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Convex hypersurfaces evolving by volume preserving curvature flows

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Introduction

In this thesis we are going to analyse some motion by curvature flows of smooth embedded hypersurfaces. Given a smooth n-dimensional oriented manifold $\mathcal{M}$ and an embedding $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$, a *curvature flow* of $F_0(\mathcal{M})$ is a solution of a system of the kind

$$\begin{cases} \partial_t F(x,t) = v(D^2F, x, t)\nu(x,t) \\ F(x,0) = F_0(x), \end{cases} \tag{1}$$

where the function v is given and depends on second derivatives of the the unknown of the problem in the sense that depends on the curvature. We take ν as the unit normal vector pointing outward. The variable x is a local coordinate on the hypersurface, while the parameter t is thought as time.
The study of curvatures flows in a smooth setting starts with G.Huisken in [35], where the mean curvature flow of convex hypersurfaces was analysed. *Mean curvature flow* corresponds to the case $v = -H$ in (1), where H is the mean curvature of the solution. It can be checked that $-H\nu = \Delta_g F$, where Δ_g is the Laplace-Beltrami operator associated to the induced metric on the solution. Then, in this particular case, the evolution law resembles to a heat equation where the "laplacian" also depends on time. This similarity suggests the parabolicity of the problem and in particular the short time existence and uniqueness of the solution. If the maximal time of existence of the solution is finite, we say that the flow develops a *singularity*.
Huisken proved in his paper that any Euclidean, convex, compact hypersurface without boundary that evolves by mean curvature flow shrinks to a point in finite time. Moreover, the asymptotic profile of the evolving hypersurface becomes more and more spherical. This kind of singularity is called *round point*.

An interesting feature of mean curvature flow is that it is the gradient flow of the area functional, i.e. the one such that the area of the hypersurface decreases most rapidly among all velocities with fixed L^2 norm. Then, mean curvature flow naturally arises in many phenomena where a surface energy, represented by the area, is involved. Its origin indeed can be retraced in physics, in particular in the modelling of the evolution of interfaces, see the paper of Mullins [50] about the motion of grain boundary in two dimensions. Other interesting applications are developing, in which the regularizing effect of curvature flows is employed in the treatment of digital data (see for instance [25], [30], [47]).
Curvature flows also have important applications in geometry. One of these is the classification of submanifolds. A key point in this perspective is to find geometric properties that are invariants of the flow, and which determine a specific asymptotic behaviour. Huisken again, studying the mean curvature flow on the sphere in [38], showed that any hypersurface of the sphere $\mathbb{S}^n$ satisfying a certain condition on the curvatures is diffeomeorphic to a sphere $\mathbb{S}^{n-1}$. Another useful technique for the classification purpose is to define a surgery procedure, that is, a controlled way to pass through singularities of the flow. Roughly speaking, surgeries consist in a proper "cutting and gluing" near singularities without losing information about topology, in order to obtain something smooth and restart the flow till the next singular time, and so on. Huisken and Sinestrari first introduced this surgery method in [41] for mean curvature flow of two-convex Euclidean hypersurfaces. The central point is that the procedure ends in finitely many steps and the initial manifold is recognized to be diffeomorphic either to $\mathbb{S}^n$ or to a finite connected sum of $\mathbb{S}^{n-1} \times \mathbb{S}^1$. More recently, Huisken and Brendle developed in [39] a surgery method for mean curvature flow of mean convex surfaces in $\mathbb{R}^3$.
Curvature flows have been also employed in obtaining alternative proofs of known geometric inequalities, see for instance [49], [57].

After [35], the motion of convex hypersurfaces under various curvature flows has been widely investigated. Looking at the result of Huisken, an aim is to find under what conditions on the velocity and the initial datum the convergence to a round point occurs.
In most cases in the literature the function v in (1) is a decreasing function of the principal curvatures of the unknown of the problem. Monotonicity is a fundamental ingredient for the local existence and uniqueness of the solution, since ensures the parabolicity of the problem. Usually, the velocity

is assumed to be a homogeneous functions of the principal curvatures. The case of homogeneity degree equal to one is better known and investigated. There are results of convergence to a round point for generic speed functions, see for instance [5], [14]. When the homogeneity degree is grater than one, similar results hold under some additional hypotheses. Typically, one have to require a *pinching condition* on the initial submanifold i.e., roughly speaking, a control between the principal curvatures of the kind

