) = r$. Therefore $M \in V_0$ iff $|M| = \mathfrak{g}^r_d$, $|M - q_1 - q_2| = \mathfrak{g}^{r-1}_{d-2}$, and $|M - q_1, M - q_2| = \mathfrak{g}^{r-1}_{d-1}$. Now since Z is hyperelliptic, $|M - q_1 - q_2| = \mathfrak{g}^{r-1}_{d-2}$ iff there exist $p_1, \dots, p_{d-2r} \in Z$ such that

$$O_Z(M - q_1 - q_2) = H_Z^{r-1}(p_1 + \dots + p_{d-2r}),$$

with $h^0(p_i + p_j) = 1$ for any $i, j = 1, \dots, d - 2r$. We have that

$$M = H_Z^{r-1}(p_1 + \dots + p_{d-2r} + q_1 + q_2),$$

then $h^0(Z, M - q_1) = h^0(Z, M - q_2) = r$ iff $h^0(Z, p_i + q_2) = h^0(Z, p_i + q_1) = 1$ for any $i = 1, \dots, d - 2r$. But on the other hand $h^0(Z, M) = r + 1$ if and only if there exists a pair of points in the set $\{p_1, \dots, p_{d-2r}, q_1, q_2\}$ linearly equivalent to H_Z. But from what we have seen this is possible only if $q_1 + q_2 \sim a + b$, which we excluded from the beginning. Hence $V_0 = \emptyset$. So let us now look at V_1. Again by [C2, Lemma 5.1.3.] we get that $M \in V_1$ if and only if either $h^0(Z, M) = h^0(Z, M - q_i) = r + 1$ for $i = 1, 2$, or $h^0(Z, M) \geq r + 2$. Let then $V_1 = V_1^1 \sqcup V_1^2$, where

$$V_1^1 = \{M \in W^r_d(Z)\, s.t.\, h^0(Z, M) = h^0(Z, M - q_i) = r + 1\}$$

and

$$V_1^2 = W_d^{r+1}(Z).$$

We easily see that

$$V_1^1 = \{M = H_Z^r(q_1 + q_2 + a_3 + \cdots + a_{d-2r}),\ s.t.\, a_3, \ldots, a_{d-2r} \in Z\},$$

hence by a parameter count we have that $\dim V_1^1 \leq d - 2r - 2$. Moreover, by [C2, Lemma 5.2.3.] we get $\dim V_1^2 = d-2r-2$, and hence $\dim V_1 \leq d-2r-2$. Now, recalling that the dimension of the fibers over V_1 is 1, we conclude that $\dim W_d^r(X) \leq d-2r-1$. We are now able to assume $\delta \geq 2$. Let $d \leq g-2$ and $g \geq 4$ (cases $g = 3$ and $d = g-1$ will be shown further on). By Lemma 1.3.1 there exists a node $n \in X_{sing}$ such that the normalization $Y \stackrel{\nu}{\longrightarrow} X$ of X at n is not hyperelliptic. Then, since $d \leq g_Y - 1$ and $g_Y \geq 3$, we can apply the inductive hypothesis and get that $\dim W_d^r(Y) \leq d - 2r - 1$ and, since $\dim W_d^{r+1}(Y) \leq d - 2r - 2$, repeating the argument about the dimension of the fibers that we explained in Case I, we are done. Now we focus on the left cases: if $g = 3$, then $d = 2$ and $r = 1$, therefore $W_2^1(X) = \emptyset$ since X is not hyperelliptic. Now let $d = g-1 = g_Y$; then we use again Serre duality to get $W_{g_Y}^r(Y) \cong W_{g_Y-2}^{r-1}(Y)$. Now, since $d \leq g_Y - 2$, induction yields that $\dim W_{g_Y-2}^{r-1}(Y) \leq (g-1) - 2r - 1$, so, applying the argument seen in Case I, we get that $\dim W_{g-1}^r(X) \leq (g-1) - 2r - 1$.

□

An interesting refinement of Martens' theorem is the classical Mumford's theorem (see [ACGH]), which characterizes the smooth not hyperelliptic curves such that the upper bound in Martens' theorem is attained. We present a proof of this theorem for irreducible stable curves.

Theorem 1.3.3 (Mumford for irreducible curves)**.** *Let X be an irreducible nodal not hyperelliptic curve of genus $g \geq 4$, and assume there exist $r, d \in \mathbb{N}$ such that $2 \leq d \leq g-2$, $d \geq 2r > 0$ and a component of $W_d^r(X)$ with dimension $d - 2r - 1$. Then X has one of the following properties:*

(i) X has a $\mathfrak{g}_3^1$.

(ii) X is a two-sheeted covering of a plane cubic curve.

(iii) X is a plane quintic.

Proof. We follow the proofs of [ACGH, Theorem (5.2)] and [Be77, Lemma (4.9)]. Assume that $\dim W_d^r(X) = d - 2r - 1$; if we impose $r-1$ general base points to the series belonging to $W_d^r(X)$, we obtain that $\dim W_{d-r+1}^1(X) \geq d-2r-1+(r-1) = (d-r+1)-3$. Now, applying theorem 1.3.2 we get that $\dim W_{d-r+1}^1(X) = (d-r+1)-3$. So we restrict ourselves to the case $r = 1$. Let d be the minimum integer such that $\dim W_d^1(X) = d-3$. We argue exactly as in Martens and Mumford theorems in [ACGH], pages 192-193 to conclude that

$$\dim W_{2d}^{d-1}(X) \geq d - 3. \tag{1.21}$$

To this situation we want to apply Theorem 1.3.2, so we have three cases:

I) $2d \leq g-1$. Then $\dim W^{d-1}_{2d} \leq 2d - 2(d-1) - 1$, so, by (1.21) we have that $d \leq 4$.

II) $g-1 < 2d < 2g-4$. Then, since

$$\dim W^{d-1}_{2d}(X) = \dim W^{g-d-2}_{2g-2d-2}(X) \geq d-3,$$

hence again by Theorem 1.3.2 we get $d \leq 4$.

III) $2d = 2g-4$. Again using residual series, we have that $\dim W^{g-d-2}_{2g-2d-2}(X) = W^0_2(X)$, therefore, using Proposition 1.3.5 we have $d \leq 5$.

Therefore in any case we have $d \leq 5$, and if $d = 5$ then $g = 7$. If $d = 3$ we get that $\dim W^1_3(X) = 0$, hence X has a $\mathfrak{g}^1_3$, and we are in case (i). Now let us assume $d \geq 4$ and X without trigonal series. Since $d \leq g-2$ we have that $g \geq 6$. Moreover the fact that $\dim W^1_d(X) = d-3$ implies that if $d = 4$, then $\dim W^1_4(X) = 1$, hence there exists a line bundle $L' \neq L \in W^1_4(X)$. By the minimality of d we have that L has no base points; so applying the base-point-free pencil trick and Clifford's theorem as in [**?**, Theorem (5.2)], we get that the map

$$v : H^0(X, L) \otimes H^0(X, L') \to H^0(X, L \otimes L')$$

is injective. Then, since the kernel of v is $H^0(X, L^{-1} \otimes L')$, we have that using Riemann-Roch's theorem $h^0(X, \omega_X L^{-1} L'^{-1}) = g-5$. Let now $p_1, \ldots, p_{g-6}$ be general points on X, since $\dim |\omega_X L^{-1} L'^{-1}| = g-6$, we have that $|\omega_X L^{-1} L'^{-1}(-p_1 - \cdots - p_{g-6})| \neq \emptyset$, so $|L'| \subseteq |\omega_X L^{-1}(-p_1 - \cdots - p_{g-6})|$. Now let us denote by $|M| = |\omega_X L^{-1}(-p_1 - \cdots - p_{g-6})|$, then $\dim |M| = 2$, hence it defines a morphism $\phi_M : X \to \mathbb{P}^2$; now $h^0(X, L') = 2$, hence L' induces a morphism $\phi_{L'} : X \to \mathbb{P}^1$, so, since $|L'| \subseteq |M|$ we can factorize $\phi_{L'}$ via ϕ_M plus a projection onto $\mathbb{P}^1$ with center a point of $\phi_M(X)$. Since $W^1_4(X)$ has positive dimension, we can assume that the center of projection is a smooth point of $\phi_M(X)$. Hence we have that $\deg L = \deg \phi_M(\deg \phi_M(X) - 1)$, and recalling that $\deg L = 4$, we have the following possibilities: $\deg \phi_M = 1$, therefore ϕ_M is birational and $\deg \phi_M(X) = 5$; so $\phi_M(X)$ is an irreducible plane quintic. Otherwise $\deg \phi_M = 2$ and $\deg \phi_M(X) = 3$ and X is a $2-$sheeted cover of a plane cubic curve. Now the last case to consider is $d = 5$, which comes up in case $d = g-2$, hence $g = 7$. We want to exclude this case. To this end it is sufficient to apply the argument in [Be77, (4.9.5)]: indeed if we take $\omega_X L^{-2}$, for L general in $W^1_d(X)$, by Proposition 1.3.5, we have that $\omega_X L^{-2} = \mathcal{O}_X(p+q)$ for general points $p, q \in X$. Following [Be77, (4.9.5)] we obtain that X is hyperelliptic, hence a contradiction. □

The image of the Abel map for irreducible nodal curves

In this paragraph we obtain some results which have been proved in section 1.2 by using different tools in order to show applications of the previous section. Let X be an irreducible nodal curve of genus g, and let $\dot{X} := X \setminus X_{\text{sing}}$ its smooth locus. We can consider $\dot{X}^d$, which is a smooth irreducible variety of dimension d, open and dense in X^d. Now let $\alpha_X^d : \dot{X}^d \to \text{Pic}^d X$ such that $(p_1, \ldots, p_d) \mapsto \mathcal{O}_X(\sum_{i=1}^{d} p_i)$; α_X^d is the Abel map of degree d. We can consider its image, and in particular its closure in $\text{Pic}^d X$, $A_d(X) := \overline{\alpha_X^d(\dot{X}^d)}$. We have that $A_d(X)$ is contained in $W_d(X)$, and it is irreducible by definition. We have the following:

Lemma 1.3.4. *Let X be a nodal irreducible curve of genus g, then for any $0 \le d \le g-1$ we have that $\dim A_d(X) = d$ and $h^0(X, L) = 1$ for the general $L \in A_d(X)$.*

Proof. By induction on $g - d$. For $d = g - 1$ the thesis follows from [C2, Theorem 3.1.2]. So we can assume $g - d \ge 2$. Let us observe that of course $\dim A_d(X) \le \min\{d, g\}$ and the fiber of α_X^d over a general L has dimension $h^0(X, L) - 1$, therefore it is sufficient to prove that there exists at least one $L \in A_d(X)$ such that $h^0(X, L) = 1$. We have that $d \le g - 2$, and in particular, if $Y \xrightarrow{\nu} X$ is the normalization of X at one node n, we have that $d \le g_Y - 1$, hence we can apply induction to Y. So $\dim A_d(Y) = d$ and $h^0(Y, M) = 1$ for the general $M \in A_d(Y)$. This means that there exists an open set $U \subset A_d(Y)$ such that for any $M \in U \cap \alpha_Y^d(\dot{Y}^d)$ we have $M = \mathcal{O}_Y(a_1 + \cdots + a_d)$ for suitable $a_i \in \dot{Y}$ and $h^0(Y, M) = 1$. Since $A_d(Y)$ is irreducible, $\dim U \cap \alpha_Y^d(\dot{Y}^d) = d$, therefore the general line bundle M in $U \cap \alpha_Y^d(\dot{Y}^d)$ will be such that $a_i \ne q_1, q_2$ for any $i = 1, \ldots, d$, where $\nu^{-1}(n) = \{q_1, q_2\}$. This is equivalent to saying that the general $M \in U \cap \alpha_Y^d(\dot{Y}^d)$ satisfies $h^0(Y, M - q_1) = h^0(Y, M - q_2) = 0$, so we can apply [C2, Lemma 2.2.3.] and get that $W_M(X) = \{L_M\}$, $h^0(X, L_M) = 1$ and $L_M \in \alpha_X^d(\dot{X}^d)$, where the notation is the one of Theorem 1.3.2, i.e. $W_M(X)$ is the fiber over M of $\rho : W_d(X) \to W_d(Y)$. So we have that there exists at least one $L \in A_d(X)$ with $h^0(X, L) = 1$, and we are done. □

Proposition 1.3.5. *Let X be an irreducible nodal curve of genus g. Then, for every $0 \le d \le g - 1$ we have that $W_d(X) = A_d(X)$. In particular $W_d(X)$ is irreducible of dimension d, and for general $L \in W_d(X)$ we have $h^0(X, L) = 1$.*

Remark 1.3.6. The irreducibility of $W_d(X)$ when X is an irreducible stable curve, is proved in Theorem 1.2.1.

Proof of proposition 1.3.5. We rearrange the proof of [C5, Proposition 25] using induction on $g-d$. If $d = g-1$ the thesis follows from [C2, Theorem 3.1.2]. Let us now assume $g-d \ge 2$ and let $Y \xrightarrow{\nu} X$ be the normalization of X at a node n, such that $\nu^{-1}(n) = \{q_1, q_2\}$. Let $\rho : W_d(X) \to W_d(Y)$ be the map induced by the pullback. By inductive hypothesis we have that $W_d(Y)$ has a unique component of dimension d, which is exactly $A_d(Y)$. Let us denote by $B \subset A_d(Y)$ the locus of M such that $h^0(Y, M) = h^0(Y, M - q_1) = h^0(Y, M - q_2) = 1$. Then of course we have that $h^0(Y, M - q_1 - q_2) = 1$, then $\dim B \le d - 2$, hence $\dim \rho^{-1}(B) \le d - 1$ since the dimension of the fibers is at most 1. By Lemma 1.3.4 there exists a dense open set $U \subset A_d(Y) \setminus B$ such that for any $M \in U$ we have $h^0(Y, M) = 1$. By [C5, Lemma 9] the fiber of ρ over M is a unique point, hence $\rho^{-1}(U)$ is irreducible of dimension d. Now by Theorem 1.3.2 we have that $\dim W_d^1(Y) \leq d-2$, therefore any other component of $W_d(X)$, if it exists, has dimension at most $d-1$. So this proves that $W_d(X)$ has a unique component of dimension d, which is exactly $A_d(X)$ by Lemma 1.3.4. □

1.3.2 k-gonal curves

In the present paragraph we want to describe some properties of curves lying in the boundary of the k-gonal locus in $\overline{\mathcal{M}_g}$, the moduli space of smooth curves of genus g. We start by studying $k=3$. Given any curve X, we will denote by $\mathfrak{g}_d^r$ a linear series of degree d such that every $L \in \mathfrak{g}_d^r$ has $h^0(X,L) \geq r+1$. Let then

$$\mathcal{M}_{g,d}^r := \{[C] \in \mathcal{M}_g \ : \ C \text{ has a } \mathfrak{g}_d^r\}.$$

If we consider $\mathcal{M}_{g,3}^1$, this turns out to be an irreducible variety of dimension $2g+1$, called the *Brill-Noether* locus (see [HMo98]). We look at its closure $\overline{\mathcal{M}_{g,3}^1}$ in $\overline{\mathcal{M}_g}$.

Definition 1.3.7. Let X be a stable curve of genus g; we say that $X \in \overline{\mathcal{M}_{g,3}^1}$ is *trigonal* if there exists a family $f : \mathcal{X} \to B$ of smooth curves in $\mathcal{M}_{g,3}^1$ having X as central fiber.

We notice that in general, if a curve X has a $\mathfrak{g}_3^1$, it does not mean that it is trigonal. In order to study this concept, we recall some definitions from [EH86]. Let C be a smooth curve with a $\mathfrak{g}_d^r$, and let $L \in \mathfrak{g}_d^r$. If $p \in C$ and s is a section of L, then we denote by $ord_p(s)$ the order of vanishing of s at p. There are exactly $r+1$ distinct integers $a_0^L(p) < a_1^L(p) < \ldots < a_r^L(p)$ which are orders of vanishing of sections of L at p.

Definition 1.3.8. If X is a curve of compact type, then a *limit* $\mathfrak{g}_d^r$ L on X is, for each irreducible component Y of X, a $\mathfrak{g}_d^r$ L_Y on Y, called the $Y-$*aspect* of L, satisfying the compatibility condition: if Y_1 and Y_2 are components of X meeting at a point p, then for $i = 0, \ldots, r$ we must have

$$a_i^{L_{Y_1}}(p) + a_{r-i}^{L_{Y_2}}(p) = d.$$

We have now the following:

Lemma 1.3.9. *Let $X = C_1 \cup C_2$ be a stable curve of compact type with C_i smooth of genus g_i, $C_1 \cap C_2 = \{p\}$ with branches $p_i \in C_i$ for $i = 1,2$, such that X is trigonal. Then one of the following holds:*

(i) C_1 and C_2 are hyperelliptic and their hyperelliptic series are $\mathcal{O}_{C_1}(2p_1)$ and $\mathcal{O}_{C_2}(2p_2)$.

(ii) C_1 and C_2 are hyperelliptic, the C_1-aspect of the $\mathfrak{g}_3^1$ on X is $\mathcal{O}_{C_1}(2p_1+r)$ and the C_2-aspect is $\mathcal{O}_{C_2}(2p_2+s)$, where $r \in C_1$, $s \in C_2$, and p_1, p_2 are base points.

(iii) C_1 and C_2 are trigonal, the C_1-aspect of the $\mathfrak{g}_3^1$ on X is $\mathcal{O}_{C_1}(3p_1)$ and the C_2-aspect is $\mathcal{O}_{C_2}(3p_2)$, where p_1 and p_2 are not base points.

(iv) *C_1 is trigonal and C_2 is hyperelliptic (up to switching the curves), the C_1-aspect of the $\mathfrak{g}^1_3$ on X is $\mathcal{O}_{C_1}(2p_1+r)$ and the C_2-aspect is $\mathcal{O}_{C_2}(2p_2)$ where $p_1 \neq r \in C_1$ and p_2 is not a base point for $\mathcal{O}_{C_2}(2p_2)$.*

Proof. Let $\mathcal{X} \to B$ be a regular one parameter smoothing of X whose generic fiber X_b is a smooth trigonal curve of genus g, and let $\mathcal{T} \in \mathrm{Pic}\mathcal{X}$ be a line bundle of degree 3. Let $T_b = \mathcal{T} \otimes \mathcal{O}_{X_b}$ be the trigonal series on X_b, then, denoting $T := \mathcal{T} \otimes \mathcal{O}_X$, by semicontinuity we get that $h^0(X,T) \geq 2$. We can assume up to twisting that $\underline{\deg}T = (3,0)$. Then we have to distinguish two cases.

Case I. Let $T_i = T_{|C_i}$ for $i = 1,2$, and assume that $T_2 = \mathcal{O}_{C_2}$. Then $h^0(C_2,T_2) = 1$, and, since $\mathcal{O}_{C_2}$ is free from base points we have $h^0(C_1,T_1) = 2$, hence T_1 is a trigonal series on C_1, which we denote by T_{C_1}. Let us assume that C_1 is not hyperelliptic, with $T = (T_{C_1}, \mathcal{O}_{C_2})$. If we twist T by $\mathcal{O}_{\mathcal{X}}(C_1)$ we get $T' = \mathcal{T} \otimes \mathcal{O}_{\mathcal{X}}(C_1) \otimes \mathcal{O}_X = (T_{C_1}(-p_1), \mathcal{O}_{C_2}(p_2))$, where p_1 and p_2 are the preimages on C_1 and C_2 of the node p via the normalization map. We have that $h^0(C_1, T_{C_1}(-p_1)) = 1$ and that $h^0(C_2, \mathcal{O}_{C_2}(p_2)) = 1$, then by 1.2.4 and 1.2.5 in [C6], p_1 is base point of $T_{C_1}(-p_1)$ and obviously p_2 of $\mathcal{O}_{C_2}(p_2)$, hence $h^0(C_1, T_{C_1}(-2p_1)) = 1$. Now let us consider $T'' = \mathcal{T} \otimes \mathcal{O}_{\mathcal{X}}(2C_1) \otimes \mathcal{O}_X = (T_{C_1}(-2p_1), \mathcal{O}_{C_2}(2p_2))$, then we have $h^0(C_2, \mathcal{O}_{C_2}(2p_2)) \leq 2$. If $h^0(C_2, \mathcal{O}_{C_2}(2p_2)) = 2$, then C_2 is hyperelliptic with hyperelliptic series $H_{C_2} = \mathcal{O}_{C_2}(2p_2)$. Moreover, since $h^0(C_1, T_{C_1}(-2p_1)) = 1$, we have that the C_2-aspect of T is $\mathcal{O}_{C_2}(3p_2)$ with p_2 a base point, and $T_{C_1} = \mathcal{O}_{C_1}(2p_1 + r)$ with $r \neq p_1$, since if it were $r = p_1$ we would have that $T''' = \mathcal{T} \otimes \mathcal{O}_{\mathcal{X}}(3C_1) \otimes \mathcal{O}_X = (\mathcal{O}_{C_1}, \mathcal{O}_{C_2}(3p_2))$, hence $\mathcal{O}_{C_1}$ should have a base point at p_1, which is impossible. So we get conclusion (iv). If, on the other hand, $h^0(C_2, \mathcal{O}_{C_2}(2p_2)) = 1$, we get (again by 1.2.4. and 1.2.5. in [C6]) that p_1 is a base point of $T_{C_1}(-2p_1)$, hence $T_{C_1} = \mathcal{O}_{C_1}(3p_1)$. We twist again obtaining $T''' = \mathcal{T} \otimes \mathcal{O}_{\mathcal{X}}(3C_1) \otimes \mathcal{O}_X = (\mathcal{O}_{C_1}, \mathcal{O}_{C_2}(3p_2))$; since $\mathcal{O}_{C_1}$ is free from base points, $h^0(C_2, \mathcal{O}_{C_2}(3p_2)) = 2$, hence C_1 and C_2 are trigonal and the trigonal series are $\mathcal{O}_{C_1}(3p_1)$ and $\mathcal{O}_{C_2}(3p_2)$, according to conclusion (iii).
So far we have assumed that C_1 is not hyperelliptic; if it is, then T_1 has a base point, hence $T_1 = H_{C_1}(q)$, where $q \in C_1$ and $q \neq p_1$ otherwise $T_2 = \mathcal{O}_{C_2}$ should have a base point in p_2. As before we consider $T' = \mathcal{T} \otimes \mathcal{O}_{\mathcal{X}}(C_1) \otimes \mathcal{O}_X = (H_{C_1}(q - p_1), \mathcal{O}_{C_2}(p_2))$, and we have $h^0(H_{C_1}(q - p_1)) = 1$ since $q \neq p_1$; then p_1 is base point for $H_{C_1}(q - p_1)$. Let $T'' = (H_{C_1}(q - 2p_1), \mathcal{O}_{C_2}(2p_2))$; if C_2 is hyperelliptic with $h^0(C_2, \mathcal{O}_{C_2}(2p_2)) = 2$, noticing moreover that $h^0(H_{C_1}(q - 2p_1)) = 1$, we obtain conclusion (i). If otherwise $h^0(C_2, \mathcal{O}_{C_2}(2p_2)) = 1$, then, since $h^0(C_1, H_{C_1}(q - 2p_1)) = 1$, we have that p_1 is a base point for $H_{C_1}(q - 2p_1)$, hence $h^0(C_1, H_{C_1}(q - 3p_1)) = 1$, therefore $H_{C_1}(q) = \mathcal{O}_{C_1}(3p_1)$, but then $H_{C_1} = \mathcal{O}_{C_1}(2p_1)$, and we would have that $q = p_1$, a contradiction.

Case II. Let us now assume that $T_2 \neq \mathcal{O}_{C_2}$. We are still in the hypothesis that $\underline{\deg}T = (3,0)$; we have that $h^0(C_2, T_2) = 0$, hence $h^0(C_1, T_1) = 2$ and p_1 is a base point for T_1, i.e. $h^0(C_1, T_1(-p_1)) = 2$, therefore C_1 is hyperelliptic and $T_1 = H_{C_1}(p_1)$. Let us consider as before the twist $T' = \mathcal{T} \otimes \mathcal{O}_{\mathcal{X}}(C_1) \otimes \mathcal{O}_X = (H_{C_1}, T_2(p_2))$, then it must be $h^0(T_2(p_2)) = 1$, indeed if it were $h^0(T_2(p_2)) = 0$ then the hyperelliptic series H_{C_1} should have a base point at p_1, which is impossible. Now we take $T'' = \mathcal{T} \otimes \mathcal{O}_{\mathcal{X}}(2C_1) \otimes \mathcal{O}_X = (H_{C_1}(-p_1), T_2(2p_2))$; $h^0(C_1, H_{C_1}(-p_1)) = 1$, and $h^0(C_2, T_2(2p_2)) \leq 2$. If $h^0(C_2, T_2(2p_2)) = 1$, then p_1 is a base point for $H_{C_1}(-p_1)$, so $H_{C_1} = \mathcal{O}_{C_1}(2p_1)$, whereas $T_2(p_2) = \mathcal{O}_{C_2}(q)$ for a $q \in C_2$, $q \neq p_2$. We look at $T''' = \mathcal{T} \otimes \mathcal{O}_{\mathcal{X}}(3C_1) \otimes \mathcal{O}_X = (\mathcal{O}_{C_1}, T_2(3p_2))$, then $h^0(C_2, T_2(3p_2)) = 2$, meaning that C_2 is trigonal and the trigonal series is $T_2(3p_2) = \mathcal{O}_{C_2}(q + 2p_2)$, so we agree with conclusion (iv). The last case to consider is when $h^0(C_2, T_2(2p_2)) = 2$, i.e. C_2 is hyperelliptic and we recall that C_1 is hyperelliptic with $H_{C_1} = \mathcal{O}_{C_1}(r + p_1)$, where $r \in C_1$, and $h^0(C_2, T_2(p_2)) = 1$, therefore $T_2(p_2) = \mathcal{O}_{C_2}(s)$ with $s \in C_2$. So we have that $T_2(2p_2) = \mathcal{O}_{C_2}(s + p_2) = H_{C_2}$, and we get conclusion (ii).

□

Using the theory of limit linear series we are able to state a vice-versa of 1.3.9, as we will see in the following. First of all we recall some fundamental facts from [EH86].

Definition 1.3.10. If X is a curve of compact type, then we say that a limit $\mathfrak{g}^r_d$ on X can be *smoothed* if there exists a family of smooth curves $f : \mathcal{X} \to B$ with central fiber X, and a $\mathfrak{g}^r_d$ denoted by L_b on the general fiber X_b, whose limit is the given limit $\mathfrak{g}^r_d$ on X.

Proposition 1.3.11 (Eisenbud-Harris). *On a curve of compact type every limit $\mathfrak{g}^1_d$ can be smoothed, and the smoothing can be done so as to preserve all ramification points away from the nodes.*

Lemma 1.3.12. *Let $X = C_1 \cup C_2$ be a stable curve of compact type with C_i smooth of genus g_i, $C_1 \cap C_2 = \{p\}$ with branches $p_i \in C_i$ for $i = 1, 2$, and assume one of the following:*

(i) C_1 and C_2 are hyperelliptic and their hyperelliptic series are $\mathcal{O}_{C_1}(2p_1)$ and $\mathcal{O}_{C_2}(2p_2)$.

(ii) C_1 and C_2 are hyperelliptic, $h^0(C_1, \mathcal{O}_{C_1}(2p_1 + r)) = 2$ and $h^0(C_2, \mathcal{O}_{C_2}(2p_2 + s)) = 2$, where $r \in C_1$ and $s \in C_2$, and p_1, p_2 are base points.

(iii) C_1 and C_2 are trigonal, $h^0(C_1, \mathcal{O}_{C_1}(3p_1)) = 2$ and $h^0(C_2, \mathcal{O}_{C_2}(3p_2)) = 2$, and p_1 and p_2 are not base points.

(iv) C_1 is trigonal and C_2 is hyperelliptic (up to switching the curves), $h^0(C_1, \mathcal{O}_{C_1}(2p_1 + r)) = 2$ and $h^0(C_2, \mathcal{O}_{C_2}(3p_2)) = 2$, where $p_1 \neq r \in C_1$ and p_2 is not a base point for $\mathcal{O}_{C_2}(3p_2)$.

Then X has a limit $\mathfrak{g}^1_3$ that can be smoothed.

Proof. In order to apply 1.3.11 we only have to prove that the $\mathfrak{g}^1_3$'s on C_1, C_2 as in the four cases above are the aspects of a limit series on X, as in 1.3.8. In the following we will denote the aspects by T_1 and T_2. So let's start by assuming (i): it is well known that a curve of compact type $X = C_1 \cup C_2$ is hyperelliptic if and only if both C_1 and C_2 are hyperelliptic and p_1 and p_2 are Weierstrass points. Then in our case X is hyperelliptic, hence trivially trigonal. Now let us assume (ii): the components of X are hyperelliptic curves, with series $H_{C_1} = \mathcal{O}_{C_1}(p_1 + r)$ and $H_{C_2} = \mathcal{O}_{C_2}(p_2 + s)$ and the aspects are $T_1 := \mathcal{O}_{C_1}(2p_1 + r)$ and $T_2 := \mathcal{O}_{C_2}(2p_2 + s)$. Let us consider T_1's global sections $H^0(C_1, T_1) = \{\sigma_0, \sigma_1\}$; since p_1 is a base point of T_1, we have that $\sigma_0(p_1) = \sigma_1(p_1) = 0$. Moreover we observe that $h^0(\mathcal{O}_{C_1}(r)) = h^0(T_1(-2p_1)) = 1$, hence there exists only one section of T_1 vanishing on p_1 with multiplicity 2. The same reasoning works for T_2 with $H^0(C_2, T_2) = \{\tau_0, \tau_1\}$, so according to definition 1.3.8 we have that the orders of vanishing of the sections are:

$$a_0^{T_1}(p_1) = 1, a_1^{T_1}(p_1) = 2, a_0^{T_2}(p_2) = 1, a_1^{T_2}(p_2) = 2.$$

Hence we have that

$$a_0^{T_1}(p_1) + a_1^{T_2}(p_2) = a_0^{T_2}(p_2) + a_1^{T_1}(p_1) = 3,$$

which tells us that T is a limit $\mathfrak{g}_3^1$ on X. Let us now turn to case (iii): C_1 and C_2 are trigonal curves with series $T_1 = \mathcal{O}_{C_1}(3p_1)$ and $T_2 = \mathcal{O}_{C_2}(3p_2)$; p_1 and p_2 are not base points, so $h^0(C_1, \mathcal{O}_{C_1}(2p_1)) = 1$, but we also know that $h^0(C_1, \mathcal{O}_{C_1}(p_1)) = 1$. Therefore there exist sections $\{\sigma_0, \sigma_1\}$ for $H^0(C_1, T_1)$ such that $\sigma_0(p_1) \neq 0$ and $\sigma_1(p_2) = 0$, and moreover $h^0(C_1, T_1(-3p_1)) = 1$, hence σ_1 vanishes at p_1 with multiplicity 3. So we get that, doing the same for T_2, $a_0^{T_1}(p_1) = 0$, $a_1^{T_1}(p_1) = 3$, $a_0^{T_2}(p_2) = 0$ and $a_1^{T_2}(p_2) = 3$, and arguing as before we have that T is a limit $\mathfrak{g}_3^1$ on X. The last case is when C_1 is trigonal with series $T_1 = \mathcal{O}_{C_1}(2p_1 + r)$, and C_2 is hyperelliptic with series $\mathcal{O}_{C_2}(2p_2)$ and C_2-aspect $T_2 = \mathcal{O}_{C_2}(3p_2)$, so p_2 is a base point for T_2. Then we have that $h^0(C_2, T_2(-p_2)) = 2$ and $h^0(C_2, T_2(-3p_2)) = 1$, therefore, if $H^0(C_2, T_2) = \{\tau_0, \tau_1\}$, we have that $ord_{p_2}(\tau_0) = a_0^{T_2}(p_2) = 1$ and $ord_{p_2}(\tau_1) = a_1^{T_2}(p_2) = 3$. Moreover, if we take $H^0(C_1, T_1) = \{\sigma_0, \sigma_1\}$, we see that since p_1 is not a base point for T_1, and since $h^0(C_1, T_1(-p_1)) = 1$, then $ord_{p_1}(\sigma_0) = a_0^{T_1}(p_1) = 0$ and $ord_{p_1}(\sigma_1) = a_1^{T_1}(p_1) = 2$. Therefore we have that

$$a_0^{T_1}(p_1) + a_1^{T_2}(p_2) = a_0^{T_2}(p_2) + a_1^{T_1}(p_1) = 3,$$

which means that as well as in the previous cases T_1 and T_2 define a limit $\mathfrak{g}_3^1$ on X. Applying 1.3.11 we are done. □

We are now going to introduce the notion of *weakly $k-$gonal* curves, using the terminology encountered in [C5] and [C6]:

Definition 1.3.13. We say that a nodal curve X is *weakly $k-$gonal* if there exists a balanced (see section 1.1) line bundle $L \in \mathrm{Pic}^k X$ such that $h^0(X, L) \geq 2$.

Lemma 1.3.14. *Let $X = C_1 \cup C_2$ be a stable curve of compact type with C_i smooth of genus g_i and $1 \leq g_1 \leq g/2$, be a not weakly hyperelliptic curve. If $g_1 \leq (g+2)/6$, then X is weakly trigonal if and only if C_2 is trigonal, otherwise, for $g_1 > (g+2)/6$, X is not weakly trigonal.*

Proof. Let $d = 3$, then the only possibilities we have are $\underline{d} = (0,3)$ or $\underline{d} = (1,2)$. By hypothesis we have that $g_1 \leq g_2$, and we see that $(1,2)$ is balanced if and only if $(g+2)/6 \leq g_1 \leq g/2$. We want to show that when $g_1 > (g+2)/6$, X is not weakly trigonal, i.e. there doesn't exist a balanced line bundle $L \in \operatorname{Pic}^3 X$ such that $h^0(X,L) \geq 2$. Indeed if such an L existed, then $\underline{\deg} L = (1,2)$, and, if we denote $L_i = L_{|C_i}$ for $i = 1,2$, we would have that $h^0(C_1, L_1) \leq 1$ hence L_1 should have a base point, but then L would have a base point too, which is not possible since we are assuming that X is not weakly hyperelliptic. Hence we look at the case $g_1 \leq (g+2)/6$, when $\underline{d} = (0,3)$ is balanced: let us suppose that C_2 is trigonal with $\mathfrak{g}^1_3$ denoted by T_{C_2}, then if we take $L = (\mathcal{O}_{C_1}, T_{C_2})$ we have that $h^0(C_1, L_1) = 1$, and $h^0(C_2, L_2) = 2$. So $h^0(X,L) \geq 2$, by 2.1.1 in [C2], and X is weakly trigonal. On the other hand, let $L \in \operatorname{Pic}^3 X$ with $h^0(X,L) = 2$; then it must be $\underline{\deg} L = (0,3)$, and since X has no $\mathfrak{g}^1_2$, L cannot have base points, so $L_1 = \mathcal{O}_{C_1}$, and $h^0(C_2, L_2) \leq 3$. If C_2 were elliptic, we would have that $h^0(C_2, L_2) = 3$, but in this case $g_1 = g_2 = 1$, hence $g = g_1 + g_2 = 2$, against the fact that $g_1 \leq (g+2)/6$. So via Clifford's theorem we get that $h^0(C_2, L_2) = 2$, hence C_2 is trigonal. ☐

From this lemma we see that a weakly trigonal curve need not be necessarily trigonal. In the following we are going to study properties of weakly trigonal semistable curves. We recall that a separating node of X is a node n such that there exist two subcurves of X, X_1 and X_2, with $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \{n\}$; we denote again by X_{sep} the set of separating nodes of X. A *separating line* of a curve X is a subcurve $C \subset X$ such that $C \cong \mathbb{P}^1$ and that C meets its complementary curve C^c in separating nodes of X. Let X_{sm} be the open set of smooth points of X.