$$\frac{\lambda_1}{\lambda_n} > C, \tag{2}$$

for a suitable constant $C > 0$, where $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ are the principal curvatures. Since (2) is generally not preserved by the flow, one formulates this condition in a different way, requiring for instance

$$\frac{K}{H^n} > C' \tag{3}$$

for some other constant $C' > 0$, where K is the Gauss curvature. The following inequalities hold in general:

$$\frac{\lambda_1}{\lambda_n} \leqslant 1, \qquad\qquad \frac{K}{H^n} \leqslant \frac{1}{n^n}$$

and it can be checked that, if the ratio in (3) is approaching $\frac{1}{n^n}$, which is the value reached only by the round spheres, then $\frac{\lambda_1}{\lambda_n}$ is approaching 1. The proof of the convergence is then usually based on the preserving and improvement of the pinching condition (3) during the flow. For some reference where this technique is employed, see [2], [26] [56] and [13], where a large class of speeds is considered. These results in higher homogeneity without requiring any pinching are only known for low dimensions and some specific speed, see [7], [53], [56] for results in low dimensions, or [12], [20], [27], [45] for powers of the Gaussian curvature.

In this thesis we study a variation of the flow (1) where a constraint on the hypersurfaces is given. This constraint produces an additional global term in the evolution equation:

$$\begin{cases} \partial_t F(x,t) = \left[v(D^2F, x, t) + h(t)\right] \nu(x,t) \\ F(x,0) = F_0(x) \end{cases} \tag{4}$$

with $h(t) = h(\mathcal{M}_t)$, where $\mathcal{M}_t = F(\mathcal{M}, t)$ is the evolving hypersurface. Flows of the kind (1) are usually said *standard*, while flows like (4) are called *constrained*. During the thesis we will omit, for brevity, the dependence of v on the derivatives of F. In our case, h comes out from the constraint on the hypersurfaces to have constant area or to enclose a region of constant volume. Many constrained flows were studied by various authors. In [36], Huisken proved the counterpart result of [35] for volume preserving mean curvature flow: the convergence to a round point in finite time is replaced by the convergence to a round sphere in infinite time. Results of convergence to a round sphere for generic velocities that are homogeneous of degree one in the principal curvature are given in [49]. For higher homogeneity, analogous results are known for general velocities under a pinching condition, see for instance [24].
The main original results in this thesis concern curvature flows that are not homogeneous of degree one in the principal curvatures. As in [8], [58] we do not employ any pinching condition, but we use the monotonicity of a suitable isoperimetric ratio of the hypersurface under the flow, which is a peculiar property of the volume/area preserving case. This property, in fact, does not hold in general for standard flows, see for instance [10] on non-convergence results for standard curvature flows of curves. Thanks to the control on the isoperimetric ratio, we get a uniform bound on the inner and outer radii of the evolving hypersurfaces. In this respect, constrained flows exhibit a better behaviour than the standard ones.
A general difficulty in studying such flows is that the constrain produces some non local extra terms in the evolution equations, making the applicability of the maximum principle more difficult. As a consequence, properties that are preserved by standard flows, does not hold any more in the constrained case. For some examples in the volume/area preserving mean curvature flow, see [23].

The thesis is articulated as follows. In the first chapter we recall some general preliminaries that will apply to any flow that we will see in the following, as theorems of existence and uniqueness of the solution for short times and the maximum principle, and we compute the evolution equations for the main geometric quantities associated to the hypersurface.

In Chapter 2 we analyse a generic volume preserving curvature flow, with velocity given by

$$\partial_t F(x,t) = [-\sigma(x,t) + h(t)]\nu(x,t)$$

where $\sigma(x,t) = E_k^\alpha(x,t)$ with $\alpha \geqslant \frac{1}{k}$ and E_k the k-th symmetric polynomial in the principal curvatures. These flows are related to the mixed volumes, which are quantities that generalize the notion of area and volume of a convex body, and that can be expressed as boundary integrals of the polynomials E_k. Using the monotonicity of a suitable mixed volume under the flow, we obtain a bound on the inner and outer radius of our hypersurface, which in turn implies a uniform upper bound on the speed and the global existence of the solution. By a further analysis, we can prove that E_k converges to its mean value in an integral sense and that the solution converges to a round sphere in the Hausdorff metric.
We can obtain a stronger result when we consider the volume preserving scalar curvature flow, corresponding to $k = 2$ and $\alpha = 1$. In this case we get additional estimates which give a uniform bound on the curvature. This allows us to show that the convergence to a round sphere is smooth and exponentially fast.

Chapter 3 is dedicated to the study of a volume/area preserving curvature flow driven by a generic function of the mean curvature:

$$\partial_t F(x,t) = [-\phi(H(x,t)) + h(t)]\nu(x,t)$$

with ϕ positive and increasing in H, but not necessarily homogeneous. Velocities of this kind have been considered sometimes in the past literature. We recall in particular the paper by Smoczyk [59] where the validity of differential Harnack inequalities was studied, and the one by Alessandroni and Sinestrari [3] where the singular profile of mean convex solutions was investigated for a particular class of functions. In our case, the additional hypotheses we put on the velocity are fairly general, being satisfied for a large class of functions as positive powers, exponentials and logarithms. We prove that any strictly convex compact hypersurface converges in infinite time to a round sphere that has the same area/volume as the initial datum. The convergence is smooth and exponentially fast. As for the previous flow, our proof does not employ any pinching condition and exploits the monotonicity of the isoperimetric ratio of the hypersurface under the flow.

In Chapter 4 we study the analogue of the flow in Chapter 3, but in the hyperbolic setting. In [37] Huisken showed, for mean curvature flow in a general ambient manifold $\mathcal{N}$, how the curvature of $\mathcal{N}$ interferes with the motion of the hypersurface. In particular, the negative sectional curvature of the ambient manifold contrasts the convergence to a round point. Inspired by the work of Cabezas-Rivas and Miquel [22], in our analysis we restrict our attention to the class of hypersurfaces which are convex by horospheres. *Convexity by horospheres* is the natural analogue, in the hyperbolic setting, of convexity and means that, at any point p of the hypersurfaces, there exists a horosphere passing trough p that encloses the hypersurface. This property translates in a condition on the curvature: denoting by $-a^2$ the sectional curvature of the hyperbolic space, any principal curvature of the hypersurface satisfies $\lambda_i \geqslant a$. Convexity by horospheres turns out to be a good choice when the ambient manifold is the hyperbolic space in the sense that, roughly speaking, this property is strong enough to offset the negative curvature of the ambient manifold and to be preserved along the flow. Also, since convexity by horospheres is stronger than the strict convexity of each hypersurface, we can relax some hypotheses of the speed function, getting some extra example of velocities not admitted in the Euclidean case. We prove that any compact hypersurface which is convex by horospheres converges smoothly and exponentially fast in infinite time to a geodesic sphere with the same area/volume as the initial datum. Also in this case, a key point is the monotonicity of the isoperimetric ratio, that allows to bound uniformly the inner and outer radii of the hypersurfaces.

We remark that for all the flows analysed in these three chapters, the exponential convergence is proved by an argument of the kind “improvement of a pinching condition”, but we do not require any extra hypotheses on the curvature. In fact, since we prove before that the hypersurfaces are smoothly approaching a sphere, then the pinching comes out “spontaneously” for times sufficiently large.

Finally, in Chapter 5 we present a partial result on entire Euclidean graphs moving by non homogeneous curvature flows. We take the main inspiration from [28, 29] where the authors study mean curvature flow of entire Euclidean graphs, and from [4], in which a generalization to velocities which are homogeneous of degree one in the principal curvatures is treated. The main difference with the previous chapters is that in this case we deal with non compact hypersurfaces, for which even the short time existence of the solution is not a priori guaranteed. By a technique taken from [4] and based on an approximation via compact hypersurfaces, we prove the long time existence of the solution for strictly convex initial data that are entire graphs of a Lipschitz function. But, instead of using some Harnack-type inequality as in [4], we employ just the dependence of the speed on the mean curvature. We show that the solution exists for any time, and is still an entire graph with Lipschitz constant uniformly bounded.