Lemma 1.3.15. *Let X be an irreducible nodal curve and $L \in \operatorname{Pic}^1 X$ such that $h^0(X,L) = 1$. Then there exists a point $p \in X_{\text{sm}}$ such that $L = \mathcal{O}_X(p)$.*

Proof. Let X be a curve as in the statement of the Lemma, and let $\widetilde{X} \stackrel{\nu}{\longrightarrow} X$ be its normalization. Let us denote by $n_1, \dots, n_\delta$ the nodes of X, and $\nu^{-1}(n_i) = \{p_i, q_i\}$ for any i. Take L any line bundle on X of degree 1 and with $h^0(X,L) = 1$. Then $\nu^*(L) \in \operatorname{Pic}\widetilde{X}$ is a line bundle of degree 1, and if $\widetilde{X} \ncong \mathbb{P}^1$ we must have that $h^0(\widetilde{X}, \nu^*(L)) = 1$. Moreover compatibility conditions give that

$$h^0(\widetilde{X}, \nu^*(L)(-p_i)) = h^0(\widetilde{X}, \nu^*(L)(-q_i)) = h^0(\widetilde{X}, \nu^*(L)(-p_i - q_i));$$

now if $\nu^*(L) = \mathcal{O}_{\widetilde{X}}(A)$ for a certain point $A \in \widetilde{X}, A \neq p_i, q_i$ for any i, then $L = \mathcal{O}_X(A)$ abusing the notation since ν is an isomorphism out of the branches of the nodes of X. On the other hand, if, say, $\nu^*(L) = \mathcal{O}_{\widetilde{X}}(p_i)$, then applying the compatibility conditions we have that $h^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(p_i - q_i)) = 1$, hence $\nu^*(L) = \mathcal{O}_{\widetilde{X}}(q_i)$, but then $\mathcal{O}_{\widetilde{X}}(p_i) \cong \mathcal{O}_{\widetilde{X}}(q_i)$, which is a contradiction since $\widetilde{X} \ncong \mathbb{P}^1$. Now if $\widetilde{X} = \mathbb{P}^1$ we have that $\nu^*(L) = \mathcal{O}_{\mathbb{P}^1}(1)$, hence $h^0(\mathcal{O}_{\mathbb{P}^1}(1)) = 2$. It suffices to assume that X has just one node n, otherwise we repeat the argument for every other node of X; let us denote by $\{p, q\}$ the branches of the node. Let s be a section which generates $H^0(X,L)$, then it comes from a section $\tilde{s} \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$

if $\tilde{s}(p) = c\tilde{s}(q)$ for a $c \in \mathbb{K}^*$. If s should vanish on the node n, then we should have that $\tilde{s}(p) = \tilde{s}(q) = 0$, but the sections of $\mathcal{O}_{\mathbb{P}^1}(1)$ are linear polynomials, so they cannot vanish in two points. This shows that, as well as in the previous case, L has a base point which is not a node. □

Proposition 1.3.16. *Let X be a semistable curve of genus g with irreducible components $C_1, \dots, C_r$, not weakly hyperelliptic and without separating nodes, and $\underline{d}$ balanced such that $|\underline{d}| = 3$. Suppose that there exists a line bundle $L \in \mathrm{Pic}^{\underline{d}}X$ with $h^0(X, L) = 2$. Then one of the following holds:*

(i) $\underline{d} = (1, 2, 0, \dots, 0)$*, C_1 is a separating line of $\overline{X \setminus C_2}$, and*

either C_2 has a hyperelliptic series H_{C_2}, with

$$L|_{C_1} = \mathcal{O}_{C_1}(1), L|_{C_2} = H_{C_2}, L|_{\overline{X\setminus\{C_1\cup C_2\}}} = \mathcal{O}_{\overline{X\setminus\{C_1\cup C_2\}}},$$

or $C_2 \cong \mathbb{P}^1$ with

$$L|_{C_1} = \mathcal{O}_{C_1}(1), L|_{C_2} = \mathcal{O}_{C_2}(2), L|_{\overline{X\setminus\{C_1\cup C_2\}}} = \mathcal{O}_{\overline{X\setminus\{C_1\cup C_2\}}}.$$

(ii) $\underline{d} = (3, 0, \dots, 0)$ *and*

if $C_1 \ncong \mathbb{P}^1$, C_1 has a trigonal series T_{C_1},

$$L|_{C_1} = T_{C_1}, L|_{Z_1} = \mathcal{O}_{Z_1}, \dots, L|_{Z_m} = \mathcal{O}_{Z_m},$$

where $Z_1, \dots, Z_m$ are the connected components of $\overline{X \setminus C_1}$, and $2 \leq \sharp(C_1 \cap Z_i) \leq 3$ for every $i = 1, \dots, m$;

if $C_1 \cong \mathbb{P}^1$, then

$$L|_{C_1} = \mathcal{O}_{C_1}(3), L|_{Z_1} = \mathcal{O}_{Z_1}, \dots, L|_{Z_m} = \mathcal{O}_{Z_m},$$

and $m = 1$ if and only if $\sharp\{C_1 \cap {C_1}^c\} = 3$.

(iii) $\underline{d} = (1, 1, 1, 0, \dots, 0)$*, $C_1 \cong C_2 \cong C_3 \cong \mathbb{P}^1$ and*

$$L|_{C_1} = \mathcal{O}_{C_1}(1), L|_{C_2} = \mathcal{O}_{C_2}(1), L|_{C_3} = \mathcal{O}_{C_3}(1), L|_{C_4} = \mathcal{O}_{C_4}, \dots, L|_{C_r} = \mathcal{O}_{C_r}.$$

Proof. Let $\underline{d} = (1, 2, 0, \dots, 0)$, and let C_1, C_2 be the two components of X such that $d_{C_1} = 1$ and $d_{C_2} = 2$. Since $\underline{d}$ is balanced on X semistable, if $C_2 \cong \mathbb{P}^1$ it must be $C_2 \cdot C_2^c \geq 3$ and X stable, indeed the degree on an exceptional component must be 1. Let us assume at first that $C_2 \ncong \mathbb{P}^1$, so we have that $h^0(C_2, L_{|C_2}) \leq 2$. If Y is a subcurve of X, we denote by $L_Y = L_{|Y}$, and $l_Y = h^0(Y, L_Y)$; let us now assume that $l_{C_2} = 2$. Then C_2 is hyperelliptic, $L_{C_2} = H_{C_2}$ the hyperelliptic series, and if we call $Z := X \setminus C_2$, we have that $h^0(X, L) \leq l_{C_2} + l_Z$. We notice that $\underline{d}_Z = (1, 0, \dots, 0)$, so we apply lemma 4.2.3 in [C6] and get either $l_Z \leq 1$ or $l_Z = 2$ and C_1 is a separating line of Z, $l_Z = 2$ and $L_{\overline{Z\setminus C_1}} = \mathcal{O}_{\overline{Z\setminus C_1}}$. So let us first suppose that $C_1 \not\cong \mathbb{P}^1$, then $l_Z \leq 1$; we cannot have $l_Z = 0$, otherwise L_{C_2} would have base points, which we don't allow being X not weakly hyperelliptic. So we have that $l_{C_2} = 2$ and $l_Z = 1$. Let us denote by $\{n_1, \dots, n_h\}$ for a suitable h the elements of $C_2 \cap Z$. Since $X_{sep} = \emptyset$, we have that $C_2 \cdot Z = h \geq 2$. Being $l_X = l_{C_2} + l_Z - 1$, via 1.2.6. in [C6] we get that, calling $\{p_i, q_i\}$ the branches of n_i on C_2 and Z respectively, $p_i \sim_{L_{C_2}} p_j$ and $q_i \sim_{L_Z} q_j$ for $i \neq j$, therefore $h^0(C_2, L_{C_2}(-p_i)) = h^0(C_2, L_{C_2}(-p_j)) = h^0(C_2, L_{C_2}(-p_i - p_j)) = 1$ for $1 \leq i \neq j \leq h$. This means that the hyperelliptic series $H_{C_2} = \mathcal{O}_{C_2}(p_i + p_j)$ for $1 \leq i \neq j \leq h$; since $C_2 \not\cong \mathbb{P}^1$ this is possible if and only if

$$C_2 \cdot Z = 2 \text{ and } H_{C_2} = \mathcal{O}_{C_2}(p_1 + p_2).$$

Moreover recall that $l_Z = 1$, then we can have $l_{C_1} = 1$ or $l_{C_1} = 0$. If $l_{C_1} = 1$, since $d_{C_1} = 1$, by Lemma 1.3.15 we would have that L_{C_1}, and then L, has a base point, which is impossible. So it must be $l_{C_1} = 0$; now let us denote by $Y = \overline{Z \setminus C_1}$, hence $\underline{d}_Y = \underline{0}$, moreover, by Lemma 1.3.17, since $C_2 \cdot Z = 2$, we have that $C_1 \cdot Y \geq 1$, so, if $q \in C_1 \cap Y$, we have that $l_Z = h^0(Y, L_Y(-q)) = 1$, but then we contradict Remark 4.1.2. in [C6]. Hence we can exclude the case $l_{C_2} = 2$ and $l_Z = 1$, and look at the one with $l_{C_2} = 2$ and $l_Z = 2$. In this case C_1 is a separating line for Z, C_2 is hyperelliptic, and $L = (\mathcal{O}_{C_1}(1), H_{C_2}, \mathcal{O}_{\overline{Z \setminus C_1}})$. Let us now suppose $l_{C_2} = 1$; then there exist points $r, s \in C_2$ such that $L_{C_2} = \mathcal{O}_{C_2}(r + s)$, i.e. L_{C_2} has base points, which is a contradiction. We study now the case when $C_2 \cong \mathbb{P}^1$. Since $d_{C_2} = 2$, then X is stable and $\sharp\{C_2 \cap Z\} \geq 3$. Moreover $l_{C_2} = 3$, hence applying 1.2.9. in [C6] we get that $h^0(X, L) \leq 3 + l_Z - 3$, hence it must be $l_Z \geq 2$. Notice that $\underline{d}_Z = (1, 0, \dots, 0)$, therefore applying 4.2.3. in [C6], the only possibility we have is that C_1 is a separating line of Z. So we get conclusion (i).

Let now $\underline{d} = (3, 0, \dots, 0)$; let us denote by C_1 the component such that $d_{C_1} = 3$. Let $X = C_1 \cup Z$, and suppose the genus $g_{C_1} \geq 2$. Then by Clifford's theorem and Remark 1.2.8. in [C6], we get that $2 = h^0(X, L) \leq h^0(C_1, L_{C_1}) \leq 3/2 + 1$, hence $l_{C_1} = 2$. If C_1 were hyperelliptic with series H_{C_1}, we would have that $L_{C_1} = H_{C_1}(p)$, with $p \in C_1$, but then L_{C_1} and hence L would have a base point, therefore C_1 must be trigonal, with series T_{C_1}. Let now $Z = Z_1 \sqcup \ldots \sqcup Z_m$ be the decomposition of Z into connected components, then $\underline{d}_{Z_i} = \underline{0}$, so either $L_{Z_i} = \mathcal{O}_{Z_i}$ for every i, or there exists at least one j such that $l_{Z_j} = 0$, but in this case L would have base points, which we exclude. Therefore $L = (T_{C_1}, \mathcal{O}_{Z_1}, \dots, \mathcal{O}_{Z_m})$.
We have to consider now the cases when $g_{C_1} \leq 1$. If $g_{C_1} = 1$, then by Riemann Roch $l_{C_1} = 3$, and, as before, $L_{C_i} = \mathcal{O}_{C_i}$ for every $i = 2, \dots, r$, so we have again $L = (T_{C_1}, \mathcal{O}_{C_2}, \dots, \mathcal{O}_{C_r})$ if we denote by T_{C_1} the trigonal series on C_1.
Let us observe that, when $g_{C_1} \geq 1$ and Z is connected, then it must be $2 \leq \sharp\{C_1 \cap Z\} \leq 3$. Indeed $\underline{d}_Z = \underline{0}$ and $l_Z \geq 1$ (otherwise L would have base points), then $L_Z = \mathcal{O}_Z$, $L_{C_1} = T_{C_1}$ is the trigonal series, and $l_Z = 1$ via Corollary 2.2.5 in [C2]. If we assume that $\sharp\{C_1 \cap Z\} > 3$, then applying Lemma 1.2.6. in [C6], there would exist at least 4 points, branches of the nodes on C_1, say $p_1, \dots, p_4 \in C_1$ such that $h^0(C_1, T_{C_1}(-p_1 - p_2 - p_3 - p_4)) = 1$, but this is impossible since $\deg(T_{C_1}(-p_1 - p_2 - p_3 - p_4)) = -1$. Then it must be $\sharp\{C_1 \cap Z\} \leq 3$,

and moreover, if $\sharp\{C_1 \cap Z\} = 3$, we would have that $T_{C_1} = \mathcal{O}_{C_1}(p_1 + p_2 + p_3)$. Of course, when Z is not connected, we can repeat this argument for every connected component of $Z = Z_1 \sqcup \ldots \sqcup Z_m$, getting that $2 \leq \sharp\{C_1 \cap Z_i\} \leq 3$ for every $i = 1, \ldots, m$. Let now $g_{C_1} = 0$; since $d_{C_1} = 3$, X must be stable and $\delta := C_1 \cdot Z \geq 3$. We have $l_{C_1} = 4$, so if $\delta = 3$, 1.2.9. yields $h^0(X, L) = 4 + l_Z - 3$, hence $l_Z = 1$, moreover Z is connected by Lemma 1.3.17. This implies, applying Corollary 2.2.5. in [C2] to Z, that $L_Z = \mathcal{O}_Z$. When $\delta > 3$, 1.2.9. yields $h^0(X, L) \leq l_{C_1} + l_Z - 4$, then $l_Z \geq 2$ and Z is not connected; indeed if it were connected, Corollary 2.2.5. would imply that $L_Z = \mathcal{O}_Z$, but then $l_Z = 1$, which contradicts the fact that $l_Z \geq 2$. Hence Z is connected if and only if $\delta = 3$. Now if Z were disconnected, then we would have $Z = Z_1 \sqcup \ldots \sqcup Z_m$ for some m, and $L_Z = (\mathcal{O}_{Z_1}, \ldots, \mathcal{O}_{Z_m})$.

It remains to study the last case, when $\underline{d} = (1, 1, 1, 0, \ldots, 0)$; let us assume at first that $C_1 \ncong \mathbb{P}^1$. As well as in the previous cases we denote by $Z = \overline{X \setminus C_1}$; via Clifford's Theorem we have that $l_{C_1} \leq 1$, but if $l_{C_1} = 1$, as $d_{C_1} = 1$, by 1.3.15 we get that L_1 has a base point, hence $C_1 \cong \mathbb{P}^1$ and $h^0(C_1, L_{C_1}) = 2$. Applying Lemma 4.2.4. in [C6] we obtain that either $l_Z \leq 2$ or $l_Z = 3$ and C_2 and C_3 separating lines of Z (recall that in our notation $d_{C_1} = d_{C_2} = d_{C_3} = 1$). So let us now consider all our possibilities: if $l_Z = 1$, via 1.2.9. in [C6] we see that $h^0(X, L) = 1$, a contradiction. If instead $l_Z = 2$, let us assume that $C_2 \ncong \mathbb{P}^1$; then $l_{C_2} = 1$, but L_{C_2} cannot have base points, therefore we must have $C_2 \cong \mathbb{P}^1$. Via 1.2.9. we have that $l_Z \leq l_{C_2} + h^0(\overline{Z \setminus C_2}, L_{\overline{Z \setminus C_2}}) - \min\{\delta, 2\}$, where $\delta = \sharp\{C_2 \cap \overline{Z \setminus C_2}\}$. If $\delta \geq 2$, then $l_Z \leq l_{C_2} + h^0(\overline{Z \setminus C_2}, L_{\overline{Z \setminus C_2}}) - 2$, hence $h^0(\overline{Z \setminus C_2}, L_{\overline{Z \setminus C_2}}) \geq 2$, and since $\underline{d}_{Z \setminus C_2} = (1, 0, \ldots, 0)$, applying 4.2.3. in [C6] we get that C_3 is a separating line of $\overline{Z \setminus C_2}$, hence C_1, C_2, C_3 are lines and $L_{\overline{Z \setminus (C_2 \cup C_3)}} = \mathcal{O}_{\overline{Z \setminus (C_2 \cup C_3)}}$. Let us now suppose that $\delta = 1$; we have that $l_Z = l_{C_2} + h^0(\overline{Z \setminus C_2}, L_{\overline{Z \setminus C_2}}) - 1$ again by 1.2.9. in [C6], but then $h^0(\overline{Z \setminus C_2}, L_{\overline{Z \setminus C_2}}) = 1$. Now we recall that $\underline{\deg} L_{\overline{Z \setminus C_2}} = (1, 0, \ldots, 0)$, with $d_{C_3} = 1$, besides, let us denote by $Y := \overline{Z \setminus C_2}$. Again we observe that it must be $C_3 \cong \mathbb{P}^1$ and $l_{C_3} = 2$, otherwise L should have a base point. Notice that in all the cases we have examined, we obtained that C_1, C_2, C_3 are lines, and if $h^0(Z, L_Z) = 3$, C_2, C_3 are separating lines of Z. Moreover, reasoning as before we get that $L = (\mathcal{O}_{C_1}(1), \mathcal{O}_{C_2}(1), \mathcal{O}_{C_3}(1), \mathcal{O}_{C_4}, \ldots, \mathcal{O}_{C_r})$. $\square$

Lemma 1.3.17. *Let X be a connected nodal curve, with $X_{\text{sep}} = \emptyset$, and let C be an irreducible component of X such that $\sharp C \cap C^c \in \{2, 3\}$. Then $\overline{X \setminus C}$ is connected.*

Proof. Assume first that $\sharp C \cap C^c = 2$. By contradiction, let us suppose that C is a disconnecting component of X; then $\overline{X \setminus C} = X_1 \sqcup X_2$. Now, since X is connected we have that $C \cdot X_1 = 1$, and $C \cdot X_2 = 1$; if we consider the node $C \cap X_1$, it is a separating node for the subcurves X_1 and $X_2 \cup C$, hence X has a separating node, a contradiction. If $\sharp C \cap C^c = 3$ we repeat the same argument noticing that if C is a disconnecting component of X we have (up to reordering), $C \cdot X_1 = 1$ and $C \cdot X_2 = 2$, so again $C \cap X_1$ is a separating node. $\square$

We are now going to investigate about the properties of irreducible nodal curves lying in $\overline{\mathcal{M}^1_{g,3}}$. In order to do it, we need some techniques introduced by Beauville in [Be77] and then used in [HM82], i.e. the construction of *admissible covers*. We recall here the definition:

Definition 1.3.18. An *admissible cover* is the datum of:

(i) a nodal curve C,

(ii) a stable curve B of genus 0,

(iii) a map $\pi : C \to B$,

such that

(a) the images and preimages of nodes are nodes,

(b) for every node x of B and every node n of C lying over it, the two branches of C near n map to the branches of B near x with the same ramification index.

This definition was introduced in order to compactify the *Hurwitz schemes* $\mathcal{H}_{d,g}$, which are parameter spaces for simply branched covers of $\mathbb{P}^1$. Roughly speaking the way $\mathcal{H}_{d,g}$ is compactified is by adding limit covers such that the base curve degenerates with the cover; when two branch points collide, one separates them adding a rational component to the cover and one to the base. Let us observe that there is another way to do that, i.e., using stable maps (see [FP02]). We recall the fundamental:

Theorem 1.3.19 (Existence of $\overline{\mathcal{H}_{d,g}}$)**.** *There exists a coarse moduli space $\overline{\mathcal{H}_{d,g}}$ for admissible covers.*

A question arising in a very natural way is: given a stable curve, how can we say whether or not it is trigonal, and, more generally, whether or not it lies in $\overline{\mathcal{M}^r_{g,d}}$? The theory of admissible covers provides an answer at least in the case of pencils, i.e. when $r = 1$. We recall that in our terminology a *k-gonal* curve is a curve lying in $\overline{\mathcal{M}^1_{g,k}}$ and admitting a regular smoothing by k-gonal smooth curves, whereas a *weakly k-gonal* curve is a curve having a $\mathfrak{g}^1_k$.

Definition 1.3.20. Let C be a stable curve. We say that a nodal curve C' is *stably equivalent* to C, if C is obtained from C' by contracting to a point all smooth rational components of C' meeting the other components of C' in only one or two points.

We are now able to state the following:

Theorem 1.3.21 (Harris-Mumford)**.** *A stable curve C is k-gonal if and only if there exists a $k-$sheeted admissible cover $C' \to B$ of a stable (pointed) curve of genus 0 whose domain C' is stably equivalent to C.*

Lemma 1.3.22. *Let X be an irreducible nodal curve of genus $g \geq 3$, not hyperelliptic, with δ nodes and let $\widetilde{X} \xrightarrow{\nu} X$ be its normalization. For every node n_i let us denote by $\nu^{-1}(n_i) = \{p_i, q_i\}$. If there exists a line bundle $T_X \in \operatorname{Pic}^3 X$ such that $h^0(X, T_X) = 2$, then:*

(i) X is trigonal,

(ii) there exists a $\mathfrak{g}^1_3$, $\mathcal{T}$, on $\widetilde{X}$ such that $\mathcal{T}$ contains the divisors $p_i + q_i + r_i$ for every $i = 1, \ldots, \delta$, where $r_i \in \widetilde{X}$.

Proof. We start by proving (i); the conclusion follows from the fact that given any irreducible nodal curve of genus g with δ nodes, we are able to construct a $3-$sheeted admissible cover $X' \to B$ which is stably equivalent to X, so applying 1.3.21 we get our conclusions. Let us now build $X' \to B$. If we take $\widetilde{X}$ to be the normalization of X, it is a smooth trigonal curve, so we call $\pi : \widetilde{X} \to \mathbb{P}^1$ the map induced by the $\mathfrak{g}^1_3$. We will build $\pi' : X' \to B$ starting from π. We denote by B_1 the target of π; we observe that since π is $(3:1)$, we can have at most points of ramification index 3. So if $x \in B_1$ is such a point, i.e. $\pi^{-1}(x) = \{3r\}$, with $r \in \widetilde{X}$ and of course $r \neq p_i, q_i$ for every i, in order to get the admissible cover we add to $\widetilde{X}$ a copy of $\mathbb{P}^1$ meeting it at r, and to B_1 a tail meeting it at x. The copy of $\mathbb{P}^1$ added to $\widetilde{X}$ is chosen so that it maps to the tail at x by a generic map $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 3, with $3-$fold branching at r. Let now $y \in B_1$ be such that for some i we have $p_i, q_i \in \pi^{-1}(y)$ with ramification, i.e., as a divisor $\pi^*(y) = 2p_i + q_i$ (or $2q_i + p_i$). Then we just add to $\widetilde{X}$ a copy of $\mathbb{P}^1$ glued at p_i, q_i and mapping to the tail at y which we add to B_1, by a generic map of degree 3 and ramification index 2 at p_i (or q_i). For every other pair $\{p_j, q_j\}$ such that π is not ramified at either p_j or q_j, we proceed as follows: let $z \in B_1$ be such that $\pi^*(z) = p_j + q_j + r_j$ for some $r_j \in \widetilde{X}$. Then we glue to $\widetilde{X}$ two copies of $\mathbb{P}^1$: one meeting $\widetilde{X}$ at p_j and q_j, and the other one meeting it at r_j. Moreover we add to the base B_1 a tail at z, such that the two copies of $\mathbb{P}^1$ upstairs map to the tail with degrees 2 and 1 respectively, yielding a $(3:1)$ map to the tail. We call X' the curve obtained from $\widetilde{X}$ by gluing copies of $\mathbb{P}^1$ as indicated, and B the curve of genus 0 obtained from B_1 by adding the tails. Gluing all the maps introduced above, we get a $(3:1)$ map $\pi' : X' \to B$ which is precisely the admissible cover we were looking for (see Figure 1.1). Let us compute the genus of X': in our construction every $\mathbb{P}^1$ intersects $\widetilde{X}$ in one or two points, and it's easy to see that the number of $\mathbb{P}^1$'s intersecting in two points are as many as the number of nodes, δ. The number of $\mathbb{P}^1$'s intersecting in only one point will be denoted by α. We have that

$$g(X') = g(\widetilde{X}) - 1 + \sharp\{(\mathbb{P}^1)'s\}(g(\mathbb{P}^1) - 1) + \sharp(nodes) + 1, \tag{1.22}$$

hence

$$g(X') = g - \delta - 1 + (\alpha + \delta)(-1) + (\alpha + 2\delta) + 1 = g. \tag{1.23}$$

This concludes the proof of part (i). To show (ii) let us suppose at first that $\widetilde{X}$ is not hyperelliptic. Let $X_1 \xrightarrow{\nu_1} X$ be the normalization of X at one node n_1 of X. Then X_1 has a $\mathfrak{g}^1_3$, $T_{X_1} := \nu_1^*(T_X)$, since we can apply Clifford's theorem being $g_{X_1} \geq 2$. Now T_{X_1} is globally generated, since if it had a base point, then X_1 would be hyperelliptic, but then so would be $\widetilde{X}$, which contradicts our assumption. Then we can apply 5.1.3. in [C2], and get that $W_{T_{X_1}}(X) = \{T_X\}$ and $h^0(X_1, T_{X_1}(-p_1 - q_1)) = h^0(X_1, T_{X_1}(-p_1)) = 1$, hence there exists $r_1 \in X_1$, which is not a node, such that $T_{X_1} = \mathcal{O}_{X_1}(p_1 + q_1 + r_1)$. If

X_1 is smooth we are done. Otherwise we proceed in the following way: let $X_2 \xrightarrow{\nu_2} X_1$ be the normalization at one node n_2 of X_1, and let $\nu_{1,2} : X_2 \longrightarrow X_1$ be the composition of ν_1 and ν_2. Now $\nu_{1,2}^*(T_X) = \nu_2^*(T_{X_1}) = \nu_2^*(\mathcal{O}_{X_1}(p_1 + q_1 + r_1)) = \mathcal{O}_{X_2}(p_1 + q_1 + r_1))$. Let $T_{X_2} = \nu_2^*(T_{X_1}) = \mathcal{O}_{X_2}(p_1 + q_1 + r_1))$; since by hypothesis $\widetilde{X} \ncong \mathbb{P}^1$, we have that $\delta \leq g - 1$, hence $p_a(\widetilde{X}) \geq 1$ and $p_a(X_2) \geq 1$, hence via Clifford's theorem we get that $h^0(X_2, T_{X_2}) = 2$. Now arguing as before we get that $T_{X_2} = \mathcal{O}_{X_2}(p_2+q_2+r_2))$ for a suitable $r_2 \in X_2$. Repeating this procedure we are done. Now it remains to examine the case when $\widetilde{X}$ is hyperelliptic: but this is impossible since $T_{\widetilde{X}}$ would have a base point, so T_X should have a base point, too, but in our hypothesis X is not hyperelliptic. □

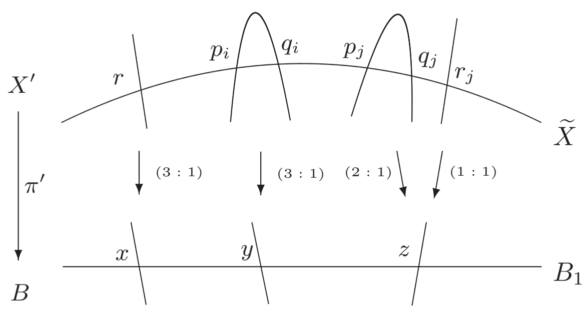

Figure 1.1: The admissible cover of X: as in the proof, x is a point with ramification index 3 at r, y has ramification index 2 at p_i and 1 at q_i (or vice-versa), and z has simple points p_j, q_j, r_j in its fiber.

Remark 1.3.23. We want to make some observations about the vice-versa of 1.3.22: of course if X is trigonal, since it has a regular smoothing where the general fiber is a smooth trigonal curve, it has a $\mathfrak{g}_3^1$, hence we can conclude that X is trigonal if and only if it has a $\mathfrak{g}_3^1$. Hence an irreducible curve $[X]$ in $\overline{\mathcal{M}_{g,3}^1}$ not trigonal is not weakly trigonal. Using the construction of the compactified Picard scheme $\overline{P_{d,g}}$ over $\overline{\mathcal{M}_g}$ in [C1], we have that $[X] \in \overline{\mathcal{M}_{g,3}^1}$ if and only if there exists a partial blow up $\widehat{X}_S$ of X at a set of nodes $S \subseteq X_{sing}$, possibly empty, and a line bundle $L_S \in \text{Pic}^3 \widehat{X}_S$, balanced on $\widehat{X}_S$ such that $h^0(\widehat{X}_S, L_S) = 2$. So let us suppose that X does not have a trigonal series. Then we can have two possibilities according to the balanced multidegrees that L_S can have. Indeed, either $\underline{\deg} L_S = (2, 1)$, where $\widehat{X}_S$ is the blow up of X in one node and we have just one exceptional component E where the degree of $L_{S|_E} = 1$; or $\underline{\deg} L_S = (1, 1, 1)$, where $\widehat{X}_S$ is the blow up of X in two points, so its components are the normalization of X in two points and the corresponding two exceptional components. In the first case, when $\underline{\deg} L_S = (2, 1)$, call X_1 the partial normalization of X at one node; then it must be hyperelliptic, and since X doesn't have any trigonal series, the preimages of the node on X_1 are not a neutral pair for the hyperelliptic series. It follows that the total normalization $\widetilde{X}$ of X_1, hence of X, is a hyperelliptic curve. In the second case, when $\underline{\deg} L_S = (1,1,1)$, if we call X_2 the partial normalization of X at two nodes, then $\deg L_{S|_{X_2}} = 1$, but since $L_{S|_{X_2}}$ must have two sections, it follows that X_2 is rational, but then X_2 is smooth (X had just two nodes) and it's the total normalization of X.

Before switching to other types of curves, we want to observe that the same techniques yield the following:

Lemma 1.3.24. *Let $k \geq 3$ and X be an irreducible nodal curve of genus g. If there exists a line bundle $L \in \mathrm{Pic}^k X$ such that $h^0(X, L) = 2$, then X is k-gonal.*

Proof. The proof is analogous to the previous one, indeed we construct a k-sheeted admissible cover for X. So we take the normalization $\widetilde{X} \xrightarrow{\nu} X$ of X, and as before it will be a smooth k-gonal curve, with induced $(k:1)$ map $\pi : \widetilde{X} \to \mathbb{P}^1$. Let us denote by n_i for $i = 1 \ldots, \delta$ the nodes of X, with $\nu^{-1}(n_i) = \{p_i, q_i\}$, and let B_1 be the target of π. We will glue copies of $\mathbb{P}^1$ to $\widetilde{X}$ and to B_1. If $x \in B_1$ is a multiple smooth point, its fiber as a divisor can be described as follows: $\pi^*(x) = m_1 x_1 + \cdots + m_l x_l$, with $\sum_{i=1}^{l} m_i = k$. Then we add to B_1 a tail at x, and for every $i = 1, \ldots, l$, we glue to $\widetilde{X}$ a copy of $\mathbb{P}^1$ at x_i, mapping to the tail by a generic map $\mathbb{P}^1 \to \mathbb{P}^1$ of degree m_i, with ramification index m_i at x_i. Let now y be a point of B_1 such that in its fiber is contained the pair $\{p_i, q_i\}$ for some $i = 1, \ldots, \delta$, p_i and q_i with multiplicities μ_1^i and μ_2^i respectively. As before, the fiber of y can be described as $\pi^*(y) = \mu_1^i p_i + \mu_2^i q_i + m_3 y_3 + \ldots + m_l y_l$, where the y_j's can possibly be preimages of other nodes, and in this case we treat them like p_i, q_i. Hence we glue to $\widetilde{X}$ a copy of $\mathbb{P}^1$ at p_i, q_i, and one at y_j for every j; the copy at p_i, q_i will map to the tail at y with degree $\mu_1^i + \mu_2^i$, with ramification index μ_1^i at p_i and μ_2^i at q_i. The copies glued at the y_j's will be mapped to the tail at y by maps of degrees m_j for every $j = 3, \ldots, l$. Iterating these steps for all the branching points of π and the nodes of X, we get a $k-$sheeted admissible cover $X' \to B$ of X just taking $\widetilde{X}$ and B_1 and adding to them all the copies of $\mathbb{P}^1$ as indicated. An easy calculation as in (1.22) and (1.23) will yield that the genus of X' is exactly g, so we are done. □

Remark 1.3.25. We notice, as in 1.3.23, that an irreducible nodal curve X is k-gonal if and only if it has a $\mathfrak{g}_k^1$. Again, any curve $X \in \overline{\mathcal{M}_{g,k}^1}$ which is not k-gonal, is not weakly k-gonal, since the limit of line bundles needs not be a line bundle itself. So, as before, we know from [C1] that if $[X] \in \overline{\mathcal{M}_{g,k}^1}$, this means that there exists a partial blow up $\widehat{X}_S$ of X at a set of nodes $S \subseteq X_{sing}$, possibly empty, and a line bundle $L_S \in \mathrm{Pic}^k \widehat{X}_S$, balanced on $\widehat{X}_S$, such that $h^0(\widehat{X}_S, L_S) = 2$. So let us suppose that our X is not weakly k-gonal, hence we can have the following possibilities: let $S_i = \{n_1, \ldots, n_i\}$, where $n_1, \ldots, n_\delta$ are the nodes of X, then there exists $i = 1, \ldots, \delta$ such that $(\widehat{X}_{S_i}, L_{S_i})$ are as above. Let us call X_{S_i} the partial normalization of X at the nodes in S_i, and E_j, for $j = 1, \ldots, i$, the exceptional components of $\widehat{X}_{S_i}$. Since L_{S_i} is balanced on $\widehat{X}_{S_i}$, we have that $\underline{\deg} L_{S_i} = (k-i, 1, \ldots, 1)$, therefore X_{S_i} must have a $\mathfrak{g}^1_{k-i}$, and its total normalization $\widetilde{X}$ must be weakly $(k-i)$-gonal as well. Let us notice that when $i = \delta$, i.e. if the total blow up of X has a $\mathfrak{g}^1_k$, then $\underline{\deg} L_{S_\delta} = (1, \ldots, 1)$, so the total normalization X_{S_δ} of X must be a copy of $\mathbb{P}^1$, and the number of nodes of X is $\delta = k$.

Curves with two components

In what follows we will give a characterization of some curves with two components lying in $\overline{\mathcal{M}^1_{g,3}}$, and we will see that some properties can be generalized to curves in $\overline{\mathcal{M}^1_{g,k}}$ for any k. First of all we need some terminology, that we recall both from [Br99] and [EM02]. Let B be the spectrum of a discrete valuation ring R. Let s denote the special point of B, and b its generic point. Let moreover $f : \mathcal{X} \to B$ be a flat projective morphism such that $X_b := \mathcal{X}(b)$ is an irreducible reduced nodal curve and the special fiber $X := \mathcal{X}(s)$ is a nodal reducible reduced curve. We assume that the scheme $\mathcal{X}$ is regular, i.e., it is a *regular smoothing* of X. Let C_1, C_2 be the irreducible components of X (that are Cartier divisors on $\mathcal{X}$). Since $\mathcal{X}$ is regular, if L_b is a line bundle on X_b, there exists a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}(b) = L_b$ and $L := \mathcal{L}(s)$ is a line bundle on X. The line bundle $\mathcal{L}$ will be called an *extension* of L_b to $\mathcal{X}$. Let us notice that twisting $\mathcal{L}$ by line bundles of the form $\mathcal{O}_{\mathcal{X}}(m_1C_1 + m_2C_2)$ with $m_1, m_2 \in \mathbb{Z}$, we get all the possible extensions of L_b to $\mathcal{X}$, since $\mathcal{L}(b) = \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(m_1C_1 + m_2C_2)(b)$. The linear system induced by $H^0(X_b, L_b)$ extends to $\mathcal{X}$ as well, giving the vector space $H^0(\mathcal{X}, \mathcal{L})$, which, restricted to the central fiber defines a vector space $H^0(X, L)$ inducing a linear system on X. We call the pair $(\mathcal{L}(s), H^0(X, L))$ a *limit linear system*, according to [EM02]. Let us now recall:

Proposition 1.3.26 (Esteves). *For every irreducible component C_i of X there is a unique (up to isomorphism) extension $\mathcal{L}_i$ of L_b to $\mathcal{X}$ with the following properties:*

(a) the canonical homomorphism

$$H^0(X, L) \to H^0(C_i, \mathcal{L}_{i|_{C_i}})$$

is injective;

(b) for every irreducible component C_j with $j \neq i$ the canonical homomorphism

$$H^0(X, L) \to H^0(C_j, \mathcal{L}_{i|_{C_j}})$$

is not identically zero.

We are now going to use these concepts in a more specific situation. Let then $f : \mathcal{X} \to B$ be a regular smoothing of X such that the generic fiber X_b is a trigonal irreducible curve (i.e. it has a $\mathfrak{g}^1_3$ by 1.3.23) with trigonal series T_b, and the central fiber X is a reducible reduced nodal curve with two components. When the curve is of compact type, i.e., the two components meet in one point, we have already given an answer in 1.3.9 and

1.3.12, following the theory of limit linear series developed by Eisenbud and Harris in [EH86]. In what follows, by analyzing the possible extensions $\mathcal{T}$'s of T_b to $\mathcal{X}$, we are going to characterize the central fiber X of the degeneration and its induced trigonal series. In what follows, we will deal only with families of curves which are regular smoothings.

Lemma 1.3.27. *Let $f : \mathcal{X} \to B$ be a family of irreducible nodal trigonal curves, and let the central fiber X be a reducible stable curve with 2 components C_1, C_2, and $\delta = 2$ nodes. Then, C_1 is trigonal and C_2 is hyperelliptic.*

Proof. Let T_b be the trigonal series of the generic curve X_b of the family. Then, since the smoothing is regular, there exists an extension $\mathcal{T}$ to $\mathcal{X}$, such that $\mathcal{T}_{|X_b} = T_b$. Following 1.3.26, we know that there exists a unique extension $\mathcal{T}_i$ with respect to C_i having properties (a) and (b) in 1.3.26. Let us denote by $T_i = \mathcal{T}_{i|X}$ for $i = 1, 2$. Then the possible multidegrees of T_1 are $\underline{\deg} T_1 \in \{(3,0),(2,1),(1,2)\}$ since $h^0(X, T_1) \leq h^0(C_1, T_{1|C_1})$ and T_1 doesn't have negative degree on C_2 because of property (b). We can assume that $g(C_1), g(C_2) \neq 0$, since X is stable and $\delta = 2$. Let us suppose that $\underline{\deg} T_1 = (3,0)$. Twisting by $\mathcal{O}_{\mathcal{X}}(C_1)$ we see that the only possibility for $\underline{\deg} T_2$ according to 1.3.26 (a) and (b), is $(1,2)$. Then, since $\deg T_{1|C_1} = 3$ and $2 = h^0(X, T_1) \leq h^0(C_1, T_{1|C_1})$, we have that C_1 is trigonal; for the same reason, since $\deg T_{2|C_2} = 2$ and $2 = h^0(X, T_2) \leq h^0(C_2, T_{2|C_2})$, we have that C_2 is hyperelliptic. Moreover, we have that $T_{1|C_2} = \mathcal{O}_{C_2}$ because of 1.3.26 (b), hence

$$h^0(X, T_1) = h^0(C_1, T_{1|C_1}) + h^0(C_2, \mathcal{O}_{C_2}) - 1.$$

Applying [C6, Lemma 1.2.6.], we get that

$$h^0(C_1, T_{1|C_1}(-p_1 - p_2)) = 1,$$

hence, by 1.3.15, $T_{1|C_1} = \mathcal{O}_{C_1}(p_1 + p_2 + a)$, where $a \in C_1$. On the other hand,

$$h^0(X, T_2) = h^0(C_1, T_{2|C_1}) + h^0(C_2, T_{2|C_2}) - 1,$$

so it follows that $h^0(C_2, T_{2|C_2}(-q_1 - q_2)) = 1$, therefore the hyperelliptic series on C_2 is the line bundle $\mathcal{O}_{C_2}(q_1 + q_2)$. Suppose now that $\underline{\deg} T_1 = (2,1)$; then, twisting by $\mathcal{O}_{\mathcal{X}}(C_2)$ we get that the only possibility for $\underline{\deg} T_1$ is $(0,3)$, i.e., we switch the curves and the argument above applies. The last case to study is when $\underline{\deg} T_1 = (1,2)$; then, by 1.3.26 (a) it follows that $(3,0)$ can't be the multidegree of T_2 with respect to C_2, hence $\mathcal{T}_1 = \mathcal{T}_2$, $T_1 = T_2$ and since $h^0(C_1, T_{1|C_1}) = 2$, then C_1 is a smooth rational curve, but we excluded the possibility $g(C_1) = 0$. □

We are now going to give a sort of vice-versa of 1.3.27:

Lemma 1.3.28. *Let X be a reducible reduced stable curve with two irreducible components C_1, C_2 and two nodes n_1, n_2 such that, if $\{p_1, q_1\}, \{p_2, q_2\}$ are the branches over the nodes, C_1 is trigonal with trigonal series $\mathcal{O}_{C_1}(p_1 + p_2 + p)$, $p \in C_1$, and C_2 is hyperelliptic with series $\mathcal{O}_{C_2}(q_1 + q_2)$. Then X is trigonal.*

Proof. Given X as in the hypothesis, we just have to construct a $3-$sheeted admissible cover of it, so applying 1.3.21 we get our conclusions. First of all we denote by $T_{C_1} = \mathcal{O}_{C_1}(p_1 + p_2 + p)$, and by $H_{C_2} = \mathcal{O}_{C_2}(q_1 + q_2)$. Then we build the admissible cover $\pi : X' \to B'$ as follows. Let us take two copies of $\mathbb{P}^1$, denoted by B_1 and B_2, such that the map $\pi_1 : C_1 \to B_1$ is the $(3 : 1)$ map induced by T_{C_1}, and the map $\pi_2 : C_2 \to B_2$ is $(2 : 1)$ induced by H_{C_2}. If we call y_1, y_2 the points of B_1, B_2 resp. such that $\pi_1^{-1}(y_1) = \{p_1, p_2, p\}$, and $\pi_2^{-1}(y_2) = \{q_1, q_2\}$, then we glue B_1 and B_2 along y_1 and y_2. We call y the point of $B_1 \cup B_2$ coming from the identification of y_1 and y_2, as in Figure 1.2. We want to complete π_2 to a $(3 : 1)$ map π_2'; let us glue a copy r of $\mathbb{P}^1$ to C_1 at p, such that r is mapped to B_2 by a generic $(1 : 1)$ map $\mathbb{P}^1 \to \mathbb{P}^1$. Gluing the last map to π_2 we obtain $\pi_2' : C_2 \sqcup r \to B_2$, which is now a $(3 : 1)$ map. Moreover, if there are triple points, i.e., if there exists some $x \in B_1$ such that its ramification index is 3, we glue to B_1 a tail at x, and to C_1 a copy of $\mathbb{P}^1$ at the corresponding multiple point (point a in Figure 1.2), mapping to the tail at x by a $(3 : 1)$ map. Then we get the cover in the following way: X' is the curve composed of C_1, C_2, r, and the copies of $\mathbb{P}^1$ glued at multiple ramification points of C_1. On the other hand B' is the union of B_1, B_2, and all the tails added at multiple points of B_1. The map π is given by π_1 on C_1, and by π_2' on $C_2 \sqcup r$, and we see that $\pi^{-1}(y) = \{n_1, n_2, p\}$, where by abuse of notation we denote via p both the point on C_1 and the node obtained by gluing r to C_1. So we are done. ☐

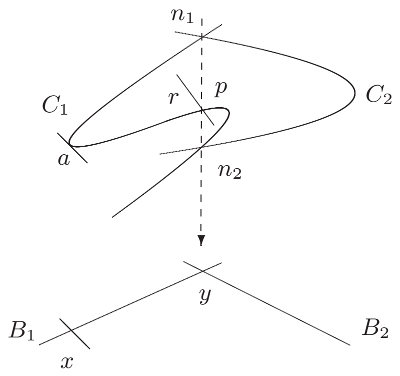

Figure 1.2: The admissible cover of X: the notation is precisely the one used in the proof of 1.3.28, in order to illustrate the construction.

Lemma 1.3.29. *Let $f : \mathcal{X} \to B$ be a family of irreducible nodal trigonal curves, and let the central fiber X be a reducible reduced stable curve with 2 irreducible components C_1, C_2, and $\delta = 3$ nodes. Then, either*

(i) C_1 is rational and C_2 is hyperelliptic,

or

(ii) C_1 and C_2 are both trigonal.

Proof. We again adopt the notation introduced in paragraph 1.3.2, and denote by n_i for $i = 1, 2, 3$ the nodes of X and by $\{p_i, q_i\}$ their branches on C_1, C_2. If T_b is the trigonal series on X_b, let $\mathcal{T}_1$ be the extension with respect to C_1 as in 1.3.26, and $\mathcal{T}_2$ be the one relatively to C_2, moreover again we denote by $T_i = \mathcal{T}_{i|_X}$. Then the multidegree $\underline{\deg} T_1 \in \{(3,0), (2,1), (1,2)\}$. If $\underline{\deg} T_1 = (3,0)$, then twisting by $\mathcal{O}_{\mathcal{X}}(C_1)$ we get that $\underline{\deg} T_2 = (0,3)$ is the only possibility according to 1.3.26 (a) and (b). So, in this case we have that both C_1 and C_2 are trigonal, since $2 = h^0(X, T_i) \leq h^0(C_i, T_{i|_{C_i}})$ for $i = 1, 2$. Moreover, 1.3.26 (b) implies that $T_{i|_{C_j}} = \mathcal{O}_{C_j}$, for $i \neq j \in \{1, 2\}$. This means that, if $g(C_1), g(C_2) \neq 0$,

$$h^0(X, T_i) = h^0(C_i, T_{i|_{C_i}}) + h^0(C_j, \mathcal{O}_{C_j}) - 1,$$

with $i \neq j \in \{1, 2\}$, then applying [C6, Lemma 1.2.6.] we get that $h^0(C_1, T_{1|_{C_1}}(-p_1 - p_2 - p_3)) = 1$ and $h^0(C_2, T_{2|_{C_2}}(-q_1 - q_2 - q_3)) = 1$, hence that

$$T_{1|_{C_1}} = \mathcal{O}_{C_1}(p_1 + p_2 + p_3),$$

and

$$T_{2|_{C_2}} = \mathcal{O}_{C_2}(q_1 + q_2 + q_3).$$

The second case we are going to develop, is when $\underline{\deg} T_1 = (1,2)$ or $(2,1)$, which is the same up to switching the two components. As we can see, since X has 3 nodes, if we twist T_1 we obtain negative degrees, against 1.3.26 (b). Hence we conclude that in this case $\mathcal{T}_1 = \mathcal{T}_2$, therefore $T_1 = T_2$, C_1 is a rational curve, and C_2 a hyperelliptic curve, with hyperelliptic series that we denote by H_{C_2}, so that $T_1 = T_2 = (\mathcal{O}_{\mathbb{P}^1}(1), H_{C_2})$. Hence we are done. □

As well as in the case of two nodes, we are going to give a vice-versa to the previous statement.

Lemma 1.3.30. *Let X be a reducible reduced stable curve with two irreducible components C_1, C_2 and 3 nodes n_1, n_2, n_3 such that $\{p_i, q_i\}$ are the branches over the nodes for $i = 1, 2, 3$, and one of the following holds:*

(i) C_1 and C_2 are trigonal with trigonal series $\mathcal{O}_{C_1}(p_1 + p_2 + p_3)$ and $\mathcal{O}_{C_2}(q_1 + q_2 + q_3)$,

(ii) C_1 is rational and C_2 is hyperelliptic, with maps $C_1 \xrightarrow{\pi_1} \mathbb{P}^1$ and $C_2 \xrightarrow{\pi_2} \mathbb{P}^1$ induced resp. by $\mathcal{O}_{\mathbb{P}^1}(1)$ and by the hyperelliptic series, such that $\pi_1(p_i) = \pi_2(q_i)$ for any i.

Then X is trigonal.

Proof. Let us assume first that C_1 and C_2 are both trigonal, with series $\mathcal{O}_{C_1}(p_1 + p_2 + p_3)$ and $\mathcal{O}_{C_2}(q_1 + q_2 + q_3)$; then we just have to build a 3−sheeted admissible cover of X. We proceed as follows: let us denote by $\pi_i : C_i \to B_i$ the map induced by the trigonal series of C_i, where B_i is a copy of $\mathbb{P}^1$; then the admissible cover $\pi : X' \to B'$ will be such that X' is the curve with irreducible components C_1, C_2 and all the copies of $\mathbb{P}^1$ glued at 3-fold branched points of C_1 and C_2 as in the proof of 1.3.28. To obtain the base B' of the cover,

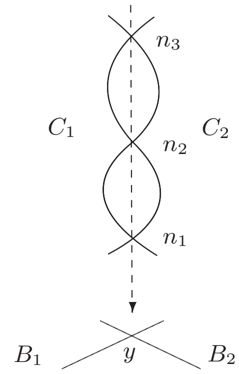

Figure 1.3: The admissible cover of X is obtained by gluing the $(3:1)$ maps induced by the trigonal series on C_1 and C_2.

we have to glue B_1 and B_2 as follows: let $y_1 \in B_1$, $y_2 \in B_2$, such that $\pi_1^{-1}(y_1) = \{p_1, p_2, p_3\}$, and $\pi_2^{-1}(y_2) = \{q_1, q_2, q_3\}$.

Then we glue B_1 and B_2 identifying y_1 with y_2; hence the base B' is the curve of genus 0 composed of B_1, B_2, and the tails glued to them at the ramification points of the series. The map π is the one given by π_1 on C_1, by π_2 on C_2, and by generic $(3:1)$ maps on the copies of $\mathbb{P}^1$ glued at multiple branch points, where the targets of these maps are precisely the tails of B'. In particular, notice that if we denote by $y \in B'$ the point coming from the gluing of y_1 and y_2, then $\pi^{-1}(y) = \{n_1, n_2, n_3\}$, (see Figure 1.3).

Now let us focus on the second case, i.e. when C_1 is rational and C_2 is hyperelliptic. Let $\pi_1 : C_1 \to \mathbb{P}^1$ be the map induced by $\mathcal{O}_{\mathbb{P}^1}(1)$, and $\pi_2 : C_2 \to \mathbb{P}^1$ the $(2:1)$ map induced by the hyperelliptic series H_{C_2} on C_2. These two maps will give us the admissible cover we are looking for. Let $x_i \in \mathbb{P}^1$ be the image via π_2 of the branch q_i; by hypothesis we know that $\pi_1(p_i) = x_i$ for every $i = 1, \ldots, 3$. We denote by B the target of π_1 and π_2; we can have two situations: q_i is either a Weierstrass point of C_2 for some $i = 1, 2, 3$, or it is not. If it is, then we build the admissible cover $\pi : X' \to B'$ as follows: fix $i \in \{1, 2, 3\}$ and suppose $x_i \in B$ is such that $\pi_2^{-1}(x_i) = 2q_i$, and therefore such that $\pi_1^{-1}(x_i) = p_i$, since p_i and q_i have the same image on B. We build X' starting from C_1 and C_2, that we consider as two disjoint curves; then we glue to C_1 and C_2 a copy of $\mathbb{P}^1$ at p_i and q_i, mapping to the tail that we glue to B at x_i via a generic $(3:1)$ map. Fix now $j \in \{1, 2, 3\}$, and suppose q_j is not a Weierstrass point of C_2, i.e., there exists $x_j \in B$ such that $\pi_2^{-1}(x_j) = \{q_j, r_j\}$, with $q_j \neq r_j \in C_2$. Notice that we have, like before, that $\pi_1^{-1}(x_j) = p_j$; now we take two copies of $\mathbb{P}^1$, then we glue one of them to C_1 and C_2 at p_j and q_j, and the other one to C_2 at r_j. The first copy will map to the tail we add to B at x_j via a generic $(2:1)$ map, and the second copy will map to the same tail at x_j via a $(1:1)$ map. Hence the admissible cover is obtained by taking X' as the union of C_1, C_2, and all the copies of $\mathbb{P}^1$ we add in correspondence of the nodes; B' is the curve having as irreducible components B and all the tails we add in correspondence of the images of the nodes. □

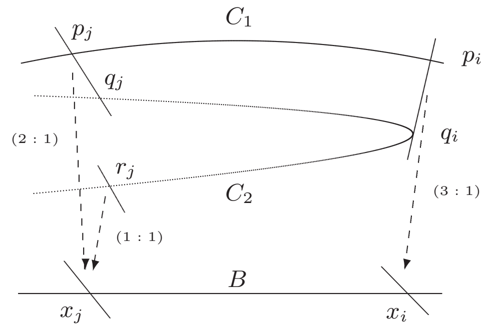

Figure 1.4: A sketch of the construction of an admissible cover of X: as in the proof of 1.3.30, we denoted by q_i a Weierstrass point of H_{C_2}, and by q_j a simple point of its. We showed how to glue copies of $\mathbb{P}^1$ to C_1 and C_2 in both the cases.

We are now going to ask ourselves what happens when we have a reducible reduced stable curve X with two components C_1 and C_2 and $\delta = k$ nodes, and X is the central fiber of a family of k-gonal irreducible curves. We assume that $k \geq 4$.

Lemma 1.3.31. *Let $f : \mathcal{X} \to B$ be a family of irreducible nodal k-gonal curves, and let the central fiber X be a reducible reduced stable curve with 2 irreducible components C_1, C_2, and $\delta = k$ nodes. Then, either*

(i) C_1 and C_2 are both k-gonal,

or

(ii) C_1 is k_1-gonal and C_2 is k_2-gonal, with $k_1 + k_2 = k$.

Proof. We proceed as in the proof of 1.3.29. Let X_b be the generic curve of $\mathcal{X}$, and let L_b be the k-gonal series on X_b. Then by 1.3.26 there exists a unique extension $\mathcal{L}_i$ to $\mathcal{X}$ of L_b with respect to C_i satisfying (a) and (b); we write $L_i = \mathcal{L}_{i|_X}$. Let us denote by n_i for $i = 1, \dots, k$ the nodes of X, and by $\{p_i, q_i\}$ their preimages on C_1 and C_2. Let us first suppose that the multidegree $\underline{\deg} L_1 = (k, 0)$; then we see that twisting by $\mathcal{O}_{\mathcal{X}}(C_1)$ we get that the only possible multidegree for L_2 is $(0, k)$. Hence by 1.3.26 we get that both C_1 and C_2 are k-gonal curves, so if we denote by $G_i^k = L_{i|_{C_i}}$, we get that $L_1 = (G_1^k, \mathcal{O}_{C_2})$, and $L_2 = (\mathcal{O}_{C_1}, G_2^k)$. Moreover, since

$$2 = h^0(X, L_1) = h^0(C_1, G_1^k) + h^0(C_2, \mathcal{O}_{C_2}) - 1$$

and

$$2 = h^0(X, L_2) = h^0(C_1, \mathcal{O}_{C_1}) + h^0(C_2, G_2^k) - 1,$$

we get that applying [C6, Lemma 1.2.6.],

$$G_1^k = \mathcal{O}_{C_1}(\sum_{i=1}^{k} p_i) \quad \text{and} \quad G_2^k = \mathcal{O}_{C_2}(\sum_{j=1}^{k} q_j).$$

Let us now assume that $\underline{\deg}L_1 = (k-1,1)$; we see that twisting we would contradict 1.3.26 (b), hence it follows that $\mathcal{L}_1 = \mathcal{L}_2$, and therefore $L_1 = L_2$. Then we have that C_2 must be rational, and C_1 must be $(k-1)$-gonal, and we write $L_1 = (G_1^{k-1}, \mathcal{O}_{\mathbb{P}^1}(1))$. Let us verify that properties (a) and (b) of 1.3.26 hold. Indeed, in all the cases before these properties were obvious; now we have to check that the maps

$$H^0(X, L_1) \xrightarrow{\varphi_1} H^0(C_1, G_1^{k-1})$$

and

$$H^0(X, L_1) \xrightarrow{\varphi_2} H^0(C_2, \mathcal{O}_{\mathbb{P}^1}(1))$$

are injective. Let us introduce the following notation: let $H^0(C_1, G_1^{k-1}) = \langle \sigma_1, \sigma_2 \rangle$ and $H^0(C_2, \mathcal{O}_{\mathbb{P}^1}(1)) = \langle \tau_1, \tau_2 \rangle$. Since $h^0(X, L_1) > 0$, there exist sections $\sigma_i \in H^0(C_1, G_1^{k-1})$ and $\tau_j \in H^0(C_2, \mathcal{O}_{\mathbb{P}^1}(1))$ such that they match on the k nodes; we denote the section of $H^0(X, L_1)$ arising from σ_i and τ_j by $\sigma_i \star \tau_j$; we want to show that $H^0(X, L_1) = \langle \sigma_1 \star \tau_1, \sigma_2 \star \tau_2 \rangle$. This would imply that the maps

$$\begin{array}{ccc} H^0(X, L_1) & \xrightarrow{\varphi_1} & H^0(C_1, G_1^{k-1}) \\ \sigma_i \star \tau_i & \mapsto & \sigma_i \end{array}$$

and

$$\begin{array}{ccc} H^0(X, L_1) & \xrightarrow{\varphi_2} & H^0(C_2, \mathcal{O}_{\mathbb{P}^1}(1)) \\ \sigma_i \star \tau_i & \mapsto & \tau_i \end{array}$$

are both isomorphisms of vector spaces. So, if by contradiction we had $H^0(X, L_1) = \langle \sigma_i \star \tau_1, \sigma_i \star \tau_2 \rangle$ for some $i \in \{1, 2\}$, then it would mean that $\sigma_i(p_j) = \tau_1(q_j) = \tau_2(q_j)$ for $j = 1, \ldots, k$, hence that the point $[\tau_1(q_j) : \tau_2(q_j)] = [1 : 1] \in \mathbb{P}^1$ has k distinct preimages, which is impossible since $\mathcal{O}_{\mathbb{P}^1}(1)$ induces a map of degree 1. If on the other hand we had, say, $H^0(X, L_1) = \langle \sigma_1 \star \tau_i, \sigma_2 \star \tau_i \rangle$ for some $i \in \{1, 2\}$, then it would imply that $\sigma_1(p_j) = \sigma_2(p_j) = \tau_i(q_j)$ for every $j = 1, \ldots, k$. Then again the point $[\sigma_1(p_j) : \sigma_2(p_j)] = [1 : 1] \in \mathbb{P}^1$ would have k distinct preimages, against the fact that the map induced by G_1^{k-1} has degree $k-1$. This proves (a), and (b) is obvious. To conclude the proof, let us just observe that if we assume $\underline{\deg}L_1 = (k_1, k_2)$ with $k_1 + k_2 = k$ and $k_1 \geq k_2$ (when $k_1 \leq k_2$ we just switch the curves), with an argument analogous to the previous one we get that properties (a) and (b) hold for $L_1 = (G_1^{k_1}, G_2^{k_2})$, hence C_1 is k_1-gonal and C_2 is k_2-gonal. □

As well as for $\delta = 2$ and $\delta = 3$, we have a vice-versa to the previous statement, so that we can completely characterize the curves with two components and k nodes lying in the closure of $\mathcal{M}^1_{g,k}$ as limits of regular smoothings.

Lemma 1.3.32. *Let X be a reducible reduced stable curve with two irreducible components C_1, C_2 and k nodes $n_1, \ldots, n_k$ such that, if $\{p_i, q_i\}$ are the branches over the nodes for $i = 1, \ldots, k$, either*

(i) C_1 and C_2 are k-gonal with $\mathfrak{g}^1_k$'s, respectively, $\mathcal{O}_{C_1}(p_1 + \cdots + p_k)$ and $\mathcal{O}_{C_2}(q_1 + \cdots + q_k)$,

or

(ii) C_1 *has a* $\mathfrak{g}^1_{k_1}$*, denoted by* $G_1^{k_1}$*, and* C_2 *has a* $\mathfrak{g}^1_{k_2}$*,* $G_2^{k_2}$*, with* $k_1 + k_2 = k$*,* $k_1, k_2 > 0$ *inducing maps* $C_1 \xrightarrow{\pi_1} \mathbb{P}^1$ *and* $C_2 \xrightarrow{\pi_2} \mathbb{P}^1$ *such that* $\pi_1(p_i) = \pi_2(q_i)$ *for any* $i = 1, \dots, k$*.*

Then X *is* k*-gonal.*

Proof. We follow the lines of the proof of 1.3.30. So let us suppose at first that C_1 and C_2 are both k-gonal with series $\mathcal{O}_{C_1}(p_1 + \cdots + p_k)$ and $\mathcal{O}_{C_2}(q_1 + \cdots + q_k)$; then we just apply the construction in the first part of 1.3.30 in order to build an admissible cover for X. Indeed, by generalizing that construction we get as k-sheeted admissible cover, a map $\pi : X' \to B'$, which we describe hereafter: let us denote by $\pi_i : C_i \to B_i$ the map induced by the k-gonal series of C_i, where B_i is a copy of $\mathbb{P}^1$. Then let $y_1 \in B_1, y_2 \in B_2$ be such that $\pi_1^{-1}(y_1) = \{p_1, \dots, p_k\}$, and $\pi_2^{-1}(y_2) = \{q_1, \dots, q_k\}$. Then we glue B_1 and B_2 identifying y_1 with y_2; hence we define the base B' to be the curve of genus 0 composed of B_1, B_2, plus the tails that we will glue to B_1 and B_2 at the multiple points of π_1 and π_2. On the other hand X' is the curve having as irreducible components C_1, C_2, and all the copies of $\mathbb{P}^1$ glued to C_1, C_2 at multiple points, and each of these copies will map to the corresponding tail of B' via a generic $(k : 1)$ map. We depicted the construction in Figure 1.5. Let us

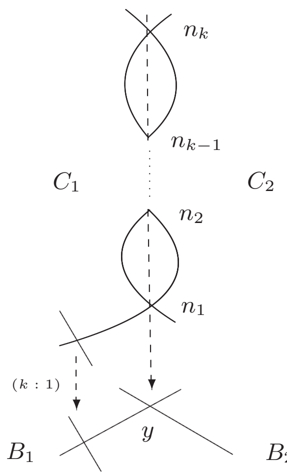

Figure 1.5: The admissible cover of X is obtained by gluing the $(k : 1)$ maps induced by the k-gonal series on C_1 and C_2.

now assume (ii). Then we construct a k-sheeted admissible cover of X as we did in the second part of 1.3.30, in order to show that $X \in \overline{\mathcal{M}^1_{g,k}}$. Let us recall that by hypothesis, $G_1^{k_1}$ and $G_2^{k_2}$ induce maps $\pi_1 : C_1 \to \mathbb{P}^1$ and $\pi_2 : C_2 \to \mathbb{P}^1$ such that $\pi_1(p_i) = \pi_2(q_i)$ for every $i = 1, \dots, k$. We denote by B the target of π_1, π_2, as in 1.3.30. We are going to build the admissible cover $\pi : X' \to B'$ from C_1 and C_2. So, let $x_i \in B$ be such that $p_i \in \pi_1^{-1}(x_i)$ and $q_i \in \pi_2^{-1}(x_i)$; moreover, let l_i^1 be the ramification index of p_i relatively to π_1, and l_i^2 the one of q_i relatively to π_2. More specifically, let $\pi_1^*(x_i) = l_i^1 p_i + \mu_i^1 r_i^1 + \cdots + \mu_i^s r_i^s$, with

$r_i^1, \ldots, r_i^s \in C_1$, and $\pi_2^*(x_i) = l_i^2 q_i + \nu_i^1 \rho_i^1 + \cdots + \nu_i^t \rho_i^t$, with $\rho_i^1, \ldots, \rho_i^t \in C_2$. Then we glue a copy of $\mathbb{P}^1$ to C_1 and C_2 at p_i and q_i (in such a way that it links the curves), mapping to the tail we glue to B at x_i via a degree $l_i^1 + l_i^2$ map; moreover we glue a copy of $\mathbb{P}^1$ to C_1 at every r_i^j for $j = 1, \ldots, s$, which is mapped to the tail at x_i by a degree μ_i^j map. Again, for each $j = 1, \ldots, t$, we glue to C_2 at ρ_i^j a copy of $\mathbb{P}^1$, mapping to the tail at x_i via a degree ν_i^j map. If there is a multiple point on B such that in its fiber there are multiple branched points of π_1 or π_2, then we proceed as follows: let $y \in B$ be such that $\pi_1^*(y) = m_1 y_1 + \cdots + m_e y_e$ and $\pi_2^*(y) = m_{e+1} y_{e+1} + \cdots + m_f y_f$, with $y_1, \ldots, y_e \in C_1$, $y_{e+1}, \ldots, y_f \in C_2$, and at least one $m_i > 2$ for some $i = 1, \ldots, f$. Then we glue to B a tail at y, and we glue to C_1 a copy of $\mathbb{P}^1$ at each y_i for $i = 1, \ldots, e$ and to C_2 one copy at each y_j, for $j = e+1, \ldots, f$, every copy mapping to the tail at y via a degree m_i map. Repeating this process for every node and every multiple point, we get that X' will be the curve having as irreducible components C_1, C_2 and all the copies of $\mathbb{P}^1$ we glue to them; B' will the curve obtained from B by gluing the tails as above; the map π is given by π_1 and π_2 when restricted to C_1 and C_2, and by the maps described above when restricted to the rational components of X', and by construction it is $(k:1)$ (see Figure 1.6).

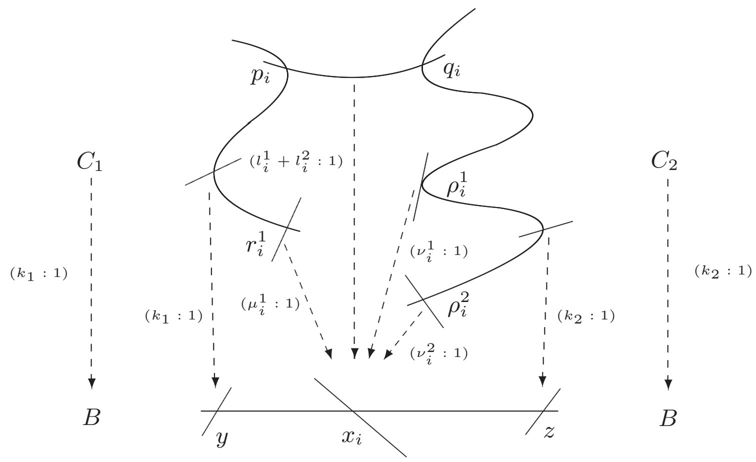

Figure 1.6: The admissible cover of X is obtained by gluing the maps π_1 and π_2 induced by the k_1 and k_2-gonal series on C_1 and C_2. Notice that x_i has the branches of a node in its fiber, while y and z in B are examples of smooth ramification points of π_1 and π_2.

□

We are now going to discuss the case when the central fiber of our degeneration is a k-gonal curve with two irreducible components meeting at $\delta > k$ nodes.

Lemma 1.3.33. *Let $f : \mathcal{X} \to B$ be a family of irreducible nodal k-gonal curves, and let the central fiber X be a reducible reduced stable curve with 2 irreducible components C_1, C_2, and $\delta > k$ nodes. Then, C_1 is k_1-gonal and C_2 is k_2-gonal, with $k_1 + k_2 = k$ and $k_1, k_2 > 0$.*

Proof. We use the notation of 1.3.31, so let X_b be the generic curve of $\mathcal{X}$, and let L_b be the k-gonal series on X_b. Then by 1.3.26 there exists a unique extension $\mathcal{L}_i$ to $\mathcal{X}$ of L_b with respect to C_i satisfying (a) and (b); we write $L_i = \mathcal{L}_{i|_X}$. Let us denote by n_i for $i = 1, \dots, \delta$ the nodes of X, and by $\{p_i, q_i\}$ their preimages on C_1 and C_2. Let us first suppose that the multidegree $\underline{\deg} L_1 = (k, 0)$; if we twist $\mathcal{L}_1$ with some line bundle of the form $\mathcal{O}_{\mathcal{X}}(n_1C_1 + n_2C_2)$, we see that since $\delta > k$, its restriction to either C_1 or C_2 would have negative degree, against property (b). But if it were $L_1 = L_2$ then we would have a contradiction since $L_{1|_{C_2}} = \mathcal{O}_{C_2}$ and it doesn't agree with property (a). So we exclude the case $\underline{\deg} L_1 = (k, 0)$, and assume that $\underline{\deg} L_1 = (k_1, k_2)$ with $k_1 + k_2 = k$ and $k_1, k_2 > 0$. Again we see that twisting is impossible, so we get $\mathcal{L}_1 = \mathcal{L}_2$, and then $L := L_1 = L_2$. Then, we repeat the argument in 1.3.31 to show that properties (a) and (b) hold for L, recalling that the number of nodes of X is δ. So we have that C_1 has a $\mathfrak{g}^1_{k_1}$ and C_2 has a $\mathfrak{g}^1_{k_2}$, which together induce L. □

Lemma 1.3.34. *Let X be a reducible reduced stable curve with two irreducible components C_1, C_2 and $\delta > k$ nodes such that C_1 has a $\mathfrak{g}^1_{k_1}$, denoted by $G_1^{k_1}$, and C_2 has a $\mathfrak{g}^1_{k_2}$, $G_2^{k_2}$, with $k_1 + k_2 = k$, $k_1 \cdot k_2 > 0$, inducing maps $C_1 \xrightarrow{\pi_1} \mathbb{P}^1$ and $C_2 \xrightarrow{\pi_2} \mathbb{P}^1$ such that $\pi_1(p_i) = \pi_2(q_i)$ for any $i = 1, \dots, \delta$. Then X is k-gonal.*

Proof. The proof is the same as in 1.3.32, i.e., we construct a k-sheeted admissible cover of X as it is shown in Figure 1.6, starting from C_1 and C_2 and their k_1-gonal and k_2-gonal series, and gluing to them copies of $\mathbb{P}^1$ at the preimages of nodes and at ramification points. □

The last case we want to study is when the number of nodes δ is such that $\delta < k$. This case requires a special attention to the twistings that will occur.

Lemma 1.3.35. *Let $f : \mathcal{X} \to B$ be a family of irreducible nodal k-gonal curves, and let the central fiber X be a reducible reduced stable curve with 2 irreducible components C_1, C_2, and $\delta < k$ nodes. Then,*

if $\delta < k < 2\delta$: either C_1 has a $\mathfrak{g}^1_k$ and C_2 has a $\mathfrak{g}^1_\delta$, or C_1 has a $\mathfrak{g}^1_{k_1}$ and C_2 has a $\mathfrak{g}^1_{k_2}$, with $k_1 + k_2 = k$ and $k_1, k_2 > 0$, or C_1 has a $\mathfrak{g}^1_{k_1}$ and C_2 has a $\mathfrak{g}^1_{(k_2-\delta)}$, with $k_1 + k_2 = k$ and $k_1, k_2 > 0$;

if $k = 2\delta$: either both C_1 and C_2 have a $\mathfrak{g}^1_\delta$, or both have a $\mathfrak{g}^1_{2\delta}$, or C_1 has a $\mathfrak{g}^1_{k_1}$ and C_2 has a $\mathfrak{g}^1_{(k_2-\delta)}$, with $k_1 + k_2 = 2\delta$ and $0 < k_1 \leq k_2$;

if $k > 2\delta$, $k = m\delta + \alpha$: either C_1 is k-gonal and C_2 has a $\mathfrak{g}^1_{j\delta}$ for some $j \in \{1, \dots, m\}$; or C_1 has a $\mathfrak{g}^1_{k-j\delta}$, and C_2 has a $\mathfrak{g}^1_{j\delta}$ for some $j \in \{1, \dots, m\}$; or if $k_1 \leq \delta \leq k_2$, $k_1 + k_2 = k$, then writing $k_2 = n\delta + \beta$, C_1 has a $\mathfrak{g}^1_{k_1}$, and C_2 has a $\mathfrak{g}^1_{k_2-j\delta}$, for some $j \in \{1, \dots, n\}$; if $\delta \leq k_1 \leq k_2$, writing $k_1 = \mu\delta + \gamma$ and $k_2 = \nu\delta + \lambda$, C_1 has a either a $\mathfrak{g}^1_{k_1}$ or a $\mathfrak{g}^1_{k_1-j\delta}$, *and C_2 has a $\mathfrak{g}^1_{k_2}$ or a $\mathfrak{g}^1_{k_2+j\delta}$, for some $j \in \{1, \dots, \mu\}$; and the last possibility is that C_1 has a $\mathfrak{g}^1_{k_1}$ and C_2 has a $\mathfrak{g}^1_{k_2-l\delta}$ for some $l \in \{1, \dots, \nu\}$.*

Proof. We use the same notation and techniques of the previous proofs, i.e., X_b will be the generic curve of $\mathcal{X}$, and L_b will be the k-gonal series on X_b. Then by 1.3.26 there exists a unique extension $\mathcal{L}_i$ to $\mathcal{X}$ of L_b with respect to C_i satisfying (a) and (b); we write $L_i = \mathcal{L}_{i|_X}$ for $i = 1, 2$. Let us denote by n_i for $i = 1, \dots, \delta$ the nodes of X, and by $\{p_i, q_i\}$ their preimages on C_1 and C_2.

Case I: let $\delta < k < 2\delta$, and suppose that the multidegree $\underline{\deg}L_1 = (k, 0)$; then we can only twist by $\mathcal{O}_{\mathcal{X}}(C_1)$, and we get $\underline{\deg}L_2 = (k-\delta, \delta)$; now we have two possibilities: either $h^0(C_1, L_{2|_{C_1}}) = 1$, or $h^0(C_1, L_{2|_{C_1}}) \geq 2$. In the first case, applying [C6, Lemma 1.2.6.] to both L_1 and L_2 we obtain that C_1 has a $\mathfrak{g}^1_k$ and C_2 has a $\mathfrak{g}^1_\delta$ such that $L_{1|_{C_1}} = \mathcal{O}_{C_1}(p_1 + \cdots + p_\delta + a_{\delta+1} + \cdots + a_k)$ and $L_{2|_{C_2}} = \mathcal{O}_{C_2}(q_1 + \cdots + q_\delta)$, with $a_i \in C_1$. In the second case, when $h^0(C_1, L_{2|_{C_1}}) \geq 2$, we have that (a) and (b) in 1.3.26 hold for both L_1 and L_2; indeed for L_1 it's obvious, whereas for L_2 we make a count of sections. We show that if $H^0(C_1, L_{2|_{C_1}}) = \langle \tau_1, \tau_2 \rangle$ and $H^0(C_2, L_{2|_{C_2}}) = \langle \sigma_1, \sigma_2 \rangle$, then it's not possible that $H^0(X, L_2) = \langle \tau_\alpha \star \sigma_i, \tau_\beta \star \sigma_i \rangle$ with $\alpha, \beta, i \in \{1, 2\}$; indeed if it were, for an argument analogous to the one in 1.3.31 we would get that the point $[1:1] \in \mathbb{P}^1$ would have δ preimages via $L_{2|_{C_1}}$, which is impossible since $k - \delta < \delta$ by hypothesis. So even in this second case when $h^0(C_1, L_{2|_{C_1}}) \geq 2$, we have that C_1 has a $\mathfrak{g}^1_k$ and C_2 has a $\mathfrak{g}^1_\delta$, but their structure is not specified. Now let us take into consideration the case when $\underline{\deg}L_1 = (k_1, k_2)$ with $k_1 \leq k_2$, and $k_1 + k_2 = k$. Then either $k_1 \leq k_2 < \delta$ or $k_1 \leq \delta \leq k_2$. In the first case we can't twist, so we have that $L_1 = L_2$ and C_1 has a $\mathfrak{g}^1_{k_1}$ and C_2 has a $\mathfrak{g}^1_{k_2}$. In the second one, twisting by $\mathcal{O}_{\mathcal{X}}(C_2)$ we obtain $\underline{\deg}L_2 = (k_1 + \delta, k_2 - \delta)$, so, when properties (a) and (b) hold we have that C_1 has a $\mathfrak{g}^1_{k_1}$ and C_2 has a $\mathfrak{g}^1_{k_2-\delta}$.

Case II: $k = 2\delta$; let $\underline{\deg}L_1 = (2\delta, 0)$, then by twisting we can have either $\underline{\deg}L_2 = (\delta, \delta)$, or $\underline{\deg}L_2 = (0, 2\delta)$. In the second case, both C_1 and C_2 have a $\mathfrak{g}^1_{2\delta}$, such that $h^0(C_1, L_{1|_{C_1}}(-p_1 - \cdots - p_\delta)) = 1$ and $h^0(C_2, L_{2|_{C_2}}(-q_1 - \cdots - q_\delta)) = 1$, by applying [C6, Lemma 1.2.6.]. If $\underline{\deg}L_2 = (\delta, \delta)$, we can have that $h^0(C_1, L_{2|_{C_1}}) = 1$ or $= 2$. If $h^0(C_1, L_{2|_{C_1}}) = 2$, since $L_{2|_{C_1}} = L_{1|_{C_1}}(-p_1 - \cdots - p_\delta)$ and $h^0(C_1, L_{1|_{C_1}}) = 2$, it means that $L_{1|_{C_1}}$ has $p_1, \dots, p_\delta$ as base points, hence C_1 and C_2 have both a $\mathfrak{g}^1_\delta$. If, on the other hand, $h^0(C_1, L_{2|_{C_1}}) = 1$, we have that C_1 has a $\mathfrak{g}^1_{2\delta}$ such that $h^0(C_1, L_{1|_{C_1}}(-p_1 - \cdots - p_\delta)) = 1$, and C_2 has a $\mathfrak{g}^1_\delta$ such that (again by [C6, Lemma 1.2.6.]) $L_{2|_{C_2}} = \mathcal{O}_{C_2}(q_1 + \cdots + q_\delta)$. Now let us study the other possibilities: if $\underline{\deg}L_1 = (k_1, k_2)$ with $k_1 + k_2 = k$, $k_i \neq 0, \delta$ for $i = 1, 2$, then, supposing without loss of generality that $k_1 < k_2$, we get that twisting by $\mathcal{O}_{\mathcal{X}}(C_2)$, the only possibility is $\underline{\deg}L_2 = (k_1 + \delta, k_2 - \delta)$. According to 1.3.26 (a) it must be $h^0(C_2, L_{2|_{C_2}}) \geq 2$, hence, since the maps in (a) are injective by an argument based on sections as the one in Case I, we can get that C_1 has a $\mathfrak{g}^1_{k_1}$ and C_2 has a $\mathfrak{g}^1_{k_2-\delta}$.

Case III: $k > 2\delta$. In this case let as usual $\underline{\deg}L_1 = (k, 0)$, then we can have more possibilities for $\underline{\deg}L_2$, indeed if $k = m\delta + \alpha$, then $\underline{\deg}L_2 \in \{(k-\delta,\delta),(k-2\delta,2\delta),\ldots,(\alpha,m\delta)\}$. Let then $\underline{\deg}L_2 = (k - j\delta, j\delta)$, fixing $j \in \{1,\ldots,m\}$; we have that either $h^0(C_1, L_{2|_{C_1}}) = 1$ or $h^0(C_1, L_{2|_{C_1}}) = 2$. If it is $h^0(C_1, L_{2|_{C_1}}) = 1$, it follows, as before, that C_1 is k-gonal with $h^0(C_1, L_{1|_{C_1}}(-p_1 - \cdots - p_\delta)) = 1$ and C_2 has a $\mathfrak{g}^1_{j\delta}$ such that $h^0(C_2, L_{2|_{C_2}}(-q_1 - \cdots - q_\delta)) = 1$. If otherwise $h^0(C_1, L_{2|_{C_1}}) = 2$, then C_1 has a $\mathfrak{g}^1_{k-j\delta}$, whereas C_2 has a $\mathfrak{g}^1_{j\delta}$. If $\underline{\deg}L_2 = (k_1, k_2)$ with $0 < k_1 \leq k_2$, $k_1 + k_2 = k$ and either $k_1 \leq \delta \leq k_2$ or $\delta \leq k_1 \leq k_2$. If $k_1 \leq \delta \leq k_2$, then let $k_2 = n\delta + \beta$, hence reasoning as before, we get that C_1 has a $\mathfrak{g}^1_{k_1}$, and C_2 has a $\mathfrak{g}^1_{k_2 - j\delta}$, for $j \in \{1,\ldots,n\}$. If otherwise $\delta \leq k_1 \leq k_2$, let us write $k_1 = \mu\delta + \gamma$ and $k_2 = \nu\delta + \lambda$: then we can twist both by $\mathcal{O}_\mathcal{X}(C_1)$ and $\mathcal{O}_\mathcal{X}(C_2)$, getting $\underline{\deg}L_2 = (k_1 - j\delta, k_2 + j\delta)$ or $\underline{\deg}L_2 = (k_1 + l\delta, k_2 - l\delta)$, for some $j \in \{1,\ldots,\mu\}$ and $l \in \{1,\ldots,\nu\}$. So, repeating the previous reasoning, we get the combinations between: C_1 has a either a $\mathfrak{g}^1_{k_1}$ or a $\mathfrak{g}^1_{k_1 - j\delta}$, and C_2 has a $\mathfrak{g}^1_{k_2}$ or a $\mathfrak{g}^1_{k_2 + j\delta}$, for some $j \in \{1,\ldots,\mu\}$; finally the last possibility is that C_1 has a $\mathfrak{g}^1_{k_1}$ and C_2 has a $\mathfrak{g}^1_{k_2 - l\delta}$ for some $l \in \{1,\ldots,\nu\}$.

□

We turn our attention to another class of curves: *graph curves*. By definition, a graph curve is a connected, projective algebraic curve which is a union of copies of $\mathbb{P}^1$, each meeting exactly three others, transversely at distinct points; it follows that a graph curve of genus g has $2g-2$ components and $3g-3$ nodes. They are obviously named after their dual representation as graphs, which are in particular trivalent ones. We want to investigate when a graph curve belongs to $\overline{\mathcal{M}^1_{g,k}}$. We consider $k = 3$.

Lemma 1.3.36. *Let X be a graph curve of genus $g \geq 4$. Then, there exists no regular smoothing of X such that the generic fiber is a trigonal irreducible nodal curve.*

Proof. Let X be a graph curve of genus $g \geq 4$. Let us suppose by contradiction that there exists a regular smoothing of X, denoted by $f : \mathcal{X} \to B$, such that the generic fiber X_b is a trigonal irreducible nodal curve of genus g. Then, according to 1.3.26, for every irreducible component C_i of X there exists a unique line bundle $\mathcal{L}_i$ on $\mathcal{X}$ satisfying properties (a) and (b). Hence suppose that $\underline{\deg}\mathcal{L}_{1|_X} = (3, 0, \ldots, 0)$; we denote by $L_1 = \mathcal{L}_{1|_X}$. We look for the other extensions $\mathcal{L}_i$ with respect to C_i for $i \neq 1$, by twisting. We observe that by 1.3.26 (b), we can't allow negative degrees, i.e. if we regard $\underline{\deg}\mathcal{L}_{i|_X}$ as a vector in $\mathbb{Z}^{2g-2}$, then its components must be all non-negative. So, fixed $i = 2, \ldots, 2g-2$, by twisting we must try to construct vectors $\underline{D}^i = \underline{\deg}L_i \in \mathbb{Z}^{2g-2}$ such that the j-th component $(\underline{D}^i)_j \geq 0$ for any $j = 1, \ldots, 2g-2$, and $(\underline{D}^i)_i \geq 1$. We want to prove that this is impossible. Indeed, let us take our vector $\underline{D}^1 = (3, 0, \ldots, 0)$; up to reordering we can assume that $C_1 \cdot C_i = 1$ for $i = 2, 3, 4$; in this case, if we twist $\mathcal{L}_1$ by $\mathcal{O}_\mathcal{X}(C_1)$ we get the multidegree $\underline{v} = (0, 1, 1, 1, 0, \ldots, 0)$, which yields $\underline{D}^2, \underline{D}^3, \underline{D}^4 = \underline{v}$. Now we perform other twistings to get more multidegrees; but if we twist by any C_j, for $j = 1, \ldots, 2g-2$, we have to add

-3 to $(\underline{v})_j$, and a $+1$ to the components of $\underline{v}$ corresponding to the C_i's intersecting C_j. So, iterating the twisting by distinct C_j's, we see that the only way to get a non-negative components multidegree is twisting by $\mathcal{O}_{\mathcal{X}}(\sum_{k=1}^{2g-2} C_k) = \mathcal{O}_{\mathcal{X}}(X)$. Hence we can't have an extension for every irreducible component of X, which gives us a contradiction. Let us notice that starting by a multidegree $\underline{v}$ whose components belong to $\{0,1\}$, we obtain the same conclusion. On the other hand, if $(\underline{\deg}L_1)_i \in \{1,2,0\}$, then we would have on X a trigonal series of degree of type $(1,2,0,\dots,0)$ up to reordering. By 1.3.16 (i), we get that C_1 would be a separating line of $X \setminus C_2$, which is impossible, since by [C6, Proposition 5.2.7.], given a graph curve, for each irreducible component C, C^c has no separating nodes. □

Remark 1.3.37. Let us notice that when $g = 3$ and $g = 4$ every smooth curve has a $\mathfrak{g}^1_3$, so in both cases $\overline{\mathcal{M}_g} = \overline{\mathcal{M}^1_{g,3}}$. If $g = 3$ in particular we get the only graph curve that can be embedded in the plane (see Figure 1.7). We want to observe that in this case X has 4 components and 6 nodes, and if we use the notation of 1.3.36 we get that the only possibility to have a regular smoothing of X with generic trigonal fiber is that $\underline{\deg}L_1 = (3,0,0,0)$ and $\underline{\deg}L_i = (0,1,1,1)$ for $i = 2,3,4$.

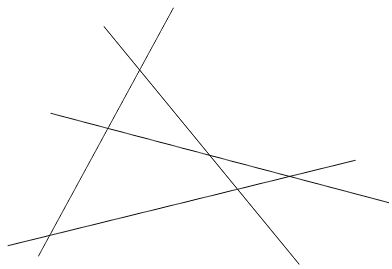

Figure 1.7: A graph curve of genus 3.

1.4 Notes on Projective Normality of reducible curves (with E. Ballico)

In this section we give some results on quadratic normality of reducible curves canonically embedded and partially extend this study to their projective normality.

1.4.1 Quadratic normality

For any reduced projective curve X and any line bundles M, N on X let

$$\mu_{M,N} : H^0(X,M) \otimes H^0(X,N) \longrightarrow H^0(X, M \otimes N); \tag{1.24}$$

denote the multiplication map. Set $\mu_M = \mu_{M,M}$. Given the dualizing sheaf ω_X on X, we are interested in studying the surjectivity of the map μ_{ω_X}. In particular, when we assume that X is canonically embedded this is equivalent to saying that X is *quadratically normal*. We have

Proposition 1.4.1. *Let X be a connected reduced curve of genus g with planar singularities and ω_X very ample. Assume that $X = A \cup B$, with A, B connected and smooth at $D := A \cap B$. If*

(i) $\mu_{\omega_A, \omega_X|_A}$ is surjective,

(ii) $\mu_{\omega_X|_B}$ is surjective,

then μ_{ω_X} is surjective.

In order to prove the proposition, we need some background material. We are going to keep the notation used in the statement of Proposition 1.4.1. Let $D := A \cap B$ be the scheme-theoretic intersection. We will view D also as a subscheme of A and B. Since both A and B are smooth at each point of the support of D, that we denote by $supp(D)$, the scheme D is a Cartier divisor of both A and B; more in general, this is true if X has only planar singularities at each point of $supp(D)$, because in this case a local equation of B in an ambient germ of a smooth surface gives a local equation of D as a subscheme of A.

Remark 1.4.2. According to the notation above, we have that

(i) It is well known that a curve with planar singularities is Gorenstein.

(ii) Since X is Gorenstein and locally planar at the points of $supp(D)$, then A and B are Gorenstein as well, so that ω_A and ω_B are both line bundles on A and B.

(iii) Since X is locally planar at the points of $supp(D)$, the adjunction formula gives $\omega_X|_A = \omega_A(D)$ and $\omega_X|_B = \omega_B(D)$. Thus $\deg(\omega_X|_A) = 2g_A - 2 + \delta$ and $\deg(\omega_X|_B) = 2g_B - 2 + \delta$, where of course g_A, g_B are the arithmetic genera of A and B, and $\delta = \deg(D)$.

Lemma 1.4.3. *Let Z be a reduced, Gorenstein and connected projective curve. Let E be an effective Cartier divisor on Z such that $E \neq 0$. Then $h^0(\mathcal{I}_E) = 0$ and $h^1(\omega_Z(E)) = 0$.*

Proof. Since Z is connected, $h^0(\mathcal{O}_Z) = 1$. Since E is effective and non-empty, we get $h^0(\mathcal{I}_E) = 0$. We apply the duality for locally Cohen-Macaulay schemes, i.e. we apply to the scheme $X := Z$ and the sheaf $F := \omega_Z(E)$ the case $r = p = 1$ of the theorem at page 1 of [AK70]. We get $h^1(\omega_Z(E)) = \dim(Ext^0(\omega_Z(E), \omega_Z))$, i.e. $h^1(\omega_Z(E)) = h^0(Hom(\omega_Z(E), \omega_Z))$. Since ω_Z is assumed to be locally free, we get $h^1(\omega_Z(E)) = h^0(Hom(\mathcal{O}_Z(E), \mathcal{O}_Z)) = 0$. □

Lemma 1.4.4. *Let X be a connected reduced curve of genus g with planar singularities and ω_X very ample. Assume that $X = A \cup B$, with A, B connected and smooth at $D := A \cap B$. For any subcurve Z of X we consider the map*

$$\rho_Z : H^0(X, \omega_X) \longrightarrow H^0(Z, \omega_X|_Z).$$

Then ρ_A and ρ_B are surjective.

Proof. To fix ideas we work on $Z = A$; let us consider the exact sequence:

$$0 \to \mathcal{I}_A \otimes \omega_X \to \omega_X \to \omega_X|_A \to 0.$$

We claim that $\mathcal{I}_A \otimes \omega_X = \omega_B$. To prove this, we notice that since X has only planar singularities, it can be embedded in a smooth surface S, where X, A and B are Cartier divisors. Thus D is a Cartier divisor of A and of B (but seldom of X). By the adjunction formula we have that

$$\omega_X = \omega_S(A + B)|_X,$$

then

$$\omega_B = \omega_S(B)|_B = \omega_S(A + B - A)|_B = (\omega_S(A + B - A)|_X)|_B$$
$$= (\omega_S(A + B)|_X \otimes \mathcal{I}_A)|_B = (\omega_X \otimes \mathcal{I}_A)|_B.$$

So the claim is proved and the previous sequence becomes

$$0 \to \omega_B \to \omega_X \to \omega_X|_A \to 0.$$

The corresponding long exact sequence in cohomology is

$$0 \to H^0(\omega_B) \to H^0(\omega_X) \to H^0(\omega_X|_A) \to H^1(\omega_B) \to H^1(\omega_X) \to H^1(\omega_X|_A) \to \cdots$$

Since $\omega_X|_A = \omega_A(D)$, by lemma 1.4.3 we have that $\dim H^1(\omega_X|_A) = 0$. Moreover, being both B and X connected, we have that $\dim H^1(\omega_B) = 1$ and $\dim H^1(\omega_X) = 1$, so the map $H^0(\omega_X) \to H^0(\omega_X|_A)$ is surjective. □

We are now able to prove proposition 1.4.1:

Proof of proposition 1.4.1. Let us consider the composition

$$H^0(\omega_X) \otimes H^0(\omega_X) \xrightarrow{\mu_{\omega_X}} H^0(\omega_X^2) \xrightarrow{\rho_B^2} H^0({\omega_X}^2|_B); \tag{1.25}$$

In order to show that μ_{ω_X} is surjective, it suffices, by a basic argument of linear algebra, to prove that

(a) $\rho_B^2 \circ \mu_{\omega_X}$ is surjective,

(b) $\mathrm{Ker}\rho_B^2 \subseteq \mathrm{Im}\mu_{\omega_X}$.

So let us show (a): we have a commutative diagram

$$\begin{array}{ccc} H^0(\omega_X) \otimes H^0(\omega_X) & \xrightarrow{\rho_B^2 \circ \mu_{\omega_X}} & H^0({\omega_X}^2|_B) \\ \downarrow{\scriptstyle \rho_B \otimes \rho_B} & \nearrow{\scriptstyle \mu_{\omega_B(D)}} & \\ H^0(\omega_X|_B) \otimes H^0(\omega_X|_B) & & \end{array} \tag{1.26}$$

where the map $\rho_B \otimes \rho_B$ is surjective by lemma 1.4.4 and $\mu_{\omega_B(D)}$ is surjective by assumption (ii). So, by the commutativity of the diagram we get (a).

In order to prove (b), we notice that

$$\mathrm{Ker}\rho_{\mathrm{B}}^2 = H^0(X, \mathcal{I}_B \otimes \omega_X^2),$$

and take

$$\mu := \mu_{\omega_X}|_{H^0(X, \mathcal{I}_B \otimes \omega_X) \otimes H^0(\omega_X)}.$$

So we have the following commutative diagram:

$$\begin{array}{ccc} H^0(\mathcal{I}_B \otimes \omega_X) \otimes H^0(\omega_X) & \xrightarrow{\mu} & H^0(\mathcal{I}_B \otimes {\omega_X}^2) \\ \downarrow{\scriptstyle id \otimes \rho_A} & & \downarrow{\scriptstyle \cong} \\ H^0(\omega_A) \otimes H^0(\omega_X|_A) & \xrightarrow{\mu_{\omega_A, \omega_X|_A}} & H^0(\omega_A \otimes \omega_X|_A) \end{array} \tag{1.27}$$

The map $id \otimes \rho_A$ is surjective by lemma 1.4.4, while $\mu_{\omega_A, \omega_X|_A}$ is surjective by assumption (i). Hence μ is surjective. Since μ is a restriction of μ_{ω_X}, we get $\mathrm{Ker}\rho_B^2 \subseteq \mathrm{Im}\mu_{\omega_X}$. □

Definition 1.4.5. Fix an integer $m > 0$; let X be a reduced and Gorenstein projective curve. We say that X is *m-connected* (resp. *numerically m-connected*) if for any decomposition $X = U \cup V$ with U, V subcurves without common irreducible components, the scheme $U \cap V$ has degree at least m (resp. $\deg \omega_X|_U - \deg \omega_U \geq m$ and $\deg \omega_X|_V - \deg \omega_V \geq m$).

Remark 1.4.6. If every point of X lying on at least two irreducible components of X is a planar singularity of X, then X is m-connected if and only if it is numerically m-connected (see [CFHR99], Remark 3.2).

Notation 1.4.7. Given a reduced curve X, we will denote by $X_{\mathrm{mult}} \subset X$ the set of points of X lying on at least two irreducible components of X and by X_{sm} the open set of smooth points of X.

Lemma 1.4.8. *Let X be a connected, reduced and Gorenstein curve of genus g with ω_X very ample. Assume that X has planar singularities at the points of X_{mult}. Then X is 3-connected.*

Proof. Let us fix any decomposition $X = U \cup V$ of X, with U, V subcurves and $\dim(U \cap V) = 0$. Set $D := U \cap V$. Since X has planar singularities at the points of $supp(D)$, D is a Cartier divisor of U. To prove the lemma it is sufficient to show the inequality

$\deg(D) \geq 3$. Assume $\deg(D) \leq 2$. Since ω_X is globally generated, X is 2-connected (see [Cat81], Theorem D). Assume, then, $\deg D = 2$. Remark 1.4.2 gives $\omega_X|_U \cong \omega_U(D)$. Since X is 2-connected and $\deg D = 2$, we easily see that U is connected. By lemma 1.4.3 we get that $\dim H^1(\omega_U(D)) = 0$. Thus Riemann-Roch gives

$$\dim H^0(\omega_U(D)) = \dim H^0(\omega_U) + 1.$$

Since D is a Cartier divisor of U, we get $\mathcal{I}_D \otimes \omega_U(D) \cong \omega_U$. Thus

$$\dim H^0(\mathcal{I}_D \otimes \omega_X|_U) = \dim H^0(\omega_X|_U) - 1,$$

hence the restriction to D of the morphism induced by $|\omega_X|$ is not very ample, contradiction. □

Definition 1.4.9. One says that a line bundle L on a curve X is *normally generated* if the maps

$$H^0(X, L)^k \to H^0(X, L^k)$$

are surjective for any $k \geq 1$.

Now we need to recall Theorem B in [F04].

Theorem 1.4.10 (Franciosi). *Let C be a connected reduced curve and let $\mathcal{H}$ be an invertible sheaf on C such that*

$$\deg \mathcal{H}|_Z \geq 2p_a(Z) + 1 \text{ for all subcurves } Z \subseteq C.$$

Then $\mathcal{H}$ is normally generated on C.

We are now able to prove the following lemma.

Lemma 1.4.11. *Let $X = A \cup B$, with $A, B \neq \emptyset$ and assume that X is Gorenstein, with planar singularities at the points of X_{mult}. Let ω_X be very ample. Then $\omega_X|_A$ and $\omega_X|_B$ are normally generated.*

Proof. Let us prove the conclusions for B. By Theorem 1.4.10 it is sufficient to prove that $\deg \omega_X|_Z \geq 2p_a(Z) + 1$ for every subcurve $Z \subseteq B$. Since $A \neq \emptyset$, we have that $Z \subsetneq X$. But since ω_X is very ample, by lemma 1.4.8 we have that X is 3-connected, hence the conclusions. □

We are now ready to prove what in the introduction we called Theorem 1. For the reader's sake we recall its terms hereafter.

Theorem 1.4.12 (Theorem 1). *Let X be a connected, reduced and Gorenstein projective curve of genus g with ω_X very ample. Assume that X has planar singularities at the points lying on at least two irreducible components, and set $X = A \cup B$ with A, B connected subcurves being smooth at $D := A \cap B$. If $A \neq \emptyset$ and the map*

$$\mu_{\omega_A, \omega_X|_A} : H^0(A, \omega_A) \otimes H^0(A, \omega_X|_A) \to H^0(X, \omega_A \otimes \omega_X|_A)$$

is surjective, then X is quadratically normal.

Proof. We recall that X is a connected, reduced and Gorenstein projective curve of genus g with ω_X very ample. By hypothesis we assume that X has planar singularities at the points of X_{mult}, and that $X = A \cup B$ with A, B connected subcurves being smooth at $D := A \cap B$. Since $\mu_{\omega_A, \omega_X|_A}$ is surjective, by proposition 1.4.1 it suffices to show that (ii) holds. But this is true by lemma 1.4.11. □

In what follows we will investigate when condition (i) of proposition 1.4.1 holds. If X is any curve, we denote by X_{sm} its smooth locus. We recall a result from [B01]; before doing this, let us introduce some notation: if L is a line bundle on a curve C globally generated and such that $\dim H^0(C, L) = r$, it induces a morphism

$$h_L : C \to \mathbb{P}^{r-1}.$$

Lemma 1.4.13 (Ballico). *Let C be an integral projective curve with $C \neq \mathbb{P}^1$ and $R \in \mathrm{Pic} C$, R globally generated and such that h_R is birational onto its image. Then the multiplication map*

$$\mu_{\omega_C, R} : H^0(C, \omega_C) \otimes H^0(C, R) \to H^0(C, \omega_C \otimes R)$$

is surjective.

More in general we have the following result.

Theorem 1.4.14. *Let A be a reduced, connected and Gorenstein projective curve such that ω_A is very ample and the map μ_{ω_A} is surjective. Let $E \subset A_{\mathrm{sm}}$ be an effective divisor on A such that $\deg E \geq 2$. Then $\mu_{\omega_A, \omega_A(E)}$ is surjective.*

Proof. Since A is connected, lemma 1.4.3 gives $H^1(\omega_A(D)) = 0$ for every effective and nonzero Cartier divisor D on A. Thus

$$\dim H^0(\omega_A(D)) = g_A + \deg D - 1$$

for every such D. We use induction on $e := \deg E$.

(a) Let us first assume $e = 2$. We check that $\omega_A(E)$ is globally generated. Set $E = p_1 + p_2$, where p_1, p_2 are smooth points for A. Since ω_A is globally generated, then $\omega_A(E)$ is globally generated outside $\{p_1, p_2\}$. We just proved that

$$\dim H^0(\omega_A(p_i)) = \dim H^0(\omega_A(p_1 + p_2)) - 1.$$

Thus there is at least one section of $\omega_A(E)$ that doesn't vanish at p_i, with $i = 1, 2$. Hence $\omega_A(E)$ is globally generated. The divisor E induces two inclusions $j : \omega_A \hookrightarrow \omega_A(E)$ and $j' : {\omega_A}^2 \hookrightarrow {\omega_A}^2(E)$, which in turn induce the linear maps $j_* : H^0(\omega_A) \longrightarrow H^0(\omega_A(E))$ and $j'_* : H^0({\omega_A}^2) \longrightarrow H^0({\omega_A}^2(E))$ which have respectively corank 1 and 2. Consider the following diagram:

$$\begin{array}{ccc} H^0(\omega_A) \otimes H^0(\omega_A) & \xrightarrow{id \otimes j_*} & H^0(\omega_A) \otimes H^0(\omega_A(E)) \\ \downarrow{\scriptstyle \mu_{\omega_A, \omega_A}} & & \downarrow{\scriptstyle \mu_{\omega_A, \omega_A(E)}} \\ H^0({\omega_A}^2) & \xrightarrow{j'_*} & H^0({\omega_A}^2(E)) \end{array} \qquad (1.28)$$

Since by hypothesis μ_{ω_A,ω_A} is surjective and

$$\dim H^0({\omega_A}^2(E)) = \dim H^0({\omega_A}^2) + 2,$$

then $j'_*(Im(\mu_{\omega_A,\omega_A}))$ is the codimension 2 linear subspace $\Gamma := H^0(\mathcal{I}_E \otimes \omega_A(E))$ of $H^0({\omega_A}^2(E))$. Since the subspace $j'_*(Im(\mu_{\omega_A,\omega_A}))$ is contained in $Im(\mu_{\omega_A,\omega_A(E)})$, in order to get the conclusions for $e = 2$ it suffices to prove the existence of two elements of $Im(\mu_{\omega_A,\omega_A(E)})$ which together with a basis of $j'_*(Im(\mu_{\omega_A,\omega_A}))$, i.e. of Γ, are linearly independent. Since $\omega_A(E)$ is globally generated, there exists $\alpha \in H^0(\omega_A(E))$ not vanishing at p_1 and p_2. Since ω_A is globally generated, there is $\beta \in H^0(\omega_A)$ not vanishing at p_1 and p_2 as well. Since ω_A is very ample, there is $\gamma \in H^0(\omega_A)$ vanishing at p_1 but not at p_2, or, in the case when $p_1 = p_2$, vanishing at p_1 with order exactly 1. Now the section $\sigma := \mu_{\omega_A,\omega_A(E)}(\gamma \otimes \alpha)$ doesn't belong to Γ; indeed, if $p_1 \neq p_2$, σ doesn't vanish at p_2, and if $p_1 = p_2$, it vanishes at p_1 with order exactly 1. Since the section $\mu_{\omega_A,\omega_A(E)}(\beta \otimes \alpha)$ does not vanish at p_1, it is not contained in the linear span of Γ and σ. Thus

$$\dim Im(\mu_{\omega_A,\omega_A(E)}) \geq dim\Gamma + 2.$$

Thus $\mu_{\omega_A,\omega_A(E)}$ is surjective in the case $e = 2$.

(b) Let now $e \geq 3$. We use induction on e. We fix a point p contained in the support of the divisor E, and set $F := E - p$. We check that $\omega_A(E)$ is globally generated, By inductive hypothesis the line bundle $\omega_A(F)$ is globally generated, hence so is $\omega_A(E)$ outside p. Since $\dim H^1(\omega_A(F)) = 0$, Riemann-Roch gives $\dim H^0(\omega_A(E)) > \dim H^0(\omega_A(F))$. Thus $\omega_A(F)$ has a section not vanishing at p. Hence $\omega_A(E)$ is globally generated. We define two inclusions: $\iota : \omega_A(F) \hookrightarrow \omega_A(E)$ and $\iota' : {\omega_A}^2(F) \hookrightarrow {\omega_A}^2(E)$, which induce the linear maps $\iota_* : H^0(\omega_A(F)) \longrightarrow H^0(\omega_A(E))$ and $\iota'_* : H^0({\omega_A}^2(F)) \longrightarrow H^0({\omega_A}^2(E))$, both having corank 1. We consider the diagram

$$\begin{array}{ccc} H^0(\omega_A) \otimes H^0(\omega_A(F)) & \xrightarrow{id \otimes u_*} & H^0(\omega_A) \otimes H^0(\omega_A(E)) \\ \downarrow{\scriptstyle \mu_{\omega_A,\omega_A(F)}} & & \downarrow{\scriptstyle \mu_{\omega_A,\omega_A(E)}} \\ H^0({\omega_A}^2(F)) & \xrightarrow{u'_*} & H^0({\omega_A}^2(E)) \end{array} \tag{1.29}$$

By the inductive hypothesis the map $\mu_{\omega_A,\omega_A(F)}$ is surjective. Thus the linear subspace $u'_*(Im(\mu_{\omega_A,\omega_A(F)}))$ has codimension 1 in $H^0({\omega_A}^2(E))$. Fix $\eta \in H^0(\omega_A)$ not vanishing at p and $\tau \in H^0(\omega_A(E))$ not vanishing at p. Since $\mu_{\omega_A,\omega_A(E)}(\eta \otimes \tau)$ does not vanish at p, it doesn't belong to $u'_*(Im(\mu_{\omega_A,\omega_A(F)}))$. Thus $\mu_{\omega_A,\omega_A(E)}$ is surjective. $\square$

1.4.2 k-normality in higher degree

We are now interested in studying the surjectivity of higher order maps, i.e. of

$$Sym^k(H^0(\omega_X)) \longrightarrow H^0(\omega_X^k)$$

when $k \geq 3$, but since $Sym^k(H^0(\omega_X))$ is a quotient of $H^0(\omega_X)^{\otimes k}$, we can equivalently study the surjectivity of

$$H^0(\omega_X)^{\otimes k} \longrightarrow H^0(\omega_X^k).$$

We observe that by applying part (b) in the proof of theorem 1.4.14 we get the following:

Proposition 1.4.15. *Let A be a reduced, connected and Gorenstein curve such that ω_A is globally generated. Fix a globally generated $R \in \mathrm{Pic}A$ such that $H^1(R) = 0$ and $\mu_{\omega_A,R}$ is surjective. Let $D \subset A_{\mathrm{sm}}$ be any effective divisor. Then $\mu_{\omega_A,R(D)}$ is surjective.*

As a corollary of theorem 1.4.14, we get the following result.

Corollary 1.4.16. *Let A be a reduced, connected and Gorenstein projective curve such that ω_A is very ample and μ_{ω_A} is surjective. Let $E \subset A_{\mathrm{sm}}$ be an effective divisor such that $\deg E \geq 2$. Then the maps $\mu_{\omega_A,\omega_A^k(kE)}$ are surjective for all $k \geq 2$.*

We are now going to give some definitions in order to state a result;

Definition 1.4.17. A *simple* $(r-1)$-secant is a configuration of $r-1$ smooth points $p_1, \dots, p_{r-1}$ on a curve $X \subset \mathbb{P}^N$, spanning a $\mathbb{P}^{r-2}$ and such that $X \cap \mathbb{P}^{r-2} = \{p_1, \dots, p_{r-1}\}$ as schemes.

Definition 1.4.18. Let R be a globally generated line bundle on a curve X, inducing a map $h_R : X \longrightarrow \mathbb{P}^r$, $r := \dim H^0(R) - 1$, which is birational onto the image. A *good* $(r-1)$-secant of R is a set $S := \{p_1, \dots, p_{r-1}\}$ such that $\dim H^0(R(-\sum_{i=1}^{r-1} p_i)) = 2$, $R(-\sum_{i=1}^{r-1} p_i)$ is still globally generated, and h_R is an embedding at each p_i.

We recall the following result from [B01]

Lemma 1.4.19 (Ballico)**.** *Let X be a one-dimensional projective locally Cohen-Macaulay scheme with $\dim H^0(\mathcal{O}_X) = 1$ and $R \in \mathrm{Pic}X$ globally generated and such that $\dim H^0(R) = 2$. Then the multiplication map*

$$\mu_{\omega_X,R} : H^0(\omega_X) \otimes H^0(R) \longrightarrow H^0(\omega_X \otimes R)$$

is surjective.

Lemma 1.4.20. *Let A be a connected, projective curve, $L, M \in \mathrm{Pic}A$, M globally generated, and such that $\dim H^0(M) = 2$ and $\dim H^1(L \otimes M^\vee) = 0$. Then $\mu_{L,M}$ is surjective.*

Proof. Obvious by the base point free pencil trick. □

Proposition 1.4.21. *Let A be a connected, Gorenstein curve with ω_A globally generated, $R \in \mathrm{Pic}A$ with R globally generated, with h_R birational onto its image and with a good $(r-1)$-secant, where $r := h^0(R) - 1$. Then the maps μ_{ω_A,R^k} are surjective for all $k \geq 1$.*

Proof. Fix a good $(r-1)$-secant set $S = \{q_1, \dots, q_{r-1}\}$. Thus the linear span $\langle h_R(q_1), \dots, h_R(q_{r-1})\rangle$ has dimension $r-2$, $h_R(A) \cap \langle h_R(q_1), \dots, h_R(q_{r-1})\rangle = \{h_R(q_1), \dots, h_R(q_{r-1})\}$ as schemes and

$$h_R^{-1}(\{h_R(q_1), \dots, h_R(q_{r-1})\}) = \{q_1, \dots, q_{r-1}\}.$$

Set $M := R(-S)$. We start by examining the case $k = 1$. Since ω_A is globally generated, we have $A \neq \mathbb{P}^1$. Since the map h_R induced by R is birational onto its image, we have $r \geq 2$. The first condition on the good $(r-1)$-secant points gives $h^0(M) = 2$. The last two conditions give that M is globally generated. Since $h^0(R) = h^0(M) + r - 1$, we also get $h^0(M(q_1)) = h^0(M)+1$. Thus there is $\eta \in H^0(M(q_1))$ such that $\eta(q_1) \neq 0$. The factorization shown in the following diagram

$$\begin{array}{ccc} H^0(\omega_A) \otimes H^0(M) & \twoheadrightarrow & H^0(\omega_A \otimes M) \\ \downarrow & & \downarrow j \\ H^0(\omega_A) \otimes H^0(M(q_1)) & \xrightarrow{\varphi} & H^0(\omega_A \otimes M(q_1)) \end{array}$$

shows that the image of φ contains a copy of $H^0(\omega_A \otimes M)$ as a hyperplane. Since q_1 is not a base point for M and ω_A is globally generated, there is $\sigma \in H^0(\omega_A) \otimes H^0(M(q_1))$ that doesn't vanish on q_1. Hence the image of σ via φ doesn't vanish on q_1, and we get the surjectivity of φ. Repeating this argument for all the points $q_1, \ldots, q_{r-1}$ adding them one by one we get that $\mu_{\omega_A,R}$ is surjective.

Now we assume $k \geq 2$ and use induction on k. The inductive assumption gives the surjectivity of the map $H^0(\omega_A) \otimes H^0(R^{k-1}) \longrightarrow H^0(\omega_A \otimes R^{k-1})$. We use the following commutative diagram:

$$\begin{array}{ccc} H^0(\omega_A) \otimes H^0(R^{k-1}) \otimes H^0(R^k) & \overset{\psi}{\twoheadrightarrow} & H^0(\omega_A \otimes R^{k-1}) \otimes H^0(R) \\ \downarrow & & \downarrow \phi \\ H^0(\omega_A) \otimes H^0(R^k) & \xrightarrow{\mu} & H^0(\omega_A \otimes R^k) \end{array}$$

It suffices to prove that ϕ is surjective, indeed, if it is, then $\phi \circ \psi$ is surjective, hence μ must be surjective. We proved that M is globally generated and $\dim H^0(M) = 2$. Moreover we notice that

$$\omega_A \otimes R^{k-1} \otimes M^\vee = \omega_A \otimes R^{k-2}(S).$$

Since $k \geq 2$ and $S \neq \emptyset$, we have that $\dim H^1(\omega_A \otimes R^{k-2}(S) = 0$. The base point free pencil trick applied to $\omega_A \otimes R^{k-1}$ and M gives the surjectivity of $\mu_{\omega_A \otimes R^{k-1},M}$. By Riemann-Roch theorem we get that

$$\dim H^0(\omega_A \otimes R^k) = \dim H^0(\omega_A \otimes R^{k-1} \otimes M) + \sharp S.$$

Arguing as in case $k = 1$ we get that the map μ_{ω_A,R^k} is surjective. □

Definition 1.4.22. We say that a line bundle L on a curve X is *k-normally generated* if the map

$$H^0(\omega_X)^{\otimes k} \longrightarrow H^0(\omega_X^k)$$

is surjective.

For instance "quadratically normal" means "linearly normal" plus "2-normally generated".

Proposition 1.4.23. *Let X be a connected, reduced, Gorenstein projective curve with planar singularities and ω_X very ample. Assume that $X = A \cup B$, with A, B connected and smooth at $D := A \cap B$. Fix $k \geq 3$; if*

(i) *ω_X is $(k-1)$-normally generated,*

(ii) *$\mu_{\omega_A, \omega_X^j|_A}$ is surjective for $1 \leq j \leq k$,*

(iii) *$\omega_X|_B$ is j-normally generated for $1 \leq j \leq k$,*

then ω_X is k-normally generated.

Proof. The proof is similar to the one of proposition 1.4.1; we just change notation slightly, denoting the multiplication maps in an easier way. We notice that in order to prove that the map

$$H^0(\omega_X)^{\otimes k} \xrightarrow{\mu_k} H^0(\omega_X^k)$$

is surjective, by factorizing we get

$$H^0(\omega_X) \otimes H^0(\omega_X)^{\otimes k-1} \xrightarrow{\mu \otimes \mu_{k-1}} H^0(\omega_X) \otimes H^0(\omega_X^{k-1}) \xrightarrow{\widetilde{\mu}} H^0(\omega_X^k),$$

so it suffices to see that the map $\widetilde{\mu}$ is surjective. We consider the diagram

$$\begin{array}{ccc} H^0(\omega_X) \otimes H^0(\omega_X^{k-1}) & \xrightarrow{\widetilde{\mu}} & H^0(\omega_X^k) \\ \downarrow \eta & & \downarrow \psi \\ H^0(\omega_X|_B) \otimes H^0(\omega_X^{k-1}|_B) & \xrightarrow{\phi} & H^0(\omega_X^k|_B) \end{array} \tag{1.30}$$

where the map $\widetilde{\mu} = \mu_{\omega_X, \omega_X^{k-1}}$. We know that ϕ is surjective by (iii), and if

(a) $\psi \circ \widetilde{\mu}$ is surjective,

(b) $\mathrm{Ker}\psi \subseteq \mathrm{Im}\widetilde{\mu}$,

then by linear algebra we get that $\widetilde{\mu}$ is surjective. In order to prove (a), by (1.30) we equivalently show that the map $\phi \circ \eta$ is surjective. We claim that η is surjective. Indeed, since ω_X is locally free we have the exact sequence

$$0 \to \mathcal{I}_B \otimes \omega_X \to \omega_X \to \omega_X|_B \to 0.$$

If we tensor by ω_X^{k-2}, we get

$$0 \to \mathcal{I}_B \otimes \omega_X^{k-1} \to \omega_X^{k-1} \to \omega_X|_B \otimes \omega_X^{k-2} \to 0,$$

which is equivalent to

$$0 \to \omega_A \otimes \omega_X^{k-2} \to \omega_X^{k-1} \to \omega_X^{k-1}|_B \to 0,$$

The corresponding long exact sequence in cohomology is

$$0 \to H^0(\omega_A \otimes \omega_X^{k-2}) \to H^0(\omega_X^{k-1}) \to H^0(\omega_X^{k-1}|_B) \to H^1(\omega_A \otimes \omega_X^{k-2}) \to$$

$$\to H^1(\omega_X^{k-1}) \to H^1(\omega_X^{k-1}|_B) \to \cdots$$

Now we consider $H^1(\omega_A \otimes \omega_X^{k-2})$; we have that $\omega_A \otimes \omega_X^{k-2} = \omega_A \otimes \omega_X^{k-2}|_A = \omega_A \otimes \omega_A^{k-2}((k-2)D)$, hence by lemma 1.4.3 we obtain that $H^1(\omega_A \otimes \omega_A^{k-2}((k-2)D)) = 0$, therefore the map

$$H^0(\omega_X^{k-1}) \to H^0(\omega_X^{k-1}|_B)$$

is surjective, and we get (a).

Now we want to prove (b). We notice that

$$\mathrm{Ker}\psi = H^0(\mathcal{I}_B \otimes \omega_X^k)$$

and set

$$\mu := \widetilde{\mu}|_{H^0(X,\mathcal{I}_B \otimes \omega_X) \otimes H^0(\omega_X^{k-1})}.$$

We have the following commutative diagram:

$$\begin{array}{ccc} H^0(\mathcal{I}_B \otimes \omega_X) \otimes H^0(\omega_X^{k-1}) & \xrightarrow{\mu} & H^0(\mathcal{I}_B \otimes {\omega_X}^k) \\ \downarrow \gamma & & \downarrow \cong \\ H^0(\omega_A) \otimes H^0(\omega_X^{k-1}|_A) & \xrightarrow{\mu_{\omega_A, \omega_X^{k-1}|_A}} & H^0(\omega_A \otimes \omega_X^{k-1}|_A) \end{array} \tag{1.31}$$

Now we have that $\mathcal{I}_B \otimes \omega_X \cong \omega_A$ and applying the previous argument to A rather than to B, we obtain that

$$H^0(\omega_X^{k-1}) \to H^0(\omega_X^{k-1}|_A)$$

is surjective, hence so is γ in (1.31). Applying hypothesis (ii) we have that μ is surjective, hence as in the proof of 1.4.1, we get that $\mathrm{Ker}\psi = \mathrm{Im}\mu \subseteq \mathrm{Im}\widetilde{\mu}$. ☐

We notice that when k grows, the hypothesis in proposition 1.4.23 can be simplified:

Proposition 1.4.24. *Let X be a connected, reduced, Gorenstein projective curve of genus g, with ω_X globally generated. Fix $k \geq 4$ and assume that ω_X is $(k-1)$-normally generated. Then ω_X is k-normally generated.*

Proof. As in the proof of 1.4.23, looking at the factorization

$$H^0(\omega_X) \otimes H^0(\omega_X)^{\otimes k-1} \xrightarrow{\mu \otimes \mu_{k-1}} H^0(\omega_X) \otimes H^0(\omega_X^{k-1}) \xrightarrow{\mu_{\omega_X, \omega_X^{k-1}}} H^0(\omega_X^k),$$

by hypothesis it suffices to prove that $\mu_{\omega_X, \omega_X^{k-1}}$ is surjective. We use Proposition 8 in [F07] in the following way: we take $\mathcal{F} := \omega_X$ and $\mathcal{H} := \omega_X^{k-1}$, so we have that $H^0(\mathcal{F})$ is globally generated. Moreover we have that

$$H^1(\mathcal{H} \otimes \mathcal{F}^{-1}) = H^1(\omega_X^{k-2}) = 0$$

if $k \geq 4$, so we get that the $\mu_{\omega_X, \omega_X^{k-1}}$ is surjective. ☐

1.4.3 Applications

In the sequel we are going to study some cases where we can apply our results.

Lemma 1.4.25. *Let Z be a connected and Gorenstein curve such that ω_Z is globally generated. Let $D \subset Z_{\mathrm{sm}}$ be an effective Cartier divisor such that $\deg(D) \geq 2$. Then $\omega_Z(D)$ is globally generated.*

Proof. Since $\omega_Z(D)$ is a line bundle, it is globally generated if and only if for every $q \in Z$ there is $s \in H^0(\omega_Z(D))$ such that $s(q) \neq 0$. Since ω_Z is assumed to be globally generated and D is effective, the sheaf $\omega_Z(D)$ is globally generated outside the finitely many points appearing in $supp(D)$. Fix $p \in supp(D)$ and set $D' := \mathcal{I}_p \otimes D$. Since $p \in X_{sm}$, D' is a Cartier divisor of degree $\deg(D) - 1$. Moreover, since $p \in supp(D)$, D' is effective. Thus Lemma 1.4.3 gives $h^1(\omega_Z(D')) = 0$. Riemann-Roch gives $h^0(\omega_Z(D)) = h^0(\omega_Z(D')) + 1$. Thus there is $s \in H^0(\omega_Z(D))$ such that $s(p) \neq 0$. □

Corollary 1.4.26. *Let X be a connected reduced curve with two irreducible non-rational components C_1, C_2 meeting at planar singularities for X and both smooth at $C_1 \cap C_2$; assume that ω_X is very ample. Then X is canonically embedded is projectively normal.*

Proof. First of all we have to prove that X is quadratically normal, so let us use the set-up of proposition 1.4.1, and set $A = C_1$, $B = C_2$. We look at hypothesis (i) and (ii) of the theorem; hypothesis (i) is verified by applying 1.4.13 to C_1. Indeed in our situation $R = \omega_X|_{C_1}$, i.e. $R = \omega_{C_1}(D)$ where D is the divisor on C_1 and C_2 corresponding to $C_1 \cap C_2$. Hence by lemma 1.4.25 we have that R is globally generated and birational onto the image, and we get (i). Concerning (ii), it suffices to apply 1.4.11, and then by 1.4.1 we obtain that X is quadratically generated. Now we want to study the 3-normal generation of X. So we look at the hypothesis of 1.4.23: we know that ω_X is quadratically normal, and of course (iii) holds by lemma 1.4.11. So it remains to prove (ii): but this is a consequence of corollary 1.4.16, indeed we have that μ_{ω_A} is surjective since A is irreducible and hence projectively normal, moreover, being ω_X very ample, $A \cdot B \geq 3$. Now when $k \geq 4$ we just apply 1.4.24 and get the conclusions. □

Remark 1.4.27. We observe that in the case of nodal connected curves with two non-rational irreducible components, the corollary above says that if the two components C_1 and C_2 meet at least at 3 points, then $X = C_1 \cup C_2$ canonically embedded is projectively normal. The corollary leaves out the curves having at least one $\mathbb{P}^1$ as a component, and in particular binary curves (i.e. a curve X is binary if it is composed of two $\mathbb{P}^1$'s meeting at $g+1$ points where g is the genus of X), but for the latter special class of curves we can use [S91] (see 1.4.30) and easily get projective normality. Concerning the class of curves $X = C_1 \cup C_2$ with $C_1 \neq \mathbb{P}^1$ and $C_2 = \mathbb{P}^1$, we get the projective normality by applying the same proof as in corollary 1.4.26, once we denote by A the component C_1. Indeed the hypothesis $C_1 \neq \mathbb{P}^1$ is used only when we apply 1.4.13 to A.

We can generalize the previous result:

Corollary 1.4.28. *Let X be a connected reduced Gorenstein curve with ω_X very ample and with planar singularities. Assume that $X = A \cup B$ with $A \neq \mathbb{P}^1$ irreducible and let B be a connected curve. Let A and B be smooth at $A \cap B$. Then ω_X is k-normally generated for any $k \geq 2$.*

Proof. The proof is straightforward once we notice that we can apply 1.4.13 to A and by Theorem 1 we get quadratic normality of X; for $k = 3$ we apply 1.4.23 since both 1.4.11 for B and 1.4.13 for A hold, and when $k \geq 4$ we apply 1.4.24. □

Corollary 1.4.29. *Let X be a connected reduced Gorenstein curve with ω_X very ample and with planar singularities. Assume that $X = A \cup B$ with A as in theorem 1.4.14 and let B be a connected curve. Let A and B be smooth at $A \cap B$. Then X canonically embedded is projectively normal.*

Proof. The proof is as in corollary 1.4.28, we just apply theorem 1.4.14 to A. □

We give now an example; before doing this, we recall an important result from [S91]:

Theorem 1.4.30 (Schreyer). *Let $X \subset \mathbb{P}^{g-1}$ be a canonical curve of genus g. If X has a simple $(g-2)$-secant, then X is projectively normal.*

Schreyer's theorem can be used in the most general setting once one is able to verify the existence of a simple $(g-2)$-secant. In [S91]pp.86 gave an example of a reducible canonically embedded curve admitting no simple $(g-2)$-secant. In the following example we show that our theorem applies to that case.

Example 1.4.31. Let $X = X_1 \cup X_2 \cup X_3 \cup X_4$, with X_i smooth of genus g_i and such that the components intersect in 6 distinct points $p_{ij} = X_i \cap X_j$ that are ordinary nodes for X. Then X has genus $g = g_1 + g_2 + g_3 + g_4 + 3$. We have that ω_X is a very ample line bundle; if $g_i = 0$ for every i we have a graph curve, and it is projectively normal, as we see in [BE91]. Hence we can assume $g_i \neq 0$ for some i, say $g_1 > 0$. Set $A := X_1$, $B := X_2 \cup X_3 \cup X_4$. Since $A \neq \mathbb{P}^1$ we can apply 1.4.13 and get that the multiplication map $\mu_{\omega_A, \omega_X|_A}$ is surjective. Sincethe conditions on the degree of $\omega_X|_B$ in 1.4.10 are satisfied, the map $\mu_{\omega_X|_B}$ is surjective and we can apply proposition 1.4.1 and get that X is quadratically normal.

1.5 Projective normality of binary curves

Definition 1.5.1. A reduced nodal curve X is called a *binary curve* if $X = C_1 \cup C_2$ where $C_i \cong \mathbb{P}^1$ and the number of nodes of X is $\delta = g + 1$, where g is the arithmetic genus of X.

Hereafter we recall some properties of binary curves that we will use several times; for the details we refer the interested reader to [C5].

Definition 1.5.2. Let X be a binary curve of genus $g \geq -1$. A multidegree $\underline{d} = (d_1, d_2)$ with $d = |\underline{d}| = d_1 + d_2$ is said to be balanced on X if for $i = 1, 2$,

$$\frac{d-g-1}{2} \leq d_i \leq \frac{d+g+1}{2}. \tag{1.32}$$

We say that $L \in \text{Pic}^d(X)$ is balanced if $\underline{\deg}L$ is balanced on X, and we say that a balanced degree $\underline{d} \in B^d(X)$, where $B^d(X)$ denotes the set of the balanced degrees on X. Let us introduce the dualizing sheaf ω_X on X; it is a line bundle on X, such that its restriction ω_{C_i} to C_i has $\deg \omega_{C_i} = g - 1$ and its total degree has the following important property, recovered from the *basic inequality* in [C1]:

$$\underline{d} \text{ is balanced} \Leftrightarrow \underline{d} + n\underline{\deg}\omega_X \text{ is balanced}. \tag{1.33}$$

In the sequel we are going to recall briefly some classical results which still hold for binary curves, namely [C5, Proposition 11], and [C5, Proposition 19]:

Theorem 1.5.3 (Riemann's theorem)**.** *Let X be a binary curve of genus g, and let $d \geq 2g-1$.*

(i) For every balanced $L \in Pic^d(X)$ we have $h^0(L) = d - g + 1$.

(ii) For every $[\widehat{L}] \in \overline{P_X^d}$ we have $h^0(\widehat{L}) = d - g + 1$.

Theorem 1.5.4. *Let X be a binary curve; its dualizing sheaf, ω_X, is very ample if and only if X is non-hyperelliptic.*

In order to prove an analogous version of Max Noether's theorem for binary curves, we follow the proof leading to Castelnuovo's bound, (see [ACGH]), which is basically a consequence of the *General Position Theorem*. This theorem in general fails when the curve X is reducible, but in our setting it still holds: indeed when X is binary and embedded in $\mathbb{P}^{g-1}$ via the dualizing sheaf ω_X, the two components C_1, C_2 are such that $\omega_{C_i} = \mathcal{O}_{\mathbb{P}^1}(g-1)$, hence both C_1 and C_2 are embedded in $\mathbb{P}^{g-1}$ as non-degenerate rational normal curves meeting each other at $\delta = g + 1$ points. This implies that the general hyperplane of $\mathbb{P}^{g-1}$ will cut on C_i an effective divisor of degree $g - 1$, whose points are linearly independent.

Proposition 1.5.5 (General Position Theorem for Canonical Binary Curves)**.** *Let $X \subset \mathbb{P}^{g-1}$ be a binary curve of genus g embedded via ω_X; then a general hyperplane meets X in $2g-2$ points, any $g-1$ of which are linearly independent.*

Proof. Let us consider the canonical embedding of X via ω_X, $X \hookrightarrow \mathbb{P}^{g-1}$, so that both C_1 and C_2 span the ambient space. Let us take $g - 4$ general points $p_1, \ldots, p_{g-4}$ on (say) C_1, and consider the projection from their span $\Lambda = \langle p_1, \ldots, p_{g-4} \rangle$, which is a linear subspace of $\mathbb{P}^{g-1}$ of dimension $g - 5$ since C_1 is a rational normal curve of degree $g - 1$. Let us denote the projection by $\pi_\Lambda : \mathbb{P}^{g-1} \dashrightarrow \mathbb{P}^3$, then the images in $\mathbb{P}^3$ of C_1, C_2 are the curves ${C_1}' = \overline{\pi_\Lambda(C_1)}$, which is a twisted cubic, and ${C_2}' = \overline{\pi_\Lambda(C_2)}$, which is a curve of degree $g - 1$. Let us notice that both of them are non-degenerate curves in $\mathbb{P}^3$. Now we want to show that if the statement of the theorem fails in $\mathbb{P}^{g-1}$, then it also fails in $\mathbb{P}^3$. Indeed, if it fails in $\mathbb{P}^{g-1}$, this means that for every hyperplane section of X there exist $g-1$ points which are linearly dependent, hence they span a $\mathbb{P}^{g-3}$; now we consider again the projection of X from $g-4$ general points on C_1, and we obtain that the projection of the $\mathbb{P}^{g-3}$ from Λ is a line in $\mathbb{P}^3$, hence the images of the $g-1$ points are collinear, so, in the hypothesis that $g \geq 4$ we get that there exists a triplet of dependent points out of a hyperplane section in $\mathbb{P}^3$. Now we claim that the statement of the theorem in $\mathbb{P}^3$ is equivalent to saying that there are at most ∞^1 trichords to X, and this is in turn equivalent to the statement: not every pair of points of $X \setminus X_{sing}$ lie on a trichord. We prove the equivalence of the last two statements as in [ACGH, Lemma, p.110]. Let U be the open set in $(\mathbb{P}^{g-1})^*$ of the hyperplanes transverse to X, and let us denote by $\{a_i, b_i\} = \nu^{-1}(n_i)$ for any node n_i of X, where ν is the normalization map, and let $\dot{C_1} = C_1 \setminus \{a_1, \dots, a_\delta\}$ and $\dot{C_2} = C_2 \setminus \{b_1, \dots, b_\delta\}$. Consider the set

$$J := \{(p_1, p_2, H) : p_1, p_2 \in H \text{ and } \overline{p_1 + p_2} \text{ is a trichord}\} \subset X \times X \times U,$$

where now $U \subset (\mathbb{P}^3)^*$, and abusing notation X denotes its projection. Since ${C_1}'$ and ${C_2}'$ are non-degenerate irreducible curves in $\mathbb{P}^3$ and the General Position Theorem (see [ACGH]) holds for both of them, we can consider the open subset of J, $J_0 = \{(p_1, p_2, H) : p_1, p_2 \in H \text{ and } \overline{p_1 + p_2} \text{ is a trichord}\} \subset \dot{C}_1' \times \dot{C}_2' \times U$. Now $\dot{C}_1' \times \dot{C}_2'$ is irreducible and the fiber dimension of $J_0 \to \dot{C_1} \times \dot{C_2}$ is always $(g-1)-2$, then, as in [ACGH, Lemma, p.110], we have that the surjectivity of $J_0 \to U$ is equivalent to $\dim J_0 \geq g-1$, which is in turn equivalent to the surjectivity of $J_0 \to \dot{C_1} \times \dot{C_2}$. This shows the equivalence of the two statements. In particular, saying that not every pair of points of $X \setminus X_{sing}$ lie on a trichord, implies that not every pair of points of X lie on a trichord. Then let us suppose by contradiction that every secant of X is trichord: arguing as in [ACGH, Lemma, p.110] we see that if the statement is false, then every two tangent lines meet in a point. But this is clearly false, indeed if we take two tangent lines to C_1, since it is a rational normal curve spanning the whole $\mathbb{P}^3$, they can be skew lines and not meet. □

In what follows we are going to state Max Noether's theorem for binary curves:

Definition 1.5.6. Let $X \subset \mathbb{P}^r$ be any embedded curve; we say that X is *linearly normal* if the linear series of hyperplane sections is complete, i.e. if $h^0(X, \mathcal{O}_X(1)) = r+1$. Moreover we say that X is *projectively normal* if the linear series cut out on X by hypersurfaces of degree l is complete for every $l \geq 1$, i.e. if the natural map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(l)) \to H^0(X, \mathcal{O}_X(l))$ is surjective.

Proposition 1.5.7. *If X is a non-hyperelliptic binary curve, its canonical model is projectively normal.*

Proof. If X is a non-hyperelliptic binary curve, by 1.5.4 we know that the dualizing sheaf ω_X gives an embedding φ of X in $\mathbb{P}^{g-1}$. Let us denote by $X_c \subset \mathbb{P}^{g-1}$ the canonical model of X, such that $X \xrightarrow{\varphi} X_c$. From definition 1.5.6 we immediately see that X_c is linearly normal since it's embedded via a complete linear series. We want to show that X_c is projectively normal. Let H be a general hyperplane in $\mathbb{P}^{g-1}$ so that $\Gamma_c = X_c \cap H$ is a general hyperplane section on X_c. For $l \geq 1$ consider the linear series $|\omega_X^l| = |\varphi^* \mathcal{O}_{X_c}(l)|$ on X and its subseries $\mathscr{D}_l$ cut out on X by hypersurfaces of degree l. We denote by $\alpha_l = \dim |\omega_X^l|$ and by $\beta_l = \dim \mathscr{D}_l$. Of course we have that, if Γ is the pullback to X of the hyperplane section Γ_c, $\mathscr{D}_{l-1} + \Gamma \subseteq \mathscr{D}_l$; hence $\mathscr{D}_l(-\Gamma) \subseteq \mathscr{D}_{l-1}$ and $\beta_l - \sharp\{$conditions imposed by Γ_c on hypersurfaces of degree l in $\mathbb{P}^{g-1}\} \geq \beta_{l-1}$. Notice that the number of conditions imposed by Γ_c on hypersurfaces of degree l in $\mathbb{P}^{g-1}$ is equal to the number of conditions imposed by Γ_c on hypersurfaces of degree l contained in the hyperplane H that cuts Γ_c on X_c. Let us call n_l this number, then applying proposition 1.5.5 and [ACGH, Lemma p.115], we get that

$$n_l \geq \min\{2g-2, l(g-2)+1\}.$$

So $\beta_l - \beta_{l-1} \geq n_l \geq \min\{2g-2, l(g-2)+1\}$; following the proof leading to [ACGH, Castelnuovo's bound p.115], if we set

$$m := \left\lfloor \frac{2g-3}{g-2} \right\rfloor = 2,$$

we get that

$$\begin{array}{rclcl} \alpha_0 & = & \beta_0 & = & 0, \\ \alpha_1 & = & \beta_1 & = & g-1, \\ \alpha_2 & \geq & \beta_2 & \geq & (g-1)+2(g-2)+1 = 3g-4. \end{array} \quad (1.34)$$

Now if we take $|\omega_X^2|$, then $\underline{\deg}\omega_X^2 = 2\underline{\deg}\omega_X$, hence by (1.33) we can apply theorem 1.5.3, and we get that

$$g = \deg \omega_X^2 - h^0(X, \omega_X^2) + 1 = 2 \deg \omega_X - \alpha_2 \leq 4g - 4 - 3g + 4 = g.$$

Therefore $\alpha_2 = 3g-4$, and $\alpha_k = \beta_k$ for any $1 \leq k \leq 2$; we are now going to prove that $\alpha_l - \alpha_{l-1} = \min\{2g-2, l(g-2)+1\}$ for any $l \geq 1$. It clearly holds for $l \leq 2$; let now $l \geq 3$: continuing (1.34) we have that $\alpha_3 \geq \beta_3 \geq 5g-6$, but we also have by theorem 1.5.3 that $\alpha_3 \leq 5g-6$, whence $\alpha_3 = \beta_3$. Iterating we get that $\alpha_l - \alpha_{l-1} = \beta_l - \beta_{l-1} = 2g-2$ for any $l \geq 3$, so in particular this implies that $\alpha_l = \beta_l$ for any $l \geq 1$. This in turn is equivalent to saying that $|\omega_X^l| = \mathscr{D}_l$, i.e., the map

$$H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(l)) \to H^0(X_c, \mathcal{O}_{X_c}(l))$$

is surjective for any l, so X_c is projectively normal. $\square$

Chapter 2

On the tropical Torelli map (with M. Melo and F. Viviani)

2.1 Preliminaries

2.1.1 Stacky fans

In order to fix notations, recall some concepts from convex geometry. A *polyhedral cone* Ξ is the intersection of finitely many closed linear half-spaces in $\mathbb{R}^n$. The *dimension* of Ξ is the dimension of the smallest linear subspace containing Ξ. Its *relative interior* $\operatorname{Int}\Xi$ is the interior inside this linear subspace, and the complement $\Xi \setminus \operatorname{Int}\Xi$ is called the *relative boundary* $\partial\,\Xi$. If $\dim \Xi = k$ then $\partial\,\Xi$ is itself a union of polyhedral cones of dimension at most $k-1$, called *faces*, obtained by intersection of Ξ with linear hyperplanes disjoint from $\operatorname{Int}\Xi$. Faces of dimensions $k-1$ and 0 are called *facets* and *vertices*, respectively. A polyhedral cone is *rational* if the linear functions defining the half-spaces can be taken with rational coefficients.

An *open polyhedral cone* of $\mathbb{R}^n$ is the relative interior of a polyhedral cone. Note that the closure of an open polyhedral cone with respect to the Euclidean topology of $\mathbb{R}^n$ is a polyhedral cone. An open polyhedral cone is rational if its closure is rational.

We say that a map $\mathbb{R}^n \to \mathbb{R}^m$ is *integral linear* if it is linear and sends $\mathbb{Z}^n$ into $\mathbb{Z}^m$, or equivalently if it is linear and can be represented by an integral matrix with respect to the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.

Definition 2.1.1. Let $\{X_k \subset \mathbb{R}^{m_k}\}_{k\in K}$ be a finite collection of rational open polyhedral cones such that $\dim X_k = m_k$. Moreover, for each such cone $X_k \subset \mathbb{R}^{m_k}$, let G_k be a group and $\rho_k : G_k \to \mathrm{GL}_{m_k}(\mathbb{Z})$ a homomorphism such that $\rho_k(G_k)$ stabilizes the cone X_k under its natural action on $\mathbb{R}^{m_k}$. Therefore G_k acts on X_k (resp. $\overline{X_k}$), via the homomorphism ρ_k, and we denote the quotient by X_k/G_k (resp. $\overline{X_k}/G_k$), endowed with the quotient topology. A topological space X is said to be a *stacky (abstract) fan* with cells $\{X_k/G_k\}_{k\in K}$ if there exist continuous maps $\alpha_k : \overline{X_k}/G_k \to X$ satisfying the following properties:

(i) The restriction of α_k to X_k/G_k is an homeomorphism onto its image;

(ii) $X = \coprod_k \alpha_k(X_k/G_k)$ (set-theoretically);

(iii) For any $j, k \in K$, the natural inclusion map $\alpha_k(\overline{X_k}/G_k) \cap \alpha_j(\overline{X_j}/G_j) \hookrightarrow \alpha_j(\overline{X_j}/G_j)$ is induced by an integral linear map $L : \mathbb{R}^{m_k} \to \mathbb{R}^{m_j}$, i.e. there exists a commutative diagram

$$\begin{array}{ccccccc} \alpha_k(\overline{X_k}/G_k) \cap \alpha_j(\overline{X_j}/G_j) & \hookrightarrow & \alpha_k(\overline{X_k}/G_k) & \twoheadleftarrow & \overline{X_k} & \hookrightarrow & \mathbb{R}^{m_k} \\ & \searrow & & & \downarrow L & & \downarrow L \\ & & \alpha_j(\overline{X_j}/G_j) & \twoheadleftarrow & \overline{X_j} & \hookrightarrow & \mathbb{R}^{m_j}. \end{array} \tag{2.1}$$

By abuse of notation, we usually identify X_k/G_k with its image inside X so that we usually write $X = \coprod X_k/G_k$ to denote the decomposition of X with respect to its cells X_k/G_k.

A stacky *subfan* of X is a closed subspace $X' \subseteq X$ that is a disjoint union of cells of X. Note that X' inherits a natural structure of stacky fan with respect to the sub-collection $\{X_k/G_k\}_{k \in K'}$ of cells that are contained in X'.

The dimension of X, denoted by $\dim X$, is the greatest dimension of its cells. We say that a cell is maximal if it is not contained in the closure of any other cell. X is said to be of *pure dimension* if all its maximal cells have dimension equal to $\dim X$. A *generic point* of X is a point contained in a cell of maximal dimension.

Assume now that X is a stacky fan of pure dimension n. The cells of dimension $n-1$ are called codimension one cells. X is said to be *connected through codimension one* if for any two maximal cells X_k/G_k and $X_{k'}/G_{k'}$ one can find a sequence of maximal cells $X_{k_0}/G_{k_0} = X_k/G_k$, $X_{k_1}/G_{k_1}, \cdots, X_{k_r}/G_{k_r} = X_{k'}/G_{k'}$ such that for any $0 \le i \le r-1$ the two consecutive maximal cells X_{k_i}/G_{k_i} and $X_{k_{i+1}}/G_{k_{i+1}}$ have a common codimension one cell in their closure.

Definition 2.1.2. Let X and Y be two stacky fans with cells $\{X_k/G_k\}_{k\in K}$ and $\{Y_j/H_j\}_{j\in J}$ where $\{X_k \subset \mathbb{R}^{m_k}\}_{k\in K}$ and $\{Y_j \subset \mathbb{R}^{m'_j}\}_{j\in J}$, respectively. A continuous map $\pi : X \to Y$ is said to be a *map of stacky fans* if for every cell X_k/G_k of X there exists a cell Y_j/H_j of Y such that

1. $\pi(X_k/G_k) \subset Y_j/H_j$;

2. $\pi : X_k/G_k \to Y_j/H_j$ is induced by an integral linear function $L_{k,j} : \mathbb{R}^{m_k} \to \mathbb{R}^{m'_j}$, i.e. there exists a commutative diagram

$$\begin{array}{ccccc} X_k/G_k & \twoheadleftarrow & X_k & \hookrightarrow & \mathbb{R}^{m_k} \\ \downarrow \pi & & \downarrow L_{k,j} & & \downarrow L_{k,j} \\ Y_j/H_j & \twoheadleftarrow & Y_j & \hookrightarrow & \mathbb{R}^{m'_j}. \end{array} \tag{2.2}$$

We say that $\pi : X \to Y$ is *full* if it sends every cell X_k/G_k of X surjectively into some cell Y_j/H_j of Y. We say that $\pi : X \to Y$ is *of degree one* if for every generic point $Q \in Y_j/H_j \subset Y$ the inverse image $\pi^{-1}(Q)$ consists of a single point $P \in X_k/G_k \subset X$ and the integral linear function $L_{k,j}$ inducing $\pi : X_k/G_k \to Y_j/H_j$ is primitive (i.e. $L_{k,j}^{-1}(\mathbb{Z}^{m'_j}) \subset \mathbb{Z}^{m_k}$).

Remark 2.1.3. The above definition of stacky fan is inspired by some definitions of polyhedral complexes present in the literature, most notably in [KKMS73, Def. 5, 6], [GM08, Def. 2.12], [AR10, Def. 5.1] and [GS07, Pag. 9].

The notions of pure-dimension and connectedness through codimension one are well-known in tropical geometry (see the Structure Theorem in [McLS]).

2.1.2 Graphs

Here we recall the basic notions of graph theory that we will need in the sequel. We follow mostly the terminology and notations of [Di97].

Throughout this paper, Γ will be a finite connected graph. By finite we mean that Γ has a finite number of vertices and edges; moreover loops or multiple edges are allowed. We denote by $V(\Gamma)$ the set of vertices of Γ and by $E(\Gamma)$ the set of edges of Γ. The *valence* of a vertex v, $\mathrm{val}(v)$, is defined as the number of edges incident to v, with the usual convention that a loop around a vertex v is counted twice in the valence of v. A graph Γ is *k-regular* if $\mathrm{val}(v) = k$ for every $v \in V(\Gamma)$.

Definition 2.1.4. A *cycle* of Γ is a subset $S \subseteq E(\Gamma)$ such that the graph $\Gamma/\ (E(\Gamma) \setminus S)$, obtained from Γ by contracting all the edges not in S, is (connected and) 2-regular.
If $\{V_1, V_2\}$ is a partition of $V(\Gamma)$, the set $E(V_1, V_2)$ of all the edges of Γ with one end in V_1 and the other end in V_2 is called a *cut*; a *bond* is a minimal cut, or equivalently, a cut $E(\Gamma_1, \Gamma_2)$ such that the graphs Γ_1 and Γ_2 induced by V_1 and V_2, respectively, are connected.

In the Example 2.1.18, the subsets $\{f_1, f_2, f_3\}$ and $\{f_4, f_5\}$ are bonds of Γ_2 while the subset $\{f_1, f_2, f_3, f_4, f_5\}$ is a non-minimal cut of Γ_2.

2.1.5. Homology theory

Consider the space of 1-chains and 0-chains of Γ with values in a finite abelian group A (we will use the groups $A = \mathbb{Z}, \mathbb{R}$):

$$C_1(\Gamma, A) := \oplus_{e \in E(\Gamma)} A \cdot e \qquad\qquad C_0(\Gamma, A) := \oplus_{v \in V(\Gamma)} A \cdot v.$$

We endow the above spaces with the A-bilinear, symmetric, non-degenerate forms uniquely determined by:

$$(e, e') := \delta_{e,e'} \qquad\qquad \langle v, v' \rangle := \delta_{v,v'},$$

where $\delta_{-,-}$ is the usual Kronecker symbol and $e, e' \in E(\Gamma)$; $v, v' \in V(\Gamma)$. Given a subspace $V \subset C_1(\Gamma, A)$, we denote by $V^{\perp}$ the orthogonal subspace with respect to the form $(\,,\,)$.

Fix now an orientation of Γ and let $s, t : E(\Gamma) \to V(\Gamma)$ be the two maps sending an oriented edge to its source and target vertex, respectively. Define two boundaries maps

$$\begin{aligned} \partial : C_1(\Gamma, A) &\longrightarrow C_0(\Gamma, A) \\ e &\mapsto t(e) - s(e) \end{aligned} \qquad \begin{aligned} \delta : C_0(\Gamma, A) &\longrightarrow C_1(\Gamma, A) \\ v &\mapsto \sum_{e\,:\,t(e)=v} e - \sum_{e\,:\,s(e)=v} e. \end{aligned}$$

It is easy to check that the above two maps are adjoint with respects to the two symmetric A-bilinear forms $(,)$ and $\langle,\rangle$, i.e. $\langle\partial(e), v\rangle = (e, \delta(v))$ for any $e \in E(\Gamma)$ and $v \in V(\Gamma)$.

The kernel of ∂ is called the first homology group of Γ with coefficients in A and is denoted by $H_1(\Gamma, A)$. Since ∂ and δ are adjoint, it follows that $H_1(\Gamma, A)^\perp = \Im(\delta)$. It is a well-known result in graph theory that $H_1(\Gamma, A)$ and $H_1(\Gamma, A)^\perp$ are free A-modules of ranks:

$$\begin{cases} \mathrm{rank_A H_1(\Gamma, A) = 1 - \#V(\Gamma) + \#E(\Gamma)}, \\ \mathrm{rank_A H_1(\Gamma, A)^\perp = \#V(\Gamma) - 1}. \end{cases}$$

The A-rank of $H_1(\Gamma, A)$ is called also the *genus* of Γ and it is denoted by $g(\Gamma)$; the A-rank of $H_1(\Gamma, A)^\perp$ is called the *co-genus* of Γ and it is denoted by $g^*(\Gamma)$.

2.1.6. Connectivity and Girth

There are two ways to measure the connectivity of a graph: the vertex-connectivity (or connectivity) and the edge-connectivity. Recall their definitions (following [Di97, Chap. 3]).

Definition 2.1.7. Let $k \geq 1$ be an integer.

1. A graph Γ is said to be *k-vertex-connected* (or simply *k-connected*) if the graph obtained from Γ by removing any set of $s \leq k-1$ vertices and the edges adjacent to them is connected.

2. The *connectivity* of Γ, denoted by $k(\Gamma)$, is the maximum integer k such that Γ is k-connected. We set $k(\Gamma) = +\infty$ if Γ has only one vertex.

3. A graph Γ is said to be *k-edge-connected* if the graph obtained from Γ by removing any set of $s \leq k-1$ edges is connected.

4. The *edge-connectivity* of Γ, denoted by $\lambda(\Gamma)$, is the maximum integer k such that Γ is k-edge-connected. We set $\lambda(\Gamma) = +\infty$ if Γ has only one vertex.

Note that $\lambda(\Gamma) \geq 2$ if and only if Γ has no separating edges; while $\lambda(\Gamma) \geq 3$ if and only if Γ does not have pairs of separating edges.

In [CV1], a characterization of 3-edge-connected graphs is given in terms of the so-called *$C1$-sets*. Recall (see [CV1, Def. 2.3.1, Lemma 2.3.2]) that a $C1$-set of Γ is a subset of $E(\Gamma)$ formed by edges that are non-separating and belong to the same cycles of Γ. The $C1$-sets form a partition of the set of non-separating edges ([CV1, Lemma 2.3.4]). In [CV1, Cor. 2.3.4], it is proved that Γ is 3-edge-connected if and only if Γ does not have separating edges and all the $C1$-sets have cardinality one.

The two notions of connectivity are related by the following relation:

$$k(\Gamma) \leq \lambda(\Gamma) \leq \delta(\Gamma),$$

where $\delta(\Gamma) := \min_{v \in V(\Gamma)}\{\mathrm{val}(v)\}$ is the valence of Γ.

Finally recall the definition of the girth of a graph.

Definition 2.1.8. The *girth* of a graph Γ, denoted by $\mathrm{girth}(\Gamma)$, is the minimum integer k such that Γ contains a cycle of length k. We set $\mathrm{girth}(\Gamma) = +\infty$ if Γ has no cycles, i.e. if it is a tree.

Note that $\mathrm{girth}(\Gamma) \geq 2$ if and only if Γ has no loops; while $\mathrm{girth}(\Gamma) \geq 3$ if and only if Γ has no loops and no multiples edges. Graphs with girth greater or equal than 3 are called simple.

Example 2.1.9. For the graph Γ_1 in the Example 2.1.18, we have that $k(\Gamma_1) = 1$ because v is a separating vertex. The $C1$-sets of Γ_1 are $\{e_1, e_2, e_3\}$ and $\{e_4, e_5\}$. We have that $\lambda(\Gamma_1) = 2$ because Γ_1 has a $C1$-set of cardinality greater than 1 and does not have separating edges. Moreover, $\mathrm{girth}(\Gamma_1) = 2$ since $\{e_4, e_5\}$ is the smallest cycle of Γ_1.

The Peterson graph Γ depicted in Figure 2.6 is 3-regular and has $k(\Gamma) = \lambda(\Gamma) = 3$. Moreover, it is easy to check that $\mathrm{girth}(\Gamma) = 5$.

2.1.10. 2-isomorphism

We introduce here an equivalence relation on the set of all graphs, that will be very useful in the sequel.

Definition 2.1.11 **([W33]).** Two graphs Γ_1 and Γ_2 are said to be 2*-isomorphic*, and we write $\Gamma_1 \equiv_2 \Gamma_2$, if there exists a bijection $\phi : E(\Gamma_1) \to E(\Gamma_2)$ inducing a bijection between cycles of Γ_1 and cycles of Γ_2, or equivalently, between bonds of Γ_1 and bonds of Γ_2. We denote by $[\Gamma]_2$ the 2-isomorphism class of a graph Γ.

This equivalence relation is called cyclic equivalence in [CV1] and denoted by $\equiv_{\mathrm{cyc}}$.

Remark 2.1.12. The girth, the connectivity, the edge-connectivity, the genus and the co-genus are defined up to 2-isomorphism; we denote them by $\mathrm{girth}([\Gamma]_2)$, $k([\Gamma]_2)$, $\lambda([\Gamma]_2)$, $g([\Gamma]_2)$ and $g^*([\Gamma]_2)$.

As a consequence of a very well known theorem of Whitney (see [W33] or [O92, Sec. 5.3]), we have the following

Fact 2.1.13. If Γ is 3-connected, the 2-isomorphism class $[\Gamma]_2$ contains only Γ.

In the sequel, graphs with girth or edge-connectivity at least 3 will play an important role. We describe here a way to obtain such a graph starting with an arbitrary graph Γ.

Definition 2.1.14. Given a graph Γ, the *simplification* of Γ is the simple graph Γ^{sim} obtained from Γ by deleting all the loops and all but one among each collection of multiple edges.

Note that the graph Γ^{sim} does not depend on the choices made in the operation of deletion. A similar operation can be performed with respect to the edge-connectivity, but the result is only a 2-isomorphism class of graphs.

Definition 2.1.15. *[CV1, Def. 2.3.6]* Given a graph Γ, a *3-edge-connectivization* of Γ is a graph, denoted by Γ^3, obtained from Γ by contracting all the separating edges and all but one among the edges of each $C1$-set of Γ.

The 2-isomorphism class of Γ^3, which is independent of all the choices made in the construction of Γ^3 (see [CV1, Lemma 2.3.8(iii)]), is called the 3-edge-connectivization class of Γ and is denoted by $[\Gamma^3]_2$.

2.1.16. Duality

Recall the following definition (see [Di97, Sec. 4.6]).

Definition 2.1.17. Two graphs Γ_1 and Γ_2 are said to be in *abstract duality* if there exists a bijection $\phi : E(\Gamma_1) \to E(\Gamma_2)$ inducing a bijection between cycles (resp. bonds) of Γ_1 and bonds (resp. cycles) of Γ_2. Given a graph Γ, a graph Γ' such that Γ and Γ' are in abstract duality is called an abstract dual of Γ and is denoted by Γ^*.

Example 2.1.18. Let us consider the graphs

The cycles of Γ_1 are $C_1 := \{e_1, e_2, e_3\}$ and $C_2 := \{e_4, e_5\}$, while the bonds of Γ_2 are $B_1 := \{f_1, f_2, f_3\}$ and $B_2 := \{f_4, f_5\}$. The bijection $\phi : E(\Gamma_1) \to E(\Gamma_2)$ sending e_i to f_i for $i = 1, \ldots, 5$ sends the cycles of Γ_1 into the bonds of Γ_2; therefore Γ_1 and Γ_2 are in abstract duality.

Not every graph admits an abstract dual. Indeed we have the following theorem of Whitney (see [Di97, Theo. 4.6.3]).

Theorem 2.1.19. *A graph Γ has an abstract dual if and only if Γ is planar, i.e. if it can be embedded into the plane.*

It is easy to give examples of planar graphs Γ admitting non-isomorphic abstract duals (see [O92, Example 2.3.6]). However it follows easily from the definition that two abstract duals of the same graph are 2-isomorphic. Therefore, using the above Theorem 2.1.19, it follows that abstract duality induces a bijection

$$\begin{aligned} \{\text{Planar graphs}\}_{/\equiv_2} &\longleftrightarrow \{\text{Planar graphs}\}_{/\equiv_2} \\ [\Gamma]_2 &\longmapsto [\Gamma]_2^* := [\Gamma^*]_2. \end{aligned} \tag{2.3}$$

Moreover, it is easy to check that the duality satisfies:

$$\text{girth}([\Gamma]_2) = \lambda([\Gamma]_2^*) \qquad g^*([\Gamma]_2) = g([\Gamma]_2^*) \qquad k([\Gamma]_2) = k([\Gamma]_2^*). \tag{2.4}$$

2.1.3 Matroids

Here we recall the basic notions of (unoriented) matroid theory that we will need in the sequel. We follow mostly the terminology and notations of [O92].

2.1.20. Basic definitions

There are several ways of defining a matroid (see [O92, Chap. 1]). We will use the definition in terms of bases (see [O92, Sect. 1.2]).

Definition 2.1.21. A *matroid* M is a pair $(E(M), \mathcal{B}(M))$ where $E(M)$ is a finite set, called the *ground set*, and $\mathcal{B}(M)$ is a collection of subsets of $E(M)$, called *bases* of M, satisfying the following two conditions:

(i) $\mathcal{B}(M) \neq \emptyset$;

(ii) If $B_1, B_2 \in \mathcal{B}(M)$ and $x \in B_1 \setminus B_2$, then there exists an element $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}(M)$.

Given a matroid $M = (E(M), \mathcal{B}(M))$, we define:

(a) The set of independent elements

$$\mathcal{I}(M) := \{I \subset E(M) \,:\, I \subset B \text{ for some } B \in \mathcal{B}(M)\};$$

(b) The set of dependent elements

$$\mathcal{D}(M) := \{D \subset E(M) \,:\, E(M) \setminus D \in \mathcal{I}(M)\};$$

(c) The set of circuits

$$\mathcal{C}(M) := \{C \in \mathcal{D}(M) \,:\, C \text{ is minimal among the elements of } \mathcal{D}(M)\}.$$

It can be derived from the above axioms, that all the bases of M have the same cardinality, which is called the *rank* of M and is denoted by $r(M)$.

Observe that each of the above sets $\mathcal{B}(M), \mathcal{I}(M), \mathcal{D}(M), \mathcal{C}(M)$ determines all the others. Indeed, it is possible to define a matroid M in terms of the ground set $E(M)$ and each of the above sets, subject to suitable axioms (see [O92, Sec. 1.1, 1.2]).

The above terminology comes from the following basic example of matroids.

Example 2.1.22. Let F be a field and A an $r \times n$ matrix of rank r over F. Consider the columns of A as elements of the vector space F^r, and call them $\{v_1, \ldots, v_n\}$. The *vector matroid* of A, denoted by $M[A]$, is the matroid whose ground set is $E(M[A]) := \{v_1, \ldots, v_n\}$

and whose bases are the subsets of $E(M[A])$ consisting of vectors that form a base of F^r. It follows easily that $\mathcal{I}(M[A])$ is formed by the subsets of independent vectors of $E(M[A])$; $\mathcal{D}(M[A])$ is formed by the subsets of dependent vectors and $\mathcal{C}(M[A])$ is formed by the minimal subsets of dependent vectors.

We now introduce a very important class of matroids.

Definition 2.1.23. A matroid M is said to be *representable* over a field F, or simply F-representable, if it is isomorphic to the vector matroid of a matrix A with coefficients in F. A matroid M is said to be *regular* if it is representable over any field F.

Regular matroids are closely related to *totally unimodular matrices*, i.e. to real matrices for which every square submatrix has determinant equal to -1, 0 or 1. We say that two totally unimodular matrices $A, B \in M_{g,n}(\mathbb{R})$ are *equivalent* if $A = XBY$ where $X \in \mathrm{GL}_g(\mathbb{Z})$ and $Y \in \mathrm{GL}_n(\mathbb{Z})$ is a permutation matrix.

Theorem 2.1.24. *(i) A matroid M of rank r is regular if and only if $M = M[A]$ for a totally unimodular matrix $A \in M_{g,n}(\mathbb{R})$ of rank r, where $n = \#E(M)$ and g is a natural number such that $g \geq r$.*

(ii) Given two totally unimodular matrices $A, B \in M_{g,n}(\mathbb{R})$ of rank r, we have that $M[A] = M[B]$ if and only if A and B are equivalent.

Proof. Part (i) is proved in [O92, Thm. 6.3.3]. Part (ii) follows easily from [O92, Prop. 6.3.13, Cor. 10.1.4], taking into account that $\mathbb{R}$ does not have non-trivial automorphisms. □

In matroid theory, there is a natural duality theory (see [O92, Chap. 2]).

Definition 2.1.25. Given a matroid $M = (E(M), \mathcal{B}(M))$, the *dual matroid* $M^* = (E(M^*), \mathcal{B}(M^*))$ is defined by putting $E(M^*) = E(M)$ and

$$\mathcal{B}(M^*) = \{B^* \subset E(M^*) = E(M) \,:\, E(M) \setminus B^* \in \mathcal{B}(M)\}.$$

It turns out that the dual of an F-representable matroid is again F-representable (see [O92, Cor. 2.2.9]) and therefore that the dual of a regular matroid is again regular (see [O92, Prop. 2.2.22]).

Finally, we need to recall the concept of simple matroid (see [O92, Pag. 13, Pag. 52]).

Definition 2.1.26. Let M be a matroid. An element $e \in E(M)$ is called a *loop* if $\{e\} \in \mathcal{C}(M)$. Two distinct elements $f_1, f_2 \in E(M)$ are called *parallel* if $\{f_1, f_2\} \in \mathcal{C}(M)$; a parallel class of M is a maximal subset $X \subset E(M)$ with the property that all the elements of X are not loops and they are pairwise parallel.

M is called *simple* if it has no loops and all the parallel classes have cardinality one.

Given a matroid, there is a standard way to associate to it a simple matroid.

Definition 2.1.27. Let M be a matroid. The *simple matroid associated* to M, denoted by $\widetilde{M}$, is the matroid whose ground set is obtained by deleting all the loops of M and, for each parallel class X of M, deleting all but one distinguished element of X and whose set of bases is the natural one induced by M.

2.1.28. Graphic and Cographic matroids

Given a graph Γ, there are two natural ways of associating a matroid to it.

Definition 2.1.29. The *graphic matroid* (or *cycle matroid*) of Γ is the matroid $M(\Gamma)$ whose ground set is $E(\Gamma)$ and whose circuits are the cycles of Γ. The *cographic matroid* (or *bond matroid*) of Γ is the matroid $M^*(\Gamma)$ whose ground set is $E(\Gamma)$ and whose circuits are the bonds of Γ.

The rank of $M(\Gamma)$ is equal to $g^*(\Gamma)$ (see [O92, Pag. 26]), and the rank of $M^*(\Gamma)$ is equal to $g(\Gamma)$, as it follows easily from [O92, Formula 2.1.8].

It turns out that $M(\Gamma)$ and $M^*(\Gamma)$ are regular matroids (see [O92, Prop. 5.1.3, Prop. 2.2.22]) and that they are dual to each other (see [O92, Sec. 2.3]). Moreover we have the following obvious

Remark 2.1.30. Two graphs Γ_1 and Γ_2 are 2-isomorphic if and only if $M(\Gamma_1) = M(\Gamma_2)$ or, equivalently, if and only if $M^*(\Gamma_1) = M^*(\Gamma_2)$. Therefore, we can write $M([\Gamma]_2)$ and $M^*([\Gamma]_2)$ for a 2-isomorphism class $[\Gamma]_2$.

We have the following characterization of abstract dual graphs in terms of matroid duality (see [O92, Sec. 5.2]).

Proposition 2.1.31. *Let Γ and Γ^* be two graphs. The following conditions are equivalent:*

(i) Γ *and* Γ^* *are in abstract duality;*

(ii) $M(\Gamma) = M^*(\Gamma^*)$*;*

(iii) $M^*(\Gamma) = M(\Gamma^*)$.

By combining Proposition 2.1.31 with Remark 2.1.30, we get the following

Remark 2.1.32. There is a bijection between the following sets

$$\{\text{Graphic and cographic matroids}\} \longleftrightarrow \{\text{Planar graphs}\}_{/\equiv_2}.$$

Moreover this bijection is compatible with the respective duality theories, namely the duality theory for matroids (definition 2.1.25) and the abstract duality theory for graphs (definition 2.1.17).

Finally, we want to describe the simple matroid associated to a graphic or to a cographic matroid, in terms of the simplification 2.1.14 and of the 3-edge-connectivization 2.1.15.

Proposition 2.1.33. *Let Γ be a graph. We have that* *(i)* $\widetilde{M(\Gamma)} = M(\Gamma^{\text{sim}})$.

(ii) $\widetilde{M^*(\Gamma)} = M^*(\Gamma^3)$*, for any* 3*-edge-connectivization* Γ^3 *of* Γ.

Proof. The first assertion is well-known (see [O92, Pag. 52]).

The second assertion follows from the fact that an edge $e \in E(\Gamma)$ is a loop of $M^*(\Gamma)$ if and only if e is a bond of Γ, i.e. if e is a separating edge of Γ; and that a pair f_1, f_2 of edges is parallel in $M^*(\Gamma)$ if and only $\{f_1, f_2\}$ is a bond of Γ, i.e. if it is a pair of separating edges of Γ. □

2.2 The moduli space M_g^{tr}

2.2.1 Tropical curves

In order to define tropical curves, we start with the following

Definition 2.2.1. A *marked graph* is a couple (Γ, w) consisting of a finite connected graph Γ and a function $w : V(\Gamma) \to \mathbb{N}_{\geq 0}$, called the weight function. A marked graph is called *stable* if any vertex v of weight zero (i.e. such that $w(v) = 0$) has valence $\text{val}(v) \geq 3$. The total weight of a marked graph (Γ, w) is

$$|w| := \sum_{v \in V(\Gamma)} w(v),$$

and the genus of (Γ, w) is equal to

$$g(\Gamma, w) := g(\Gamma) + |w|.$$

We will denote by $\underline{0}$ the identically zero weight function.

Remark 2.2.2. It is easy to see that there is a finite number of stable marked graphs of a given genus g.

Definition 2.2.3. A *tropical curve* C is the datum of a triple (Γ, w, l) consisting of a stable marked graph (Γ, w), called the combinatorial type of C, and a function $l : E(\Gamma) \to \mathbb{R}_{>0}$, called the length function. The genus of C is the genus of its combinatorial type.

See 2.4.4 for an example of a tropical curve.

Remark 2.2.4. The above definition generalizes the definition of (equivalence class of) tropical curves given by Mikhalkin-Zharkov in [MZ07, Prop. 3.6]. More precisely, tropical curves with total weight zero in our sense are the same as compact tropical curves up to tropical modifications in the sense of Mikhalkin-Zharkov.

A *specialization* of a tropical curve is obtained by letting some of its edge lengths go to 0, i.e. by contracting some of its edges (see [Mi6, Sec.3.1.D]). The weight function of the specialized curve changes according to the following rule: if we contract a loop e around a vertex v then we increase the weight of v by one; if we contract an edge e between two distinct vertices v_1 and v_2 then we obtain a new vertex with weight equal to $w(v_1) + w(v_2)$. We write $C \rightsquigarrow C'$ to denote that C specializes to C'; if (Γ, w) (resp. (Γ', w')) are the combinatorial types of C (resp. C'), we write as well $(\Gamma, w) \rightsquigarrow (\Gamma', w')$. Note that a specialization preserves the genus of the tropical curves.

2.2.2 Construction of M_g^{tr}

Given a marked graph (Γ, w), its automorphism group $\text{Aut}(\Gamma, w)$ is the subgroup of $S_{|E(\Gamma)|} \times S_{|V(\Gamma)|}$ consisting of all pairs of permutations (ϕ, ψ) such that $w(\psi(v)) = w(v)$ for any $v \in V(\Gamma)$ and, for a fixed orientation of Γ, we have that $\{s(\phi(e)), t(\phi(e))\} = \{\psi(s(e)), \psi(t(e))\}$ for any $e \in E(\Gamma)$, where $s, t : E(\Gamma) \to V(\Gamma)$ are the source and target maps corresponding to the chosen orientation. Note that this definition is independent of the orientation. There is a natural homomorphism

$$\rho_{(\Gamma,w)} : \text{Aut}(\Gamma, w) \to S_{|E(\Gamma)|} \subset GL_{|E(\Gamma)|}(\mathbb{Z})$$

induced by the projection of $\text{Aut}(\Gamma, w) \subset S_{|E(\Gamma)|} \times S_{|V(\Gamma)|}$ onto the second factor followed by the inclusion of $S_{|E(\Gamma)|}$ into $GL_{|E(\Gamma)|}(\mathbb{Z})$ as the subgroup of the permutation matrices.

The group $\text{Aut}(\Gamma, w)$ acts on $\mathbb{R}^{|E(\Gamma)|}$ via the homomorphism $\rho_{(\Gamma,w)}$ preserving the open rational polyhedral cone $\mathbb{R}_{>0}^{|E(\Gamma)|}$ and its closure $\mathbb{R}_{\geq 0}^{|E(\Gamma)|}$. We denote the respective quotients by

$$C(\Gamma, w) := \mathbb{R}_{>0}^{|E(\Gamma)|} / \text{Aut}(\Gamma, w) \quad \text{and} \quad \overline{C(\Gamma, w)} := \mathbb{R}_{\geq 0}^{|E(\Gamma)|} / \text{Aut}(\Gamma, w)$$

endowed with the quotient topology. When Γ is such that $E(\Gamma) = \emptyset$ and $V(\Gamma)$ is just one vertex of weight g, we set $C(\Gamma, w) := \{0\}$. Note that $C(\Gamma, w)$ parametrizes tropical curves of combinatorial type equal to (Γ, w).

Observe that, for any specialization $i : (\Gamma, w) \rightsquigarrow (\Gamma', w')$, we get a natural continuous map

$$\bar{i} : \mathbb{R}_{\geq 0}^{|E(\Gamma')|} \hookrightarrow \mathbb{R}_{\geq 0}^{|E(\Gamma)|} \twoheadrightarrow \overline{C(\Gamma, w)},$$

where $\overline{C(\Gamma, w)}$ is endowed with the quotient topology. Note that, if i is a nontrivial specialization, the image of the map $\bar{i}$ is contained in $\overline{C(\Gamma, w)} \setminus C(\Gamma, w)$, so it does not meet the locus of $\overline{C(\Gamma, w)}$ parametrizing tropical curves of combinatorial type (Γ, w).

We are now ready to define the moduli space of tropical curves of fixed genus.

Definition 2.2.5. We define M_g^{tr} as the topological space (with respect to the quotient topology)

$$M_g^{\text{tr}} := \left(\coprod \overline{C(\Gamma, w)} \right)_{/\sim}$$

where the disjoint union (endowed with the disjoint union topology) runs through all stable marked graphs (Γ, w) of genus g and $\sim$ is the equivalence relation generated by the following binary relation $\approx$: given two points $p_1 \in \overline{C(\Gamma_1, w_1)}$ and $p_2 \in \overline{C(\Gamma_2, w_2)}$, $p_1 \approx p_2$ iff there exists a stable marked graph (Γ, w) of genus g, a point $q \in \mathbb{R}_{\geq 0}^{|E(\Gamma)|}$ and two specializations $i_1 : (\Gamma_1, w_1) \rightsquigarrow (\Gamma, w)$ and $i_2 : (\Gamma_2, w_2) \rightsquigarrow (\Gamma, w)$ such that $\bar{i}_1(q) = p_1$ and $\bar{i}_2(q) = p_2$.

From the definition of the above equivalence relation $\sim$, we get the following

Remark 2.2.6.

(i) Let $p_1, p_2 \in \coprod \overline{C(\Gamma, w)}$ such that $p_1 \sim p_2$. If there exist two stable marked graphs (Γ_1, w_1) and (Γ_2, w_2) such that $p_1 \in C(\Gamma_1, w_1)$ and $p_2 \in C(\Gamma_2, w_2)$, then $(\Gamma_1, w_1) = (\Gamma_2, w_2)$ and $p_1 = p_2$.

(ii) Let $p \in \coprod \overline{C(\Gamma, w)}$. Then there exists a stable marked graph (Γ', w') and $p' \in C(\Gamma', w')$ such that $p \sim p'$.

Example 2.2.7. In the following figure we represent all stable marked graphs corresponding to tropical curves of genus 2. The arrows represent all possible specializations.

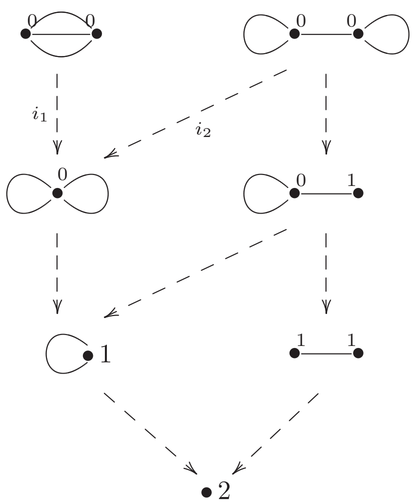

Figure 2.1: Specializations of tropical curves of genus 2.

The cells corresponding to the two graphs on the top of Figure 2.1 are $\mathbb{R}^3_{\geq 0}/S_3$ and $\mathbb{R}^3_{\geq 0}/S_2$, respectively. According to Definition 2.2.5, M_2^{tr} corresponds to the topological space obtained by gluing $\mathbb{R}^3_{\geq 0}/S_3$ and $\mathbb{R}^3_{\geq 0}/S_2$ along the points of $(\mathbb{R}^3_{\geq 0}/S_3) \setminus (\mathbb{R}^3_{>0}/S_3)$ and of $(\mathbb{R}^3_{\geq 0}/S_2) \setminus (\mathbb{R}^3_{>0}/S_3)$ that correspond to common specializations of those graphs according to the above diagram. For instance, the specializations i_1 and i_2 induce the maps

$$\begin{array}{rccc} \bar{i}_1 : & \mathbb{R}^2_{\geq 0} & \to & \mathbb{R}^3_{\geq 0}/S_3 \\ & (a_1, a_2) & \mapsto & [(a_1, a_2, 0)] \end{array} \quad \text{and} \quad \begin{array}{rccc} \bar{i}_2 : & \mathbb{R}^2_{\geq 0} & \to & \mathbb{R}^3_{\geq 0}/S_2, \\ & (a_1, a_2) & \mapsto & [(a_1, 0, a_2)] \end{array}$$

where in $\mathbb{R}^3_{\geq 0}/S_2$ the second coordinate corresponds to the edge of the graph connecting the two vertices. So, a point $[(x_1, x_2, x_3)] \in \mathbb{R}^3_{\geq 0}/S_3$ will be identified with a point $[(y_1, y_2, y_3)] \in \mathbb{R}^3_{\geq 0}/S_2$ via the maps $\bar{i}_1$ and $\bar{i}_2$ if $y_2 = 0$ and if there exists $\sigma \in S_3$ such that $(y_1, y_3) = (x_{\sigma(1)}, x_{\sigma(2)})$ and $x_{\sigma(3)} = 0$.

Theorem 2.2.8. *The topological space M_g^{tr} is a stacky fan with cells $C(\Gamma, w)$, as (Γ, w) varies through all stable marked graphs of genus g. In particular, its points are in bijection with tropical curves of genus g.*

Proof. Let us prove the first statement, by checking the conditions of Definition 2.1.1. Consider the maps $\alpha_{(\Gamma,w)} : \overline{C(\Gamma, w)} \to M_g^{\mathrm{tr}}$ naturally induced by $\overline{C(\Gamma, w)} \hookrightarrow \coprod \overline{C(\Gamma', w')} \twoheadrightarrow M_g^{\mathrm{tr}}$. The maps $\alpha_{(\Gamma,w)}$ are continuous by definition of the quotient topology and the restriction of $\alpha_{(\Gamma,w)}$ to $C(\Gamma, w)$ is a bijection onto its image by Remark 2.2.6(i). Moreover, given an open subset $U \subseteq C(\Gamma, w)$, $\alpha_{(\Gamma,w)}(U)$ is an open subset of M_g^{tr} since its inverse image on $\coprod \overline{C(\Gamma', w')}$ is equal to U. This proves that the maps $\alpha_{(\Gamma,w)}$ when restricted to $C(\Gamma, w)$ are homeomorphisms onto their images, and condition 2.1.1(i) is satisfied.

From Remark 2.2.6(ii), we get that

$$M_g^{\mathrm{tr}} = \bigcup_{(\Gamma,w)} \alpha_{(\Gamma,w)}(C(\Gamma, w)) \tag{2.5}$$

and the union is disjoint by Remark 2.2.6(i); thus condition 2.1.1(ii) is satisfied.

Let us check the condition 2.1.1(iii). Let (Γ, w) and (Γ', w') be two stable marked graphs of genus g and set $\alpha := \alpha_{(\Gamma,w)}$ and $\alpha' := \alpha_{(\Gamma',w')}$. By definition of M_g^{tr}, the intersection of the images of $\overline{C(\Gamma, w)}$ and $\overline{C(\Gamma', w')}$ in M_g^{tr} is equal to

$$\alpha(\overline{C(\Gamma, w)}) \cap \alpha'(\overline{C(\Gamma', w')}) = \coprod_i \alpha_i(C(\Gamma_i, w_i)),$$

where (Γ_i, w_i) runs over all common specializations of (Γ, w) and (Γ', w'). We have to find an integral linear map $L : \mathbb{R}^{|E(\Gamma)|} \to \mathbb{R}^{|E(\Gamma')|}$ making the following diagram commutative

$$\begin{array}{ccccccc}
\coprod_i \alpha_i(C(\Gamma_i, w_i)) & \hookrightarrow & \alpha(\overline{C(\Gamma, w)}) & \twoheadleftarrow & \mathbb{R}_{\geq 0}^{|E(\Gamma|} & \hookrightarrow & \mathbb{R}^{|E(\Gamma)|} \\
 & \searrow & & & \downarrow L & & \downarrow L \\
 & & \alpha'(\overline{C(\Gamma', w')}) & \twoheadleftarrow & \mathbb{R}_{\geq 0}^{|E(\Gamma')|} & \hookrightarrow & \mathbb{R}^{|E(\Gamma')|}.
\end{array} \tag{2.6}$$

To this aim, observe that, since (Γ_i, w_i) are specializations of both (Γ, w) and (Γ', w'), there are orthogonal projections $f_i : \mathbb{R}^{|E(\Gamma)|} \twoheadrightarrow \mathbb{R}^{|E(\Gamma_i)|}$ and inclusions $g_i : \mathbb{R}^{|E(\Gamma_i)|} \hookrightarrow \mathbb{R}^{|E(\Gamma')|}$. We define L as the composition

$$L : \mathbb{R}^{|E(\Gamma)|} \xrightarrow{\oplus f_i} \oplus_i \mathbb{R}^{|(E(\Gamma_i)|} \xrightarrow{\oplus g_i} \mathbb{R}^{|E(\Gamma)|}.$$

It is easy to see that L is an integral linear map making the above diagram (2.6) commutative, and this concludes the proof of the first statement.

The second statement follows from (2.5) and the fact, already observed before, that $C(\Gamma, w)$ parametrizes tropical curves of combinatorial type (Γ, w). ☐

We now prove that M_g^{tr} is of pure dimension and connected through codimension one. To that aim, we describe the maximal cells and the codimension one cells of M_g^{tr}.

Proposition 2.2.9.

(i) The maximal cells of M_g^{tr} are exactly those of the form $C(\Gamma, \underline{0})$ where Γ is 3-regular. In particular, M_g^{tr} is of pure dimension $3g-3$.

(ii) M_g^{tr} is connected through codimension one.

(iii) The codimension one cells of M_g^{tr} are of the following two types:

(a) $C(\Gamma, \underline{0})$ where Γ has exactly one vertex of valence 4 and all other vertices of valence 3;

(b) $C(\Gamma, w)$ where Γ has exactly one vertex v of valence 1 and weight 1, and all the other vertices of valence 3 and weight 0.

Each codimension one cell of type (b) lies in the closure of exactly one maximal cell, while each codimension one cell of type (a) lies in the closure of one, two or three maximal cells.

Proof. First of all, observe that given a stable marked graph (Γ, w) of genus g we have

$$3|V(\Gamma)| \leq \sum_{v \in V(\Gamma)} [\mathrm{val}(v) + 2w(v)] = 2|E(\Gamma)| + 2|w|, \tag{2.7}$$

and the equality holds if and only if every $v \in V(\Gamma)$ is such that either $w(v) = 0$ and $\mathrm{val}(v) = 3$ or $w(v) = \mathrm{val}(v) = 1$. By substituting the formula for the genus $g = g(\Gamma, w) = g(\Gamma) + |w| = 1 + |E(\Gamma)| - |V(\Gamma)| + |w|$ in inequality (2.7), we obtain

$$|E(\Gamma)| \leq 3g - 3 - |w|. \tag{2.8}$$

Let us now prove part (i). If Γ is 3-regular and $w \equiv 0$, then $g(\Gamma) = g(\Gamma, w) = g$ and an easy calculation gives that $|E(\Gamma)| = 3g-3$. Therefore $\dim(C(\Gamma, \underline{0})) = 3g-3$, which is the maximal possible dimension of the cells of M_g^{tr} according to the above inequality (2.8). Hence $C(\Gamma, \underline{0})$ is maximal. On the other hand, every stable marked graph (Γ', w') can be obtained by specializing a stable marked graph $(\Gamma, \underline{0})$ with Γ a 3-regular graph (see for example [CV1, Appendix A.2]), which concludes the proof of part (i).

Let us prove part (ii). It is well-known (see the appendix of [HT80] for a topological proof, [Ts96, Thm. II] for a combinatorial proof in the case of simple graphs and [C8, Thm 3.3] for a combinatorial proof in the general case) that any two 3-regular graphs Γ_1 and Γ_2 of genus g can be obtained one from the other via a sequence of twisting operations as the one shown in the top line of Figure 2.2 below. In each of these twisting operations, the two graphs Γ_1 and Γ_2 specialize to a common graph Γ (see Figure 2.2) that has one vertex of valence 4 and all the others of valence 3. By what will be proved below, $C(\Gamma, \underline{0})$ is a codimension one cell. Therefore the two maximal dimensional cells $C(\Gamma_1, \underline{0})$ and $C(\Gamma_2, \underline{0})$ contain a common codimension one cell $C(\Gamma, \underline{0})$ in their closures, which concludes the proof of part (ii).

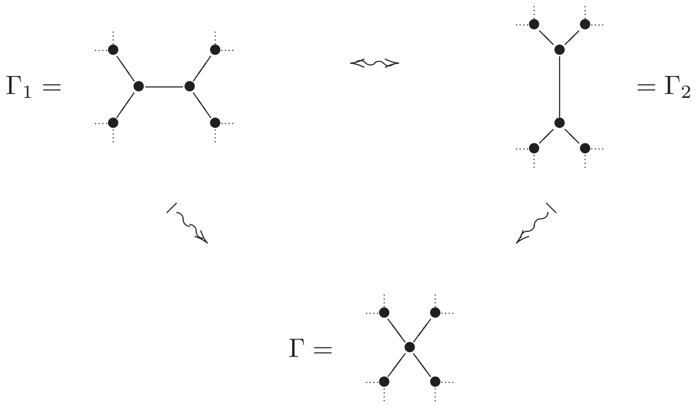

Figure 2.2: The 3-regular graphs Γ_1 and Γ_2 are twisted. They both specialize to Γ. $C(\Gamma_1, \underline{0})$ and $C(\Gamma_2, \underline{0})$ are maximal dimensional cells containing the codimension one cell $C(\Gamma, \underline{0})$ in their closures.

Let us prove part (iii). Let $C(\Gamma, w)$ be a codimension one cell of $M_g^{\rm tr}$, i.e. such that $|E(\Gamma)| = 3g-4$. According to the inequality (2.8), there are two possibilities: either $|w| = 0$ or $|w| = 1$. In the first case, i.e. $|w| = 0$, using the inequality in (2.7), it is easy to check that there should exist exactly one vertex v such that $\mathrm{val}(v) = 4$ and all the other vertices should have valence equal to 3, i.e. we are in case (a). In the second case, i.e. $|w| = 1$, all the inequalities in (2.7) should be equalities and this implies that there should be exactly one vertex v such that $\mathrm{val}(v) = w(v) = 1$ and all the other vertices have weight equal to zero and valence equal to 3, i.e. we are in case (b).

For a codimension one cell of type (a), $C(\Gamma, \underline{0})$, there can be at most three maximal cells $C(\Gamma_i, \underline{0})$ ($i = 1, 2, 3$) containing it in their closures, as we can see in Figure 2.3. Note, however, that it can happen that some of the Γ_i's are isomorphic, and in that case the number of maximal cells containing $C(\Gamma, \underline{0})$ in their closure is strictly smaller than 3.

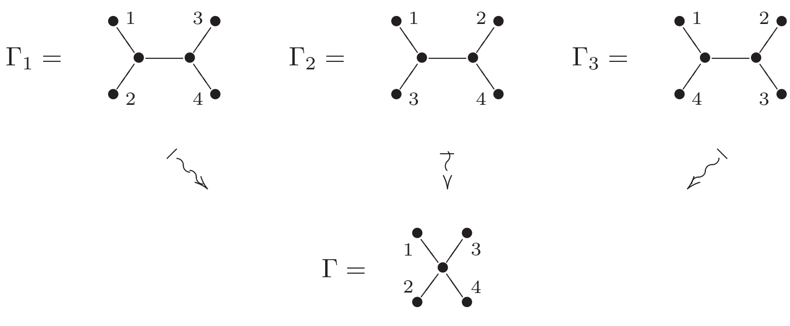

Figure 2.3: The codimension one cell $C(\Gamma, \underline{0})$ is contained in the closure of the three maximal cells $C(\Gamma_i, \underline{0})$, $i = 1, 2, 3$.

For a codimension one cell $C(\Gamma, w)$ of type (b), there is only one maximal cell $C(\Gamma', \underline{0})$

containing it in its closure, as we can see in Figure 2.4 below. □

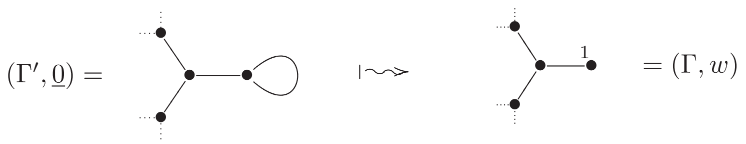

Figure 2.4: The codimension one cell $C(\Gamma, w)$ is contained in the closure of the maximal dimensional cell $C(\Gamma', \underline{0})$.

2.3 The moduli space A_g^{tr}

2.3.1 Tropical abelian varieties

Definition 2.3.1. A principally polarized *tropical abelian variety* A of dimension g is a g-dimensional real torus $\mathbb{R}^g/\Lambda$, where Λ is a lattice of rank g in $\mathbb{R}^g$ endowed with a flat semi-metric induced by a positive semi-definite quadratic form Q on $\mathbb{R}^g$ such that the null space $\mathrm{Null}(Q)$ of Q is defined over $\Lambda \otimes \mathbb{Q}$, i.e. it admits a basis with elements in $\Lambda \otimes \mathbb{Q}$. Two tropical abelian varieties $(\mathbb{R}^g/\Lambda, Q)$ and $(\mathbb{R}^g/\Lambda', Q')$ are isomorphic if there exists $h \in GL(g, \mathbb{R})$ such that $h(\Lambda) = \Lambda'$ and $hQh^t = Q'$.

From now on, we will drop the attribute principally polarized as all the tropical abelian varieties that we will consider are of this kind.

Remark 2.3.2. The above definition generalizes the definition of tropical abelian variety given by Mikhalkin-Zharkov in [MZ07, Sec. 5]. More precisely, tropical abelian varieties endowed with positive definite quadratic forms in our sense are the same as (principally polarized) tropical abelian varieties in the sense of Mikhalkin-Zharkov.

Remark 2.3.3. Every tropical abelian variety $(\mathbb{R}^g/\Lambda, Q)$ can be written in the form $(\mathbb{R}^g/\mathbb{Z}^g, Q')$. In fact, it is enough to consider $Q' = hQh^t$, where $h \in GL(g, \mathbb{R})$ is such that $h(\Lambda) = \mathbb{Z}^g$. Moreover, $(\mathbb{R}^g/\mathbb{Z}^g, Q) \cong (\mathbb{R}^g/\mathbb{Z}^g, Q')$ if and only if there exists $h \in \mathrm{GL}_g(\mathbb{Z})$ such that $Q' = hQh^t$, i.e., if and only if Q and Q' are arithmetically equivalent. Therefore, from now on we will always consider our tropical abelian varieties in the form $(\mathbb{R}^g/\mathbb{Z}^g, Q)$, where Q is uniquely defined up to arithmetic equivalence.

2.3.2 Definition of A_g^{tr} and $A_g^{\mathrm{tr},\Sigma}$

Let us denote by $\mathbb{R}^{\binom{g+1}{2}}$ the vector space of quadratic forms in $\mathbb{R}^g$ (identified with $g \times g$ symmetric matrices with coefficients in $\mathbb{R}$), by Ω_g the cone in $\mathbb{R}^{\binom{g+1}{2}}$ of positive definite quadratic forms and by Ω_g^{rt} the cone of positive semi-definite quadratic forms with rational null space (the so-called rational closure of Ω_g, see [N80, Sec. 8]).

The group $\mathrm{GL}_g(\mathbb{Z})$ acts on $\mathbb{R}^{\binom{g+1}{2}}$ via the usual law $h\cdot Q := hQh^t$, where $h\in \mathrm{GL}_g(\mathbb{Z})$ and Q is a quadratic form on $\mathbb{R}^g$. This action naturally defines a homomorphism $\rho : \mathrm{GL}_g(\mathbb{Z}) \to \mathrm{GL}_{\binom{g+1}{2}}(\mathbb{Z})$. Note that the cones Ω_g or Ω_g^{rt} are preserved by the action of $\mathrm{GL}_g(\mathbb{Z})$.

Remark 2.3.4. It is well-known (see [N80, Sec. 8]) that a positive semi-definite quadratic form Q in $\mathbb{R}^g$ belongs to Ω_g^{rt} if and only if there exists $h\in \mathrm{GL}_g(\mathbb{Z})$ such that

$$hQh^t = \begin{pmatrix} Q' & 0 \\ 0 & 0 \end{pmatrix}$$

for some positive definite quadratic form Q' in $\mathbb{R}^{g'}$, with $0\le g'\le g$.

Definition 2.3.5. We define A_g^{tr} as the topological space (with respect to the quotient topology)

$$A_g^{\mathrm{tr}} := \Omega_g^{\mathrm{rt}} / \mathrm{GL}_g(\mathbb{Z}).$$

The space A_g^{tr} parametrizes tropical abelian varieties as it follows from Remark 2.3.3. However, in order to endow A_g^{tr} with the structure of stacky fan, we need to specify some extra-data, encoded in the following definition (see [N80, Lemma 8.3] or [FC90, Chap. IV.2]).

Definition 2.3.6. A $\mathrm{GL}_g(\mathbb{Z})$*-admissible decomposition* of Ω_g^{rt} is a collection $\Sigma = \{\sigma_\mu\}$ of rational polyhedral cones of Ω_g^{rt} such that:

1. If σ is a face of $\sigma_\mu\in\Sigma$ then $\sigma\in\Sigma$;

2. The intersection of two cones σ_μ and σ_ν of Σ is a face of both cones;

3. If $\sigma_\mu\in\Sigma$ and $h\in \mathrm{GL}_g(\mathbb{Z})$ then $h\cdot\sigma_\mu\cdot h^t\in\Sigma$.

4. $\#\{\sigma_\mu\in\Sigma \mod \mathrm{GL}_g(\mathbb{Z})\}$ is finite;

5. $\cup_{\sigma_\mu\in\Sigma}\sigma_\mu = \Omega_g^{\mathrm{rt}}$.

Each $\mathrm{GL}_g(\mathbb{Z})$-admissible decomposition of Ω_g^{rt} gives rise to a structure of stacky fan on A_g^{tr}. In order to prove that, we need first to set some notations.

Let $\Sigma = \{\sigma_\mu\}$ be a $\mathrm{GL}_g(\mathbb{Z})$-admissible decomposition of Ω_g^{rt}. For each $\sigma_\mu\in\Sigma$ we set $\sigma_\mu^0 := \mathrm{Int}(\sigma_\mu)$; we denote by $\langle\sigma_\mu\rangle$ the smallest linear subspace of $\mathbb{R}^{\binom{g+1}{2}}$ containing σ_μ and we set $m_\mu := \dim_{\mathbb{R}}\langle\sigma_\mu\rangle$. Consider the stabilizer of σ_μ^0 inside $\mathrm{GL}_g(\mathbb{Z})$

$$\mathrm{Stab}(\sigma_\mu^0) := \{h\in \mathrm{GL}_g(\mathbb{Z}) \;:\; \rho(h)\cdot\sigma_\mu^0 = h\cdot\sigma_\mu^0\cdot h^t = \sigma_\mu^0\}.$$

The restriction of the homomorphism ρ to $\mathrm{Stab}(\sigma_\mu^0)$ defines a homomorphism

$$\rho_\mu : \mathrm{Stab}(\sigma_\mu^0) \to \mathrm{GL}(\langle\sigma_\mu\rangle, \mathbb{Z}) = \mathrm{GL}_{m_\mu}(\mathbb{Z}).$$

By definition, the image $\rho_\mu(\mathrm{Stab}(\sigma_\mu^0))$ acts on $\langle\sigma_\mu\rangle = \mathbb{R}^{m_\mu}$ and stabilizes the cone σ_μ^0, defining an action of $\mathrm{Stab}(\sigma_\mu^0)$ on σ_μ^0. Note that $\mathrm{GL}_g(\mathbb{Z})$ naturally acts on the set of quotients $\{\sigma_\mu^0/\,\mathrm{Stab}(\sigma_\mu^0)\}$; we will denote by $\{[\sigma_\mu^0/\,\mathrm{Stab}(\sigma_\mu^0)]\}$ the (finite) orbits of this action.

Theorem 2.3.7. *Let Σ be a $\mathrm{GL}_g(\mathbb{Z})$-admissible decomposition of Ω_g^{rt}. The topological space A_g^{tr} can be endowed with the structure of a stacky fan with cells $[\sigma_\mu^0/\,\mathrm{Stab}(\sigma_\mu^0)]$, which we denote by $A_g^{\mathrm{tr},\Sigma}$.*

Proof. Fix a set $\mathcal{S} = \{\sigma_\mu^0/\,\mathrm{Stab}(\sigma_\mu^0)\}$ of representatives for the orbits $[\sigma_\mu^0/\,\mathrm{Stab}(\sigma_\mu^0)]$. For each element $\sigma_\mu^0/\,\mathrm{Stab}(\sigma_\mu^0) \in \mathcal{S}$, consider the continuous map

$$\alpha_\mu : \frac{\sigma_\mu}{\mathrm{Stab}(\sigma_\mu^0)} \to A_g^{\mathrm{tr}},$$

induced by the inclusion $\sigma_\mu \hookrightarrow \Omega_g^{\mathrm{rt}}$. By the definition of A_g^{tr} it is clear that α_μ sends $\sigma_\mu^0/\,\mathrm{Stab}(\sigma_\mu^0)$ homeomorphically onto its image and also that

$$\bigcup \alpha_\mu\left(\frac{\sigma_\mu^0}{\mathrm{Stab}(\sigma_\mu^0)}\right) = A_g^{\mathrm{tr}},$$

where the union runs through all the elements of $\mathcal{S}$. Therefore the first two conditions of definition 2.1.1 are satisfied. Let us check the condition 2.1.1(iii). Consider two elements $\{\sigma_{\mu_1}^0/\,\mathrm{Stab}(\sigma_{\mu_1}^0)\}$ and $\{\sigma_{\mu_2}^0/\,\mathrm{Stab}(\sigma_{\mu_2}^0)\}$ of $\mathcal{S}$. Clearly, the intersection of the images of $\sigma_{\mu_1}/\,\mathrm{Stab}(\sigma_{\mu_1}^0)$ and $\sigma_{\mu_2}/\,\mathrm{Stab}(\sigma_{\mu_2}^0)$ in A_g^{tr} can be written in the form

$$\alpha_{\mu_1}\left(\frac{\sigma_{\mu_1}}{\mathrm{Stab}(\sigma_{\mu_1}^0)}\right) \cap \alpha_{\mu_2}\left(\frac{\sigma_{\mu_2}}{\mathrm{Stab}(\sigma_{\mu_2}^0)}\right) = \coprod_i \alpha_{\nu_i}\left(\frac{\sigma_{\nu_i}^0}{\mathrm{Stab}(\sigma_{\nu_i}^0)}\right),$$

where $\sigma_{\nu_i}^0/\,\mathrm{Stab}(\sigma_{\nu_i}^0)$ are the elements of $\mathcal{S}$ such that there exist elements $h_{i1}, h_{i2} \in \mathrm{GL}_g(\mathbb{Z})$ such that $h_{i1}\sigma_{\nu_i}h_{i1}^t$ is a face of the cone σ_{μ_1} and $h_{i2}\sigma_{\nu_i}h_{i2}^t$ is a face of the cone σ_{μ_2}. Note that the above elements h_{i1} and h_{i2} are not unique, but we will fix a choice for them in what follows. We have to find an integral linear map $L : \langle\sigma_{\mu_1}\rangle = \mathbb{R}^{m_{\mu_1}} \to \langle\sigma_{\mu_2}\rangle = \mathbb{R}^{m_{\mu_2}}$ making the following diagram commutative

$$\begin{array}{ccccccc}
\coprod_i \alpha_{\nu_i}\left(\frac{\sigma_{\nu_i}^0}{\mathrm{Stab}(\sigma_{\nu_i}^0)}\right) & \hookrightarrow & \alpha_{\mu_1}\left(\frac{\sigma_{\mu_1}}{\mathrm{Stab}(\sigma_{\mu_1}^0)}\right) & \twoheadleftarrow & \sigma_{\mu_1} & \hookrightarrow & \langle\sigma_{\mu_1}\rangle = \mathbb{R}^{m_{\mu_1}} \\
 & \searrow & & & \downarrow L & & \downarrow L \\
 & & \alpha_{\mu_2}\left(\frac{\sigma_{\mu_2}}{\mathrm{Stab}(\sigma_{\mu_2}^0)}\right) & \twoheadleftarrow & \sigma_{\mu_2} & \hookrightarrow & \langle\sigma_{\mu_2}\rangle = \mathbb{R}^{m_{\mu_2}}.
\end{array} \tag{2.9}$$

Consider the integral linear maps

$$\begin{cases} \pi_i : \langle\sigma_{\mu_1}\rangle = \mathbb{R}^{m_{\mu_1}} \stackrel{\widetilde{\pi_i}}{\twoheadrightarrow} \langle\rho(h_{i1})(\sigma_{\nu_i})\rangle \stackrel{\rho(h_{i1}^{-1})}{\longrightarrow} \langle\sigma_{\nu_i}\rangle := \mathbb{R}^{m_{\nu_i}}, \\ \gamma_i : \langle\sigma_{\nu_i}\rangle = \mathbb{R}^{m_{\nu_i}} \stackrel{\rho(h_{i2})}{\longrightarrow} \langle\rho(h_{i2})(\sigma_{\nu_i})\rangle \stackrel{\widetilde{\gamma_i}}{\hookrightarrow} \langle\sigma_{\mu_2}\rangle = \mathbb{R}^{m_{\mu_2}}, \end{cases}$$

where $\widetilde{\pi_i}$ is the orthogonal projection of $\langle\sigma_{\mu_1}\rangle$ onto its subspace $\langle\rho(h_{i1})(\sigma_{\nu_i})\rangle$ and $\widetilde{\gamma_i}$ is the natural inclusion of $\langle\rho(h_{i2})(\sigma_{\nu_i})\rangle$ onto $\langle\sigma_{\mu_2}\rangle$. We define the following integral linear map

$$L : \mathbb{R}^{m_{\mu_1}} \stackrel{\oplus_i\pi_i}{\longrightarrow} \bigoplus \mathbb{R}^{m_{\nu_i}} \stackrel{\oplus_i\gamma_i}{\longrightarrow} \mathbb{R}^{m_{\mu_2}}.$$

It is easy to see that L is an integral linear map making the above diagram (2.9) commutative, and this concludes the proof. □

2.3.3 Voronoi decomposition: $A_g^{\mathrm{tr,V}}$

Some $\mathrm{GL}_g(\mathbb{Z})$-admissible decompositions of Ω_g^{rt} have been studied in detail in the reduction theory of positive definite quadratic forms (see [N80, Chap. 8] and the references there), most notably:

(i) The perfect cone decomposition (also known as the first Voronoi decomposition);

(ii) The central cone decomposition;

(iii) The Voronoi decomposition (also known as the second Voronoi decomposition or the L-type decomposition).

Each of them plays a significant (and different) role in the theory of the toroidal compactifications of the moduli space of principally polarized abelian varieties (see [I67], [A02], [SB06]).

Example 2.3.8. In Figure 2.5 we illustrate a section of the 3-dimensional cone Ω_2^{rt}, where we represent just some of the infinite Voronoi cones (which for $g = 2$ coincide with the perfect cones and with the central cones). For $g = 2$, there is only one $\mathrm{GL}_g(\mathbb{Z})$-equivalence class of maximal dimensional cones, namely the principal cone σ^0_{prin} (see section 2.5.1). Therefore, all the maximal cones in the picture will be identified in the quotient $A_g^{tr,V}$.

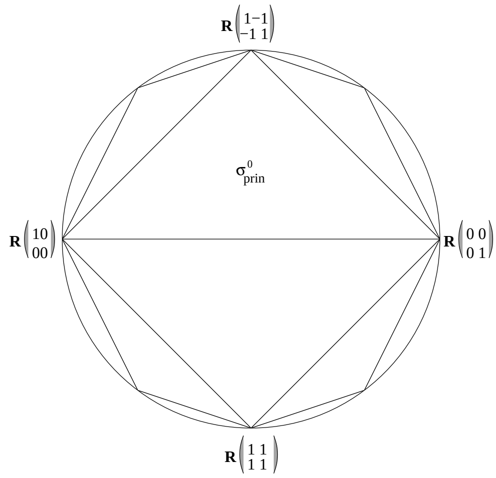

Figure 2.5: A section of Ω_2^{rt} and its Voronoi decomposition.

Let us focus our attention on the Voronoi decomposition, since it is the one that better fits in our setting. It is based on the so-called *Dirichlet-Voronoi polytope* $\mathrm{Vor}(Q) \subset \mathbb{R}^g$

associated to a positive semi-definite quadratic form $Q \in \Omega_g^{\rm rt}$. Recall (see for example [N80, Chap. 9] or [V03, Chap. 3]) that if $Q \in \Omega_g$, then $\mathrm{Vor}(Q)$ is defined as

$$\mathrm{Vor}(Q) := \{x \in \mathbb{R}^g \ : \ Q(x) \leq Q(v - x) \text{ for all } v \in \mathbb{Z}^g\}. \tag{2.10}$$

More generally, if $Q = h \begin{pmatrix} Q' & 0 \\ 0 & 0 \end{pmatrix} h^t$ for some $h \in \mathrm{GL}_g(\mathbb{Z})$ and some positive definite quadratic form Q' in $\mathbb{R}^{g'}$, $0 \leq g' \leq g$ (see Remark 2.3.4), then $\mathrm{Vor}(Q) := h^{-1}\mathrm{Vor}(Q')(h^{-1})^t \subset h^{-1}\mathbb{R}^{g'}(h^{-1})^t$. In particular, the smallest linear subspace containing $\mathrm{Vor}(Q)$ has dimension equal to the rank of Q.

Definition 2.3.9. The Voronoi decomposition $V = \{\sigma_P\}$ is the $\mathrm{GL}_g(\mathbb{Z})$-admissible decomposition of $\Omega_g^{\rm rt}$ whose open cones $\sigma_P^0 := \mathrm{Int}(\sigma_P)$ are parametrized by Dirichlet-Voronoi polytopes $P \subset \mathbb{R}^g$ in the following way

$$\sigma_P^0 := \{Q \in \Omega_g^{\rm rt} \ : \ \mathrm{Vor}(Q) = P\}.$$

Remark 2.3.10. The polytopes $P \subset \mathbb{R}^g$ that appear as Dirichlet-Voronoi polytopes of quadratic forms in Ω_g are of a very special type: they are *parallelohedra*, i.e. the set of translates of the form $v + P$ for $v \in \mathbb{Z}^g$ form a face-to-face tiling of $\mathbb{R}^g$ (see for example [McM80] or [V03, Chap. 3]). Indeed, it has been conjectured by Voronoi ([Vor08]) that all the parallelohedra are affinely isomorphic to Dirichlet-Voronoi polytopes (see [DG1] for an account on the state of the conjecture).

The natural action of $\mathrm{GL}_g(\mathbb{Z})$ on the cones σ_P^0 corresponds to the natural action of $\mathrm{GL}_g(\mathbb{Z})$ on the set of all Dirichlet-Voronoi polytopes $P \subset \mathbb{R}^g$. We denote by $[P]$ (resp. $[\sigma_P^0]$) the equivalence class of P (resp. σ_P^0) under this action. We set also $C([P]) := [\sigma_P^0/\mathrm{Stab}(\sigma_P^0)]$.

Definition 2.3.11. $A_g^{\rm tr,V}$ is the stacky fan associated to the Voronoi decomposition $V = \{\sigma_P\}$. Its cells are the $C([P])$'s as $[P]$ varies among the $\mathrm{GL}_g(\mathbb{Z})$-equivalence classes of Dirichlet-Voronoi polytopes in $\mathbb{R}^g$.

In order to describe the maximal cells and codimension one cells of $A_g^{\rm tr,V}$ (in analogy with Proposition 2.2.9), we need to introduce some definitions. A Dirichlet-Voronoi polytope $P \subset \mathbb{R}^g$ is said to be *primitive* if it is of dimension g and the associated face-to-face tiling of $\mathbb{R}^g$ (see Remark 2.3.10) is such that at each vertex of the tiling, the minimum number, namely $g + 1$, of translates of P meet (see [V03, Sec. 2.2]). A Dirichlet-Voronoi polytope $P \subset \mathbb{R}^g$ is said to be *almost primitive* if it is of dimension g and the associated face-to-face tiling of $\mathbb{R}^g$ (see Remark 2.3.10) is such that there is exactly one vertex, modulo translations by $\mathbb{Z}^g$, where $g + 2$ translates of P meet and at all the other vertices of the tiling only $g + 1$ translates of P meet.

The properties of the following Proposition are the translation in our language of well-known properties of the Voronoi decomposition (see the original [Vor08] or [V03] and the references there). Unfortunately, the results we need are often stated in terms of the Delaunay decomposition, which is the dual of the tiling of $\mathbb{R}^g$ by translates of the Dirichlet-Voronoi polytope (see for example [N80, Chap. 9] or [V03, Sec. 2.1]). So, in our proof we will assume that the reader is familiar with the Delaunay decomposition, limiting ourselves to translate the above properties in terms of the Delaunay decomposition and to explain how they follow from known results about the Voronoi decomposition.

Proposition 2.3.12.

(i) The maximal cells of $A_g^{\mathrm{tr,V}}$ are exactly those $C([P])$ such that P is primitive. $A_g^{\mathrm{tr,V}}$ is of pure dimension $\binom{g+1}{2}$.

(ii) The codimension one cells of $A_g^{\mathrm{tr,V}}$ are exactly those of the form $C([P])$ such that P is almost-primitive. $A_g^{\mathrm{tr,V}}$ is connected through codimension one.

(iii) Every codimension one cell of $A_g^{\mathrm{tr,V}}$ lies in the closure of one or two maximal cells.

Proof. The Dirichlet-Voronoi polytopes $P \subset \mathbb{R}^g$ that are primitive correspond to Delaunay decompositions that are triangulations, i. e. such that every Delaunay polytope is a simplex (see [V03, Sec. 3.2]). The Dirichlet-Voronoi polytopes $P \subset \mathbb{R}^g$ that are almost-primitive correspond to the Delaunay decompositions that have exactly one Delaunay repartitioning polytope, in the sense of [V03, Sec. 2.4], and all the other Delaunay polytopes are simplices. Two maximal cells that have a common codimension one cell in their closure are usually called bistellar neighbors (see [V03, Sec. 2.4]). With this in mind, all the above properties follow from the (so-called) *Main Theorem of Voronoi's reduction theory* (see [Vor08] or [V03, Thm. 2.5.1]).

□

2.3.4 Zonotopal Dirichlet-Voronoi polytopes: A_g^{zon}

Among all the Dirichlet-Voronoi polytopes, a remarkable subclass is represented by the *zonotopal* ones. Recall (see [Z95, Chap. 7]) that a zonotope is a polytope that can be realized as a Minkowski sum of segments, or equivalently, that can be obtained as an affine projection of an hypercube.

Remark 2.3.13. Voronoi's conjecture has been proved for zonotopal parallelohedra (see [McM75], [E99], [DG2], [V04]): every zonotopal parallelohedron is affinely equivalent to a zonotopal Dirichlet-Voronoi polytope. Therefore, there is a bijection

$$\left\{\begin{array}{l}\text{Zonotopal parallelohedra}\\ \text{in } \mathbb{R}^g\end{array}\right\}_{/\mathrm{aff}} \longleftrightarrow \left\{\begin{array}{l}\text{Zonotopal Dirichlet-Voronoi}\\ \text{polytopes in } \mathbb{R}^g\end{array}\right\}_{/\,\mathrm{GL}_g(\mathbb{Z})}$$

There is a close (and well-known) relation between zonotopal Dirichlet-Voronoi polytopes $P \subset \mathbb{R}^g$ up to $\mathrm{GL}_g(\mathbb{Z})$-action and regular matroids M of rank at most g. We need to review this correspondence in detail because it is crucial for the sequel of the paper and also because we need to fix the notations we are going to use. Consider first the following

Construction 2.3.14. Let $A \in M_{g,n}(\mathbb{R})$ be a totally unimodular matrix of rank $r \leq g$. Consider the linear map $f_{A^t} : \mathbb{R}^g \to \mathbb{R}^n$, $x \mapsto A^t \cdot x$, where A^t is the transpose of A. For any n-tuple $\underline{l} = (l_1, \ldots, l_n) \in \mathbb{R}^n_{>0}$, consider the positive definite quadratic form $|| \cdot ||_{\underline{l}}$ on $\mathbb{R}^n$ given on $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ by

$$||y||_{\underline{l}} := l_1 y_1^2 + \ldots + l_n y_n^2,$$

and its pull-back $Q_{A,\underline{l}}$ on $\mathbb{R}^g$ via f_{A^t}, i.e.

$$Q_{A,\underline{l}}(x) := ||A^t \cdot x||_{\underline{l}}, \tag{2.11}$$

for $x \in \mathbb{R}^g$. Clearly $Q_{A,\underline{l}}$ has rank equal to r and belongs to Ω_g^{rt}. As $\underline{l}$ varies in $\mathbb{R}^n_{>0}$, the semi-positive definite quadratic forms $Q_{A,\underline{l}}$ form an open cone in Ω_g^{rt} which we denote by $\sigma^0(A)$. Its closure in Ω_g^{rt}, denoted by $\sigma(A)$, consists of the quadratic forms $Q_{A,\underline{l}} \in \Omega_g^{\text{rt}}$, where $\underline{l}$ varies in $\mathbb{R}^n_{\geq 0}$. The faces of $\sigma(A)$ are easily seen to be of the form $\sigma(A \setminus I)$ for some $I \subset \{1, \ldots, n\}$, where $A \setminus I$ is the totally unimodular matrix obtained from A by deleting the column vectors v_i with $i \in I$.

Considering the column vectors $\{v_1, \ldots, v_n\}$ of A as elements of $(\mathbb{R}^g)^*$, we define the following zonotope of $\mathbb{R}^g$:

$$Z_A := \{x \in \mathbb{R}^g \; : \; -1/2 \leq v_i(x) \leq 1/2 \text{ for } i = 1, \cdots, n\} \subset \mathbb{R}^g. \tag{2.12}$$

Its polar polytope (see [Z95, Sec. 2.3]) $Z_A^* \subset (\mathbb{R}^g)^*$ is given as a Minkowski sum:

$$Z_A^* := \left[-\frac{v_1}{2}, +\frac{v_1}{2}\right] + \ldots + \left[-\frac{v_n}{2}, +\frac{v_n}{2}\right] \subset (\mathbb{R}^g)^*. \tag{2.13}$$

Clearly the linear span of Z_A has dimension r.

Finally, if M is a regular matroid of rank $r(M) \leq g$, write $M = M[A]$, where $A \in M_{g,n}(\mathbb{R})$ is a totally unimodular matrix of rank r (see Theorem 2.1.24(i)). Note that if $A = XBY$ for a matrix $X \in \text{GL}_g(\mathbb{Z})$ and a permutation matrix $Y \in \text{GL}_n(\mathbb{Z})$, then $\sigma^0(A) = X\sigma^0(B)X^t$ and $Z_A = X \cdot Z_B$. Therefore, according to Theorem 2.1.24(ii), the $\text{GL}_g(\mathbb{Z})$-equivalence class of $\sigma^0(A)$, $\sigma(A)$ and of Z_A depends only on the matroid M and therefore we will set $[\sigma^0(M)] := [\sigma^0(A)]$, $[\sigma(M)] = [\sigma(A)]$ and $[Z_M] := [Z_A]$. The matroid $M \setminus I = M[A \setminus I]$ for a subset $I \subset E(M) = \{v_1, \cdots, v_n\}$ is called the *deletion* of I from M (see [O92, Pag. 22]).

Lemma 2.3.15. *Let A be as in 2.3.14. Then Z_A is a Dirichlet-Voronoi polytope whose associated cone is given by $\sigma^0(A)$, i.e. $\sigma^0_{Z_A} = \sigma^0(A)$.*

Proof. Let us first show that $\text{Vor}(Q_{A,\underline{l}}) = Z_A$ for any $\underline{l} \in \mathbb{R}^n_{>0}$, i.e. that Z_A is a Dirichlet-Voronoi polytope and that $\sigma^0(A) \subset \sigma^0_{Z_A}$. Assume first that A has maximal rank $r = g$ or, equivalently, that $f_{A^t} : \mathbb{R}^g \to \mathbb{R}^n$ is injective. By definitions (2.10) and (2.11), we get that

$$\text{Vor}(Q_{A,\underline{l}}) = \{x \in \mathbb{R}^g \; : \; ||f_{A^t}(x)||_{\underline{l}} \leq ||f_{A^t}(\lambda - x)||_{\underline{l}} \text{ for all } \lambda \in \mathbb{Z}^g\}. \tag{*}$$

The total unimodularity of A and the injectivity of f_{A^t} imply that the map $f_{A^t}:\mathbb{R}^g\to\mathbb{R}^n$ is integral and primitive, i.e. $f_{A^t}(x)\in\mathbb{Z}^n$ if and only if $x\in\mathbb{Z}^g$. Therefore, from (*) we deduce that

$$\mathrm{Vor}(Q_{A,\underline{l}})=f_{A^t}^{-1}(\mathrm{Vor}(||\cdot||_{\underline{l}}). \qquad (**)$$

Since $||\cdot||_{\underline{l}}$ is a diagonal quadratic form on $\mathbb{R}^n$, it is easily checked that

$$\mathrm{Vor}(||\cdot||_{\underline{l}})=\left[-\frac{e_1}{2},\frac{e_1}{2}\right]+\cdots+\left[-\frac{e_n}{2},\frac{e_n}{2}\right], \qquad (***)$$

where $\{e_1,\cdots,e_n\}$ is the standard basis of $\mathbb{R}^n$. Combining (**) and (***), and using the fact that $f_{A^t}(x)=(v_1(x),\cdots,v_n(x))$, we conclude. The general case $r\leq g$ follows in a similar way after replacing $\mathbb{R}^g$ with $\mathbb{R}^g/\mathrm{Ker}(f_{A^t})$. We leave the details to the reader.

In order to conclude that $\sigma^0(A)=\sigma^0_{Z_A}$, it is enough to show that the rays of σ_{Z_A} are contained in $\sigma(A)$. By translating the results of [ER94, Sec. 3] into our notations, we deduce that the rays of σ_{Z_A} are all of the form $\sigma_{Z(A)_i}$ for the indices i such that $v_i\neq 0$, where

$$Z(A)_i:=Z(A)\bigcap_{j\neq i}\{v_j^*=0\}.$$

By what we already proved, we have the inclusion $\sigma(v_i):=\sigma(A\setminus\{i\}^c)\subset\sigma_{Z(A)_i}$, where $\{i\}^c:=\{1,\cdots,n\}\setminus\{i\}$. Since both the cones are one dimensional, we deduce that $\sigma(A\setminus\{i\}^c)=\sigma_{Z(A)_i}$, which shows that all the rays of σ_{Z_A} are also rays of $\sigma(A)$.

□

Theorem 2.3.16.

(i) Given a regular matroid M of rank $r(M)\leq g$, $[Z_M]$ is the $\mathrm{GL}_g(\mathbb{Z})$-equivalence class of a zonotopal Dirichlet-Voronoi polytope and every such class arises in this way.

(ii) If M_1 and M_2 are two regular matroids, then $[Z_{M_1}]=[Z_{M_2}]$ if and only if $[\sigma(M_1)]=[\sigma(M_2)]$ if and only if $\widetilde{M_1}=\widetilde{M_2}$.

(iii) If M is simple, then any representative $\sigma(M)$ in $[\sigma(M)]$ is a simplicial cone of dimension $\#E(M)$ whose faces are of the form $\sigma(M\setminus I)\in[\sigma(M\setminus I)]$ for some uniquely determined $I\subset E(M)$.

Proof. The first assertion of (i) follows from the previous Lemma 2.3.15 together with the fact that each representative $Z_A\in[Z_M]$ is zonotopal by definition (see 2.3.14). The second assertion is a well-known result of Shephard and McMullen ([S74], [McM75] or also [DG2, Thm. 1]).

Consider part (ii). By definition 2.3.9 and what remarked shortly after, $[\sigma(M_1)]=[\sigma(M_2)]$ if and only if $[Z_{M_1}]=[Z_{M_2}]$. Let us prove that $[Z_M]=[Z_{\widetilde{M}}]$. Write $M=M[A]$ as in 2.3.14. From Definitions 2.1.22 and 2.1.27, it is straightforward to see that $\widetilde{M}=M[\widetilde{A}]$, where $\widetilde{A}$ is the totally unimodular matrix obtained from A by deleting the zero columns and, for each set S of proportional columns, deleting all but one distinguished column of S. From the definition (2.12), it follows easily that $Z_A=Z_{\widetilde{A}}$, which proves that $[Z_M]=[Z_{\widetilde{M}}]$.

To conclude part (ii), it remains to prove that if M_1 and M_2 are simple regular matroids such that $[Z_{M_1}] = [Z_{M_2}]$, then $M_1 = M_2$. We are going to use the poset of flats $\mathcal{L}(M)$ of a matroid M (see [O92, Sec. 1.7]). In the special case (which will be our case) where $M = M[A]$ for some matrix $A \in M_{g,n}(F)$ over some field F, whose column vectors are denoted as usual by $\{v_1, \cdots, v_n\}$, a flat (see [O92, Sec. 1.4]) is a subset $S \subset E(M) = \{1, \cdots, n\}$ such that

$$\mathrm{span}(v_i : i \in S) \subsetneq \mathrm{span}(v_k, v_i : i \in S),$$

for any $k \notin S$. $\mathcal{L}(M)$ is the poset of flats endowed with the natural inclusion. It turns out that (see [O92, Pag. 58]) for two matroids M_1 and M_2, we have

$$\mathcal{L}(M_1) \cong \mathcal{L}(M_2) \Leftrightarrow \widetilde{M_1} = \widetilde{M_2}. \tag{*}$$

Moreover, in the case where M is a regular and simple matroid, $\mathcal{L}(M)$ is determined by the $GL_g(\mathbb{Z})$-equivalence class $[Z_M]$. Indeed, writing $M = M[A]$ as in 2.3.14, Z_M determines, up to the natural action of $GL_g(\mathbb{Z})$, a central arrangement $\mathcal{A}_M$ of non-trivial and pairwise distinct hyperplanes in $(\mathbb{R}^g)^*$, namely those given by $H_i := \{v_i = 0\}$ for $i = 1, \cdots, n$. Denote by $\mathcal{L}(\mathcal{A}_M)$ the intersection poset of $\mathcal{A}_M$, i.e. the poset of linear subspaces of $(\mathbb{R}^g)^*$ that are intersections of some of the hyperplanes H_i, ordered by inclusion. Clearly $\mathcal{L}(\mathcal{A}_M)$ depends only on the $GL_g(\mathbb{Z})$-equivalence class $[Z_M]$. It is easy to check that the map

$$\begin{aligned} \mathcal{L}(M) &\longrightarrow \mathcal{L}(\mathcal{A}_M)^{\mathrm{opp}} \\ S &\mapsto \bigcap_{i \in S} H_i, \end{aligned} \tag{**}$$

is an isomorphism of posets, where $\mathcal{L}(\mathcal{A}_M)^{\mathrm{opp}}$ denotes the opposite poset of $\mathcal{L}(\mathcal{A}_M)$. Now we can conclude the proof of part (ii). Indeed, if M_1 and M_2 are regular and simple matroids such that $[Z_{M_1}] = [Z_{M_2}]$ then $\mathcal{L}(\mathcal{A}_{M_1}) \cong \mathcal{L}(\mathcal{A}_{M_2})$ which implies that $\mathcal{L}(M_1) \cong \mathcal{L}(M_2)$ by (**) and hence $M_1 = M_2$ by (*).

Finally consider part (iii). Write $M = M[A]$ as in 2.3.14 and consider the representative $\sigma(A) \in [\sigma(M)]$. From [ER94, Thm. 4.1], we known that $\sigma(A)$ is simplicial. We have already observed in 2.3.14 that all the faces of $\sigma(A)$ are of the form $\sigma(A \setminus I)$ for $I \subset E(M) = \{v_1, \cdots, v_n\}$ and that $\sigma(A \setminus I) \in [\sigma(M \setminus I)]$ by definition of deletion of I from M. In particular, the rays of $\sigma(A)$ are all of the form $\sigma(v_i) := \sigma(A \setminus \{v_i\}^c)$ for some $v_i \in E(M)$, where $\{v_i\}^c := E(M) \setminus \{v_i\}$. The hypothesis that M is simple (see 2.1.26) is equivalent to the fact that the matrix A has no zero columns and no parallel columns. This implies that all the faces $\sigma(v_i)$ are 1-dimensional and pairwise distinct. Since $\sigma(A)$ is a simplicial cone, its dimension is equal to the number of rays, i.e. to $n = \#E(M)$. The fact that each face of $\sigma(A)$ is of the form $\sigma(A \setminus I)$ for a unique $I \subset E(M)$ follows from the fact that in a simplicial cone each face is uniquely determined by the rays contained in it.

□

From Theorem 2.3.16, it follows that the class of all open Voronoi cones σ_Z^0 such that $Z \subset \mathbb{R}^g$ is a zonotopal Dirichlet-Voronoi polytope is stable under the action of $\mathrm{GL}_g(\mathbb{Z})$ and under the operation of taking faces of the closures $\sigma_Z = \overline{\sigma_Z^0}$. Therefore the collection of zonotopal Voronoi cones, i.e.

$$\mathrm{Zon} := \{\sigma_Z \subset \Omega_g^{\mathrm{rt}} \ : \ Z \subset \mathbb{R}^g \text{ is zonotope}\},$$

is a $\mathrm{GL}_g(\mathbb{Z})$-admissible decomposition of a closed subcone of Ω_g^{rt}, i.e. Zon satisfies all the properties of Definition 2.3.6 except the last one. Therefore we can give the following

Definition 2.3.17. A_g^{zon} is the stacky subfan of $A_g^{\mathrm{tr,V}}$ whose cells are of the form $C([Z])$, where $[Z]$ varies among the $\mathrm{GL}_g(\mathbb{Z})$-equivalence classes of zonotopal Dirichlet-Voronoi polytopes in $\mathbb{R}^g$.

A_g^{zon} has dimension $\binom{g+1}{2}$ but it is not pure-dimensional if $g \geq 4$ (see Example 2.5.11 or [DV99] for the list of maximal zonotopal cells for small values of g). There is indeed only one zonotopal cell of maximal dimension $\binom{g+1}{2}$, namely the one corresponding to the principal cone (see section 2.5.1 below). Using the notations of 2.3.14, given a regular matroid M of rank at most g, we set $C(M) := C([Z_M])$. From Theorem 2.3.16, we deduce the following useful

Corollary 2.3.18. *The cells of A_g^{zon} are of the form $C(M)$, where M is a simple regular matroid of rank at most g.*

We want to conclude this section on zonotopal Dirichlet-Voronoi polytopes (and hence on zonotopal parallelohedra by remark 2.3.13) by mentioning the following

Remark 2.3.19. Zonotopal parallelohedra $Z \subset \mathbb{R}^g$ are also closely related to other geometric-combinatorial objects:

- (i) Lattice dicings of $\mathbb{R}^g$ (see [ER94]);
- (ii) Venkov arrangements of hyperplanes of $\mathbb{R}^g$ (see [E99]);
- (iii) Regular oriented matroids of rank at most g, up to reorientation (see [BVSWZ99, Sec. 2.2, 6.9]).

2.4 The tropical Torelli map

2.4.1 Construction of the tropical Torelli map t_g^{tr}

We begin by defining the Jacobian of a tropical curve.

Definition 2.4.1. Let $C = (\Gamma, w, l)$ be a tropical curve of genus g and total weight $|w|$. The *Jacobian* $\mathrm{Jac}(C)$ of C is the tropical abelian variety of dimension g given by the real torus $(H_1(\Gamma, \mathbb{R}) \oplus \mathbb{R}^{|w|})/(H_1(\Gamma, \mathbb{Z}) \oplus \mathbb{Z}^{|w|})$ together with the semi-positive quadratic form $Q_C = Q_{(\Gamma,w,l)}$ which vanishes identically on $\mathbb{R}^{|w|}$ and is given on $H_1(\Gamma, \mathbb{R})$ as

$$Q_C\left(\sum_{e\in E(\Gamma)} \alpha_e \cdot e\right) = \sum_{e\in E(\Gamma)} \alpha_e^2 \cdot l(e). \tag{2.14}$$

Remark 2.4.2. Note that the above definition is independent of the orientation chosen to define $H_1(\Gamma, \mathbb{Z})$. Moreover, after identifying the lattice $H_1(\Gamma, \mathbb{Z}) \oplus \mathbb{Z}^{|w|}$ with $\mathbb{Z}^g$ (which amount to chose a basis of $H_1(\Gamma, \mathbb{Z})$), we can (and will) regard the arithmetic equivalence class of Q_C as an element of Ω_g^{rt}.

Remark 2.4.3. The above definition of Jacobian is a generalization of the definition of Mikhalkin-Zharkov (see [MZ07, Sec. 6]). More precisely, the Jacobian of a tropical curve of total weight zero in our sense is the same as the Jacobian of Mikhalkin-Zharkov.

Example 2.4.4. In Figure 2.6 below, the so-called Peterson graph is regarded as a tropical curve C of genus 6 with identically zero weight function and with length function $l(e_i) := l_i \in \mathbb{R}_{>0}, i = 1, \dots, 15$.

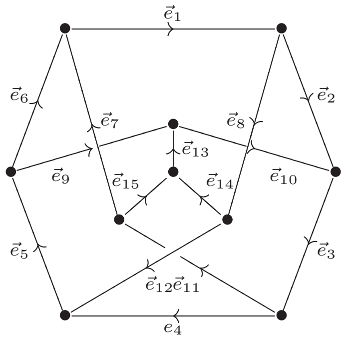

Figure 2.6: The Peterson graph Γ endowed with an orientation.

Fix an orientation of the edges as shown in the figure and consider the basis B for the space $H_1(\Gamma, \mathbb{R}) = \mathbb{R}^6$ formed by the cycles $C_1, \dots, C_6$, where $C_1 = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5, \vec{e}_6\}$, $C_2 = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_{11}, \vec{e}_7\}$, $C_3 = \{\vec{e}_1, \vec{e}_8, \vec{e}_{12}, \vec{e}_5, \vec{e}_6\}$, $C_4 = \{\vec{e}_3, \vec{e}_{11}, \vec{e}_{15}, \vec{e}_{13}, \vec{e}_{10}\}$, $C_5 = \{\vec{e}_5, \vec{e}_9, -\vec{e}_{13}, -\vec{e}_{14}, \vec{e}_{12}\}$ and $C_6 = \{\vec{e}_1, \vec{e}_8, \vec{e}_{14}, -\vec{e}_{15}, \vec{e}_7\}$. Then the tropical Jacobian $J(C)$ of C is the real torus $H_1(\Gamma, \mathbb{R})/H_1(\Gamma, \mathbb{Z}) = \mathbb{R}^6/\mathbb{Z}^6$ endowed with the positive definite quadratic form Q_C which is represented in the basis B by the following matrix:

$$\begin{pmatrix} \sum_{i=1}^6 l_i & \frac{l_1+l_2+l_3}{2} & \frac{l_1+l_5+l_6}{2} & \frac{l_3}{2} & \frac{l_5}{2} & \frac{l_1}{2} \\ \frac{l_1+l_2+l_3}{2} & l_1+l_2+l_3+l_{11}+l_7 & \frac{l_1}{2} & \frac{l_3+l_{11}}{2} & 0 & \frac{l_1+l_7}{2} \\ \frac{l_1+l_5+l_6}{2} & \frac{l_1}{2} & l_1+l_5+l_6+l_8+l_{12} & 0 & \frac{l_5+l_{12}}{2} & \frac{l_1+l_8}{2} \\ \frac{l_3}{2} & \frac{l_3+l_{11}}{2} & 0 & l_3+l_{10}+l_{11}+l_{13}+l_{15} & \frac{-l_{13}}{2} & \frac{-l_{15}}{2} \\ \frac{l_5}{2} & 0 & \frac{l_5+l_{12}}{2} & \frac{-l_{13}}{2} & l_5+l_9+l_{12}+l_{13}+l_{14} & \frac{-l_{14}}{2} \\ \frac{l_1}{2} & \frac{l_1+l_7}{2} & \frac{l_1+l_8}{2} & \frac{-l_{15}}{2} & \frac{-l_{14}}{2} & l_1+l_7+l_8+l_{14}+l_{15} \end{pmatrix}$$

Consider now the map (called tropical Torelli)

$$\begin{aligned} t_g^{\mathrm{tr}} : M_g^{\mathrm{tr}} &\to A_g^{\mathrm{tr},\mathrm{V}} \\ C &\mapsto \mathrm{Jac}(C). \end{aligned}$$

Theorem 2.4.5. *The above map $t_g^{\mathrm{tr}} : M_g^{\mathrm{tr}} \to A_g^{\mathrm{tr},\mathrm{V}}$ is a map of stacky fans.*

Proof. Let us first prove that t_g^{tr} is a continuous map. The map t_g^{tr} restricted to the closure of one cell $\overline{C(\Gamma, w)}$ of M_g^{tr} is clearly continuous since the quadratic form Q_C on $H_1(\Gamma, \mathbb{R})$ depends continuously on the lengths $l \in \mathbb{R}_{\geq 0}^{|E(\Gamma)|}$. The continuity of t_g^{tr} follows then from the fact that M_g^{tr} is a quotient of $\coprod \overline{C(\Gamma, w)}$ with the induced quotient topology.

Lemma 2.4.6 below implies that $t_g^{\text{tr}}(C(\Gamma, w)) \subset C\left(\widetilde{M^*(\Gamma)}\right)$. It remains to see that this map $t_g^{\text{tr}} : C(\Gamma, w) \to C\left(\widetilde{M^*(\Gamma)}\right)$ is induced by an integral linear function $L_{(\Gamma,w)}$ between $\mathbb{R}^{|E(\Gamma)|}$ and the space $\mathbb{R}^{\binom{g(\Gamma)+1}{2}}$ of symmetric matrices on $H_1(\Gamma, \mathbb{R})$. We define

$$\begin{aligned} L_{(\Gamma,w)} : \mathbb{R}^{|E(\Gamma)|} &\longrightarrow \mathbb{R}^{\binom{g(\Gamma)+1}{2}}, \\ l &\mapsto Q_{(\Gamma,w,l)}, \end{aligned} \tag{2.15}$$

where $Q_{(\Gamma,w,l)}$ is defined by (2.14) above . Clearly $L_{(C,\Gamma)}$ is an integral linear map that induces the map $t_g^{\text{tr}} : C(\Gamma, \mathbb{Z}) \to C\left(\widetilde{M^*(\Gamma)}\right)$. This concludes the proof. □

Lemma 2.4.6. *The map t_g^{tr} sends the cell $C(\Gamma, w)$ of M_g^{tr} surjectively onto the cell $C\left(\widetilde{M^*(\Gamma)}\right)$ of $A_g^{\text{tr,V}}$.*

Proof. We use the construction in 2.3.14. Fixing an orientation of Γ, a basis of $H_1(\Gamma, \mathbb{Z})$ and an order of the edges of Γ, we get a natural inclusion

$$H_1(\Gamma, \mathbb{Z}) \cong \mathbb{Z}^{g(\Gamma)} \hookrightarrow \mathbb{Z}^n \cong C_1(\Gamma, \mathbb{Z}).$$

The transpose of the integral matrix representing this inclusion, call it $A^*(\Gamma) \in M_{g(\Gamma),n}(\mathbb{Z})$, is well-known to be totally unimodular and such that $M^*(\Gamma) = M[A^*(\Gamma)]$ (see for example [Z95, Ex. 6.4]).

Now given a length function $l : E(\Gamma) \to \mathbb{R}_{>0}$, consider the n-tuple $\underline{l} \in \mathbb{R}_{>0}^n$ whose entries are the real positive numbers $\{l(e)\}_{e \in E(\Gamma)}$ with respect to the order chosen on $E(\Gamma)$. Comparing definitions (2.11) and (2.14), we deduce that $Q_{A^*(\Gamma),\underline{l}} = Q_{(\Gamma,w,l)}$. The conclusion now follows from Lemma 2.3.15 and Theorem 2.3.16. □

2.4.2 Tropical Schottky

In this subsection, we want to prove a Schottky-type theorem, i.e. we describe the image of the map t_g^{tr}.

We need to recall the following result (see [O92, 3.1.1, 3.1.2, 3.2.1] for a proof).

Lemma 2.4.7. *Let Γ be a graph. For any subset $I \subset E(\Gamma) = E(M^*(\Gamma))$, we have that*

$$M(\Gamma) \setminus I = M(\Gamma \setminus I) \tag{2.16}$$

$$M^*(\Gamma) \setminus I = M^*(\Gamma / I) \tag{2.17}$$

where $\Gamma \setminus I$ (resp. Γ/I) is the graph obtained from Γ by deleting (resp. contracting) the edges in I and, for a matroid M and $I \subset E(M)$, we denote by $M \setminus I$ the matroid obtained from M by deleting I.

From formula (2.17) and Theorem 2.3.16(iii), we deduce that the collection of cographic cones

$$\mathrm{Cogr} := \{\sigma_Z \subset \Omega_g^{\mathrm{rt}} \ : \ [\sigma_Z] = [\sigma(M)] \text{ for a cographic matroid } M\}$$

is closed under taking faces of the cones, and therefore it defines a $\mathrm{GL}_g(\mathbb{Z})$-admissible decomposition of a closed subcone of Ω_g^{rt}, i.e. Cogr satisfies all the properties of Definition 2.3.6 except the last one. Therefore we can give the following

Definition 2.4.8. A_g^{cogr} is the stacky subfan of $A_g^{\mathrm{zon}} \subset A_g^{\mathrm{tr,V}}$ whose cells are of the form $C(M)$, where M is a simple cographic matroid of rank at most g.

The following Proposition summarizes some important properties of A_g^{cogr} (compare with Propositions 2.2.9 and 2.3.12).

Proposition 2.4.9.

(i) The cells of A_g^{cogr} are of the form $C\left(M^*([\Gamma]_2)\right)$, where $[\Gamma]_2$ varies among the 2-isomorphism classes of 3-edge-connected graphs of genus at most g.

(ii) A_g^{cogr} has pure dimension $3g-3$ and its maximal cells are of the form $C\left(M^*(\Gamma)\right)$, where Γ is 3-regular and 3-(edge)-connected.

(iii) A_g^{cogr} is connected through codimension one.

(iv) All the codimension one cells of A_g^{cogr} lie in the closure of one, two or three maximal cells of A_g^{cogr}.

Proof. Part (i) follows by combining Definition 2.4.8, Remark 2.1.30 and Proposition 2.1.33.

According to Theorem 2.3.16(iii), a cell $C(M^*([\Gamma]_2))$ of A_g^{cogr} is of maximal dimension if and only if Γ has the maximum number of edges, and this happens precisely when Γ is 3-regular in which case $\#E(\Gamma) = \dim C(M^*([\Gamma]_2)) = 3g-3$. On the other hand, using the fact that every 3-edge-connected graph of genus g is the specialization of a 3-regular and 3-edge-connected graph (see [CV1, Prop. A.2.4]), formula (2.17) and Theorem 2.3.16(iii) give that every cell of A_g^{cogr} is the face of some maximal dimensional cell, i.e. A_g^{cogr} is of pure dimension $3g-3$. To conclude the proof of part (ii), it is enough to recall that a 3-edge-connected and 3-regular graph Γ is also 3-connected (see for example [CV1, Lemma A.1.2]) and that $[\Gamma]_2 = \{\Gamma\}$ according to Fact 2.1.13.

Using the same argument as in the beginning of the proof of Proposition 2.2.9, it is easy to see that the codimension one cells of A_g^{cogr} are of the form $C(M^*([\Gamma]_2))$, where $[\Gamma]_2$ varies among the 2-equivalence classes of genus g graphs having one vertex of valence 4 and all the others of valence 3 (it is easy to see that this property is preserved under 2-isomorphism). The same proof as in Proposition 2.2.9 gives now part (iv) while part (iii) follows from [C8, Thm. 3.3]: any two 3-regular and 3-(edge)-connected graphs of the same genus are 3-linked, i.e. they can be obtained one from the other via a sequence of twisting operations as in Figure 2.2 in such a way that each intermediate graph is also 3-edge-connected.

□

From the above Proposition 2.4.9 and Lemma 2.4.6, we deduce the following tropical Schottky theorem.

Theorem 2.4.10. *The tropical Torelli map $t_g^{\rm tr}$ is full and its image is equal to the stacky subfan $A_g^{\rm cogr} \subset A_g^{\rm tr,V}$.*

Remark 2.4.11. It is known (see Example 2.5.11 or [V03, Chap. 4]) that $A_g^{\rm cogr} = A_g^{\rm tr,V}$ if and only if $g \leq 3$. Therefore $t_g^{\rm tr} : M_g^{\rm tr} \to A_g^{\rm tr,V}$ is surjective if and only if $g \leq 3$. This has to be compared with the fact that the classical Torelli map $t_g : M_g \to A_g$ is dominant if and only if $g \leq 3$.

2.4.3 Tropical Torelli

In [CV1, Thm. 4.1.9], the authors determine when two tropical curves C and C' of total weight zero (i.e. tropical curves up to tropical modifications in the sense of Mikhalkin-Zharkov) are such that $\mathrm{Jac}(C) \cong \mathrm{Jac}(C')$. Indeed, we show here that the same result extends easily to the more general case of tropical curves (with possible non-zero weight). We first need the following definitions.

Definition 2.4.12. Two tropical curves $C = (\Gamma, w, l)$ and $C' = (\Gamma', w', l')$ are 2-isomorphic, and we write $C \equiv_2 C'$, if there exists a bijection $\phi : E(\Gamma) \to E(\Gamma')$, commuting with the length functions l and l', that induces a 2-isomorphism between Γ and Γ'. We denote by $[C]_2$ the 2-isomorphism equivalence class of a tropical curve C.

Similarly to definition 2.1.15, we have the following

Lemma - Definition 2.4.13. Let $C = (\Gamma, l, w)$ a tropical curve. A 3-edge-connectivization of C is a tropical curve $C^3 = (\Gamma^3, l^3, w^3)$ obtained in the following manner:

(i) Γ^3 is a 3-edge-connectivization of Γ in the sense of definition 2.1.15, i.e. Γ^3 is obtained from Γ by contracting all the separating edges of Γ and, for each $C1$-set S of Γ, all but one the edges of S, which we denote by e_S;

(ii) w^3 is the weight function on Γ^3 induced by the weight function w on Γ in the way explained in 2.2.1 viewing Γ^3 as a specialization of Γ;

(iii) l^3 is the length function on Γ^3 given by

$$l^3(e_S) = \sum_{e \in S} l(e),$$

for each C1-set S of Γ.

The 2-isomorphism class of C^3 is well-defined; it will be called the 3-edge-connectivization class of C and denoted by $[C^3]_2$.

It is now easy to extend [CV1, Thm. 4.1.9] to the case of tropical curves.

Theorem 2.4.14. *Let C and C' be two tropical curves of genus g. Then $t_g^{\rm tr}(C) = t_g^{\rm tr}(C')$ if and only if $[C^3]_2 = [C'^3]_2$. In particular $t_g^{\rm tr}$ is injective on the locus of 3-connected tropical curves.*

Proof. Note that $[C^3]_2 = [C'^3]_2$ if and only if the 3-edge-connectivizations (in the sense of definition [CV1, Def. 4.1.7]) of the underlying metric graphs (Γ, l) and (Γ', l') are cyclically equivalent (in the sense of [CV1, Def. 4.1.6]), or in symbols $[(\Gamma^3, l^3)]_{\rm cyc} = [(\Gamma'^3, l'^3)]_{\rm cyc}$.

On the other hand, from the definition 2.4.1, it follows that ${\rm Jac}(C) \cong {\rm Jac}(C')$ if and only if the Albanese tori (in the sense of definition [CV1, 4.1.4]) of the underlying metric graphs (Γ, l) and (Γ', l') are isomorphic, or in symbols ${\rm Alb}(\Gamma, l) \cong {\rm Alb}(\Gamma', l')$.

With these two re-interpretations, the first assertion of the Theorem follows from [CV1, Thm 4.1.10]. The second assertion follows from the first and Fact 2.1.13. $\square$

Finally we can prove a tropical analogous of the classical Torelli theorem which was conjectured by Mikhalkin-Zharkov in [MZ07, Sec. 6.4] and proved in [CV1, Thm. A.2.1] assuming the existence of the relevant moduli spaces (see [CV1, Assumptions 1, 2, 3]). However, since the conjectural properties that these moduli spaces were assumed to have in [CV1] are slightly different from the properties of the moduli spaces $M_g^{\rm tr}$ and $A_g^{\rm tr, V}$ that we have constructed here, we give a new proof of this result.

Theorem 2.4.15. *The tropical Torelli map $t_g^{\rm tr} : M_g^{\rm tr} \to A_g^{\rm tr, V}$ is of degree one onto its image.*

Proof. The image of $t_g^{\rm tr}$ is equal to $A_g^{\rm cogr}$ according to Theorem 2.4.10. Therefore, we have to prove that $t_g^{\rm tr} : M_g^{\rm tr} \to A_g^{\rm cogr}$ satisfies the two conditions of Definition 2.1.2.

Proposition 2.4.9 and Theorem 2.4.14 give that a generic point of $A_g^{\rm cogr}$ is of the form ${\rm Jac}(C)$ for a unique tropical curve $C = (\Gamma, w, l)$, whose underlying graph Γ is 3-regular and 3-connected. This proves that the first condition of Definition 2.1.2 is satisfied.

It remains to prove that the integral linear function $L_{(\Gamma,w)}$, defined in (2.15), is primitive for a tropical curve $C = (\Gamma, w, l)$ whose underlying graph Γ is 3-regular and 3-connected. So suppose that the quadratic form $Q_{(\Gamma,w,l)}$ on $H_1(\Gamma, \mathbb{R})$ is integral, i.e. that the associated symmetric bilinear form (which, by abuse of notation, we denote by $Q_{(\Gamma,w,l)}(-,-)$) takes integral values on $H_1(\Gamma, \mathbb{Z})$; we have to show that the length function l takes integral values. Since Γ is 3-edge-connected by hypothesis, every edge of Γ is contained in a $C1$-set and all the $C1$-sets of Γ have cardinality one (see 2.1.6). Therefore, using [CV1, Lemma 3.3.1], we get that for every edge $e \in E(\Gamma)$ there exist two cycles Δ_1 and Δ_2 of Γ such that the intersection of their supports is equal to $\{e\}$. By definition 2.14, these two cycles define two elements C_1 and C_2 of $H_1(\Gamma, \mathbb{Z})$ (with respect to any chosen orientation of Γ) such that $Q_{(\Gamma,w,l)}(C_1, C_2) = l(e)$. Since $Q_{(\Gamma,w,l)}(-,-)$ takes integral values on $H_1(\Gamma, \mathbb{Z})$ by hypothesis, we get that $l(e) \in \mathbb{Z}$, q.e.d. $\square$

2.5 Planar tropical curves and the principal cone

2.5.1 A_g^{gr} and the principal cone

Another important stacky subfan of A_g^{zon} (other than A_g^{cogr}) is formed by the zonotopal cells that correspond to graphic matroids. Indeed, from formula (2.16) and Theorem 2.3.16(iii), it follows that the collection of graphic cones

$$\mathrm{Gr} := \{\sigma_Z \subset \Omega_g^{\mathrm{rt}} \ : \ [\sigma_Z] = [\sigma(M)] \text{ for a graphic matroid } M\}$$

is closed under taking faces of the cones, and therefore it defines a $\mathrm{GL}_g(\mathbb{Z})$-admissible decomposition of a closed subcone of Ω_g^{rt}, i.e. Gr satisfies all the properties of Definition 2.3.6 except the last one. Therefore we can give the following

Definition 2.5.1. A_g^{gr} is the stacky subfan of $A_g^{\mathrm{zon}} \subset A_g^{\mathrm{tr,V}}$ whose cells are of the form $C(M)$, where M is a simple graphic matroid of rank at most g.

By combining Corollary 2.3.18, Remark 2.1.30 and Proposition 2.1.33, we get the following

Remark 2.5.2. The cells of A_g^{gr} are of the form $C(M([\Gamma]_2))$, where $[\Gamma]_2$ varies among the 2-isomorphism classes of simple graphs of cogenus at most g.

A_g^{gr} is closely related to the so-called principal cone (Voronoi's principal domain of the first kind), see [N80, Chap. 8.10] and [V03, Chap. 2.3]. It is defined as

$$\sigma^0_{\mathrm{prin}} := \{Q = (q_{ij}) \in \Omega_g \ : \ q_{ij} < 0 \text{ for } i \neq j, \ \sum_j q_{ij} > 0 \text{ for all } i.\}$$

It is well-known that $\mathrm{Stab}(\sigma^0_{\mathrm{prin}}) = S_{g+1}$ (see [V03, Sec. 2.3]) and we will denote by $C_{\mathrm{prin}} := [\sigma^0_{\mathrm{prin}}/\,\mathrm{Stab}(\sigma^0_{\mathrm{prin}})]$ the cell of $A_g^{\mathrm{tr,V}}$ corresponding to the principal cone σ^0_{prin}, and call it the principal cell.

The following result is certainly well-known (see for example [V03, Sec. 3.5.2]), but we include a proof here by lack of a proper reference.

Lemma 2.5.3. *The $\mathrm{GL}_g(\mathbb{Z})$-equivalence class $[\sigma^0_{\mathrm{prin}}]$ of the principal cone is equal to $[\sigma^0(M(K_{g+1}))]$, where K_{g+1} is the complete simple graph on $(g+1)$-vertices. Therefore $C_{\mathrm{prin}} = C(M(K_{g+1}))$ in $A_g^{\mathrm{tr,V}}$.*

Proof. Call $\{v_1, \cdots, v_{g+1}\}$ the vertices of K_{g+1} and e_{ij} (for $i < j$) the unique edge of K_{g+1} joining v_i and v_j. Choose the orientation of K_{g+1} such that if $i < j$ then $s(e_{ij}) = e_i$ and $t(e_{ij}) = e_j$. It can be easily checked that the elements $\{\delta(v_1), \cdots, \delta(v_g)\}$ form a basis for $\Im(\delta) = H_1(K_{g+1}, \mathbb{Z})^{\perp}$. Consider the transpose of the integral matrix, call it $A(K_{g+1})$, that gives the inclusion $H_1(K_{g+1}, \mathbb{Z})^{\perp} \hookrightarrow C_1(K_{g+1}, \mathbb{Z})$ with respect to the basis $\{\delta(v_1), \cdots, \delta(v_g)\}$ and $\{e_{ij}\}_{i<j}$. In other words

$$A(K_{g+1})^t \cdot \delta(v_k) = \sum_{i<k} e_{ik} - \sum_{k<j} e_{kj}. \qquad (*)$$

Observe that $A(K_{g+1}) \in M_{g,n}(\mathbb{Z})$ where $n = \binom{g+1}{2} = \#E(K_{g+1})$. It is well-known (see [O92, Prop. 5.1.2, 5.1.3]) that $A(K_{g+1})$ is totally unimodular and that $M(K_{g+1}) = M[A(K_{g+1})]$.

We now apply the construction in 2.3.14 to this matrix $A(K_{g+1})$. For a n-tuple $\underline{l} = (l_{ij})_{i<j} \in \mathbb{R}^n_{>0}$ (setting $l_{j,i} = l_{i,j}$ if $i < j$), consider the quadratic form $Q_{A(K_{g+1}),\underline{l}}$ of formula (2.11). For the associated bilinear symmetric form, which we denote $Q_{A(K_{g+1}),\underline{l}}(-,-)$ (by an abuse of notation), we can compute, using (*) above, that (for $i \neq j$)

$$\begin{cases} Q_{A(K_{g+1}),\underline{l}}(\delta(v_i), \delta(v_i)) = \sum_{1 \leq k \neq i \leq g} l_{k,i} + l_{i,g+1}, \\ Q_{A(K_{g+1}),\underline{l}}(\delta(v_i), \delta(v_j)) = -l_{i,j}. \end{cases}$$

This easily implies that $\sigma^0(A(K_{g+1})) = \sigma^0_{\text{prin}}$, which concludes the proof since, as observed before, $[\sigma^0(A(K_{g+1}))] = [\sigma^0(M(K_{g+1}))]$. □

From the previous Lemma, we deduce the following

Proposition 2.5.4. *The stacky subfan A_g^{gr} of $A_g^{\text{zon}} \subset A_g^{\text{tr,V}}$ coincides with the closure inside A_g^{zon} (or $A_g^{\text{tr,V}}$) of the principal cell C_{prin}. In particular it has pure dimension equal to $\binom{g+1}{2}$ and C_{prin} is the unique maximal cell.*

Proof. Consider the closure, call it $\overline{C_{\text{prin}}}$, of C_{prin} inside $A_g^{\text{tr,V}}$. Note that $C_{\text{prin}} \subset A_g^{\text{gr}}$, because of the above Lemma 2.5.3, and therefore we get that $\overline{C_{\text{prin}}} \subset A_g^{\text{gr}}$. In order to prove equality, consider a cell of A_g^{gr}, which, according to Remark 2.5.2, is of the form $C(M([\Gamma]_2))$, for a simple graph Γ of cogenus at most g. Such a graph can be obtained by K_{g+1} by deleting some edges and therefore, using Theorem 2.3.16(iii) and formula (2.16), we get that $C(M([\Gamma]_2))$ is a face of the closure of $C(M(K_{g+1})) = C_{\text{prin}}$, and hence it belongs to $\overline{C_{\text{prin}}}$, q.e.d. □

Remark 2.5.5. The principal cone σ^0_{prin} has many important properties, among which we want to mention the following

(i) C_{prin} is the unique zonotopal cell of maximal dimension $\binom{g+1}{2}$ (see [V03, Sec. 3.5.3] and the references there);

(ii) The Dirichlet-Voronoi polytope associated to $[\sigma^0_{\text{prin}}]$ is the permutahedron of dimension g (see [Z95, Ex. 0.10]), which is an extremal Dirichlet-Voronoi polytope in the sense that it has the maximum possible number of d-dimensional faces among all Dirichlet-Voronoi polytopes of dimension g (see [V03, Sec. 3.3.2] and the references there);

(iii) σ^0_{prin} is the unique Voronoi cone that is also a perfect cone (see [D72]).

2.5.2 Tropical Torelli map for planar tropical curves

We begin with the following

Definition 2.5.6. We say that a tropical curve $C = (\Gamma, w, l)$ (resp. a stable marked graph (Γ, w)) is *planar* if the underlying graph Γ is planar.

Note that the specialization of a planar tropical curve is again planar. Therefore it makes sense to give the following

Definition 2.5.7. $M_g^{\rm tr,pl}$ is the stacky subfan of $M_g^{\rm tr}$ consisting of planar tropical curves.

It is straightforward to check that any planar tropical curve can be obtained as a specialization of a 3-regular planar tropical curve. Therefore we get the following

Remark 2.5.8. $M_g^{\rm tr,pl}$ is of pure dimension $3g-3$ with cells $C(\Gamma, w) \subset \mathbb{R}^{|w|}$, for planar stable marked graphs (Γ, w) of genus g. A cell $C(\Gamma, w)$ of $M_g^{\rm tr,pl}$ is maximal if and only if Γ is 3-regular.

We want now to describe the image of $M_g^{\rm tr,pl}$ under the map $t_g^{\rm tr}$. With that in mind, we consider the locus inside $A_g^{\rm zon}$ formed by the zonotopal cells corresponding to matroids that are at the same time graphic and cographic. Indeed, from formulas (2.16), (2.17) and Theorem 2.3.16(iii), it follows that the collection of cones

$$\text{Gr-cogr} := \{\sigma_Z \ : \ [\sigma_Z] = [\sigma(M)] \text{ for a graphic and cographic matroid } M\}$$

is a $\mathrm{GL}_g(\mathbb{Z})$-admissible decomposition of a closed subcone of $\Omega_g^{\rm rt}$, i.e. Gr-cogr satisfies all the properties of Definition 2.3.6 except the last one. Therefore we can give the following

Definition 2.5.9. $A_g^{\rm gr,cogr}$ is the stacky subfan of $A_g^{\rm zon} \subset A_g^{\rm tr,V}$ whose cells are of the form $C(M)$, where M is a simple graphic and cographic matroid of rank at most g.

Equivalently, $A_g^{\rm gr,cogr}$ is the intersection of $A_g^{\rm cogr}$ and $A_g^{\rm gr}$ inside $A_g^{\rm zon}$. Using Corollary 2.3.18, Proposition 2.1.31, Remark 2.1.32 and Proposition 2.1.33, we get the following

Remark 2.5.10. The cells of $A_g^{\rm gr,cogr}$ are of the form

$$C(M([\Gamma]_2)) = C(M^*([\Gamma]_2^*)),$$

for $[\Gamma]_2$ planar and simple and $[\Gamma]_2^*$ the dual 2-isomorphism class as in (2.3) (which is therefore planar and 3-edge-connected by (2.4)).

Example 2.5.11. We have defined several stacky subfans of $A_g^{\rm tr,V}$, namely:

$$A_g^{\rm gr,cogr} \subset A_g^{\rm cogr}, A_g^{\rm gr} \subset A_g^{\rm zon} \subset A_g^{\rm tr,V}.$$

For $g = 2, 3$, they are all equal and they have a unique maximal cell, namely the principal cell $C_{\rm prin}$ associated to the principal cone $\sigma_{\rm prin}^0$ (see [V03, Chap. 4.2, 4.3]). However, for $g \geq 4$, all the above subfans are different. For example, for $g = 4$, we have that (see [V03, Chap. 4.4]):

(i) $A_4^{\rm tr,V}$ has 3 maximal cells (of dimension 10), one of which is $C_{\rm prin}$;

(ii) A_4^{zon} has two maximal cells: C_{prin} of dimension 10 and $C(M^*([K_{3,3}]_2))$ of dimension 9, where $K_{3,3}$ is the complete bipartite graph on $(3,3)$-vertices;

(iii) A_4^{cogr} has two maximal cells (of dimension 9): $C(M^*([K_{3,3}]_2))$ and $C(M^*([K_5-1]_2^*))$, where K_5-1 is the (planar) graph obtained by the complete simple graph K_5 on 5 vertices by deleting one of its edges;

(iv) A_4^{gr} has a unique maximal cell (of dimension 10), namely C_{prin};

(v) $A_4^{\text{gr,cogr}}$ has a unique maximal cell (of dimension 9): $C(M^*([K_5-1]_2^*)) = C(M([K_5-1]_2))$.

Finally, we point out that A_g^{zon} becomes quickly much smaller than $A_g^{\text{tr,V}}$ as g grows: $A_5^{\text{tr,V}}$ has 222 maximal cells while A_5^{zon} only 4; $A_6^{\text{tr,V}}$ has more than $250,000$ maximal cells (although the exact number is still not known) while A_6^{zon} only 11 (see [V03, Chap. 4.5, 4.6] and [DV99, Sec. 9]).

Now, we can prove the main result of this section.

Theorem 2.5.12. *The following diagram*

$$\begin{array}{ccc} M_g^{\text{tr,pl}} & \hookrightarrow & M_g^{\text{tr}} \\ \downarrow t_g^{\text{tr}} & & \downarrow t_g^{\text{tr}} \\ A_g^{\text{gr,cogr}} & \hookrightarrow & A_g^{\text{cogr}}. \end{array}$$

is cartesian. In particular, the map $t_g^{\text{tr}}: M_g^{\text{tr,pl}} \to A_g^{\text{gr,cogr}}$ *is full and of degree one.*

Proof. The fact that the diagram is cartesian follows from Lemma 2.4.6 together with the fact that $M^*(\Gamma)$ is graphic if and only if Γ is planar (see 2.1.32). The last assertion follows from the first and the Theorems 2.4.10, 2.4.15. □

2.5.3 Relation with the compactified Torelli map: Namikawa's conjecture

In this last subsection, we use the previous results to give a positive answer to a problem posed by Namikawa ([N80, Problem (9.31)(i)]) concerning the compactified (classical) Torelli map.

We need to recall first some facts about the classical Torelli map and its compactification. Denote by $\mathcal{M}_g$ the coarse moduli space of smooth and projective curves of genus g, by $\mathcal{A}_g$ the coarse moduli space of principally polarized abelian varieties of dimension g. The classical Torelli map

$$\mathrm{t}_g : \mathcal{M}_g \to \mathcal{A}_g,$$

sends a curve X into its polarized Jacobian $(\mathrm{Jac}(X), \Theta_X)$.

It was known to Mumford and Namikawa (see [N76, Sec. 18], or also [A04, Thm. 4.1]) that the Torelli map extends to a regular map (called the compactified Torelli map)

$$\overline{\mathrm{t}}_g : \overline{\mathcal{M}_g} \to \overline{\mathcal{A}_g}^V \tag{2.18}$$

from the Deligne-Mumford moduli space $\overline{\mathcal{M}_g}$ of stable curves of genus g (see [DM69]) to the toroidal compactification $\overline{\mathcal{A}_g}^V$ of $\mathcal{A}_g$ associated to the (second) Voronoi decomposition (see [AMRT75], [N80] or [FC90, Chap. IV]). The above map $\overline{\mathrm{t}}_g$ admits also a modular interpretation (see [A04]), which was used in [CV2] to give a description of its fibers.

The moduli space $\overline{\mathcal{M}_g}$ admits a stratification into locally closed subsets parametrized by stable weighted graphs (Γ, w) of genus g (see definition 2.2.1). Namely, for each stable weighted graph (Γ, w) we can consider the locally closed subset $S_{(\Gamma,w)} \subset \overline{\mathcal{M}_g}$ formed by stable curves of genus g whose weighted dual graph is isomorphic to (Γ, w). Observe that, given a stable curve X with weighted dual graph (Γ, w), any smoothing of X at a subset S of nodes of X has weighted dual graph equal to the specialization of (Γ, w) obtained by contracting the edges corresponding to the nodes of S (see 2.2.1). From this remark, we deduce that:

$$C(\Gamma, w) \subset \overline{C(\Gamma', w')} \Leftrightarrow \overline{S_{(\Gamma,w)}} \supset S_{(\Gamma',w')}. \tag{2.19}$$

Similarly, from the general theory of toroidal compactifications of bounded symmetric domains (see [AMRT75] or [N80]), it follows that $\overline{\mathcal{A}_g}^V$ admits a stratification into locally closed subsets $S_{C([P])}$, parametrized by the cells $C([P])$ of $A_g^{\mathrm{tr,V}}$. We have also that

$$C([P]) \subset C([P']) \Leftrightarrow \overline{S_{C([P])}} \supset S_{C([P'])}. \tag{2.20}$$

The compactified Torelli map respects the toroidal structures of $\overline{\mathcal{M}_g}$ and $\overline{\mathcal{A}_g}^V$ (see [A04, Thm. 4.1]); more precisely, we have that (compare with Lemma 2.4.6):

$$\overline{\mathrm{t}}_g(S_{(\Gamma,w)}) \subset S_{C(\widetilde{M^*(\Gamma)})}. \tag{2.21}$$

Given a stacky subfan N of M_g^{tr} (in the sense of definition 2.1.1), consider the union of all the strata $S_{(\Gamma,w)}$ of $\overline{\mathcal{M}_g}$ such that $C(\Gamma, w) \in N$, and call it U_N. Similarly for any stacky subfan of $A_g^{\mathrm{tr,V}}$. It is easily checked, using formulas (2.19) and (2.20), that such a U_N is an open subset of $\overline{\mathcal{M}_g}$ (resp. $\overline{\mathcal{A}_g}^V$) containing $\mathcal{M}_g$ (resp. $\mathcal{A}_g$), and thus it is a partial compactification of $\mathcal{M}_g$ (resp. $\mathcal{A}_g$).

In particular we define $\mathcal{M}_g^{\mathrm{pl}} \subset \overline{\mathcal{M}_g}$ as the open subset corresponding to the stacky subfan $M_g^{\mathrm{tr,pl}} \subset M_g^{\mathrm{tr}}$ and $\mathcal{A}_g^{\mathrm{gr,cogr}} \subset \mathcal{A}_g^{\mathrm{cogr}} \subset \overline{\mathcal{A}_g}^V$ as the two open subsets corresponding to the two stacky subfans $A_g^{\mathrm{gr,cogr}} \subset A_g^{\mathrm{cogr}} \subset A_g^{\mathrm{tr,V}}$.

Observe that from formula (2.21) it follows that the compactified Torelli map $\overline{\mathrm{t}}_g$ takes values in $\mathcal{A}_g^{\mathrm{cogr}}$. Finally we can state the main result of this subsection.

Corollary 2.5.13. *Given a stable curve X, we have that $\overline{\mathrm{t}}_g(X) \in \mathcal{A}_g^{\mathrm{gr,cogr}}$ if and only if the dual graph Γ_X of X is planar.*

Proof. From formula (2.21), it follows that $t_g^{\rm tr}(X) \in S_{C(\widetilde{M^*(\Gamma_X)})}$. Therefore $t_g^{\rm tr}(X) \in \mathcal{A}_g^{\rm gr,cogr}$ if and only if $\widetilde{M^*(\Gamma_X)}$ is a graphic matroid. By the definition 2.1.27 of the simplification of a matroid, it follows easily that $\widetilde{M^*(\Gamma_X)}$ is a graphic matroid if and only if $M^*(\Gamma_X)$ is a graphic matroid. By combining Proposition 2.1.31 and Theorem 2.1.19, we finally get that $M^*(\Gamma_X)$ is a graphic matroid if and only if Γ_X is planar. □

The part if of the above Corollary was proved (using analytic techniques) by Namikawa in [N73, Thm. 5]. The converse was posed as a problem in [N80, Problem (9.31)(i)].

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