Contents

CHAPTER 1

Preliminaries and tools

1.1 Notation, basic definitions and formulas

In this section we recall some definitions and facts on Riemannian geometry, in particular with regard to embedded hypersurfaces.
From now on we are going to consider $F : \mathcal{M} \to \mathbb{R}^{n+1}$ to be an embedded orientable hypersurface of the Euclidean space without boundary, with local coordinates $(x^1, \cdots, x^n)$. We endow $\mathcal{M}$ with the induced metric $g = (g_{ij})$ given by

$$g_{ij}(x) = \left(\frac{\partial F(x)}{\partial x^i}, \frac{\partial F(x)}{\partial x^j} \right)$$

where $(\cdot, \cdot)$ is the standard Euclidean inner product. The measure on $\mathcal{M}$ is given in terms of g by $d\mu = \sqrt{\det g_{ij}}, dx$. The inverse of g_{ij} will be written as $g^{-1} = (g^{ij})$.
We denote by ∇ the Levi-Civita connection uniquely associated to g via the Christoffel symbols

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

The *second fundamental form* of $\mathcal{M}$ is the $(0, 2)$ symmetric tensor $A = (h_{ij})$ defined by

$$h_{ij}(x) = - \left(\frac{\partial^2 F(x)}{\partial x^i \partial x^j}, \nu(x) \right).$$

By definition of A, the following Gauss-Weingarten relations hold:

$$\frac{\partial^2 F}{\partial x^i \partial x^j} = \Gamma^k_{ij} \frac{\partial F}{\partial x^k} - h_{ij}\nu, \qquad \frac{\partial \nu}{\partial x^j} = h_{jl} g^{lm} \frac{\partial F}{\partial x^m}.$$

The eigenvalues of the second fundamental form are called *principal curvatures* and denoted by $\lambda_1, \ldots, \lambda_n$. The trace of A is the *mean curvature* $H = \lambda_1 + \cdots + \lambda_n$. We say that the hypersurface is *strictly convex* if all λ_i's are positive, while is *mean convex* if $H > 0$.

Let $a = \{a^{ij}\}$ be a positive definite $(2,0)$ tensor. We can consider the associated operator $\Delta_a = a^{ij}\nabla_i \nabla_j$, acting on functions or tensors on $\mathcal{M}$. If f is a smooth function on $\mathcal{M}$, then Δ_a is given in coordinates by

$$a^{ij}\nabla_i \nabla_j f = a^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k} \right).$$

When $a^{ij} = g^{ij}$, we recover the Laplace-Beltrami operator of $\mathcal{M}$.

As usual, we always sum on repeated indices, and we lower or lift tensor indices via g, e.g. the *Weingarten operator* is given by

$$h^i_j = h_{kj} g^{ik}.$$

Given tensors $T = (T^{i_1 \ldots i_s}_{j_1 \ldots j_r})$ and $S = (S^{i_1 \ldots i_s}_{j_1 \ldots j_r})$ on $\mathcal{M}$, we use brackets to denote their inner product

$$\langle T, S \rangle = T^{i_1 \ldots i_s}_{j_1 \ldots j_r} S^{j_1 \ldots j_r}_{i_1 \ldots i_s}.$$

In particular, the square of the norm is given by

$$|T|^2 = T^{i_1 \ldots i_s}_{j_1 \ldots j_r} T^{j_1 \ldots j_r}_{i_1 \ldots i_s}.$$

There holds the following relation between the square norm of the second fundamental form and the square of the mean curvature, due to arithmetic reasons:

$$|A|^2 \geqslant \frac{H^2}{n}.$$

The equality is achieved if all the principal curvatures are equal to each other. Also, if the hypersurface is convex, the norm of the second fundamental form is bounded by the mean curvature:

$$|A| \leqslant H.$$

Given a point $q \in \mathbb{R}^{n+1}$, the *support function* of $\mathcal{M}$ with respect to q is

$$u_q(x) := (F(x) - q, \nu(x)).$$

The subscript q will be omitted whenever there will be no ambiguity.

1.2 Curvature flows

A *geometric flow* is a motion of a Riemannian manifold given by some partial differential equation that involves geometric quantities associated to the manifold. If these quantities concern the curvature we say that the manifold evolves by a *curvature flow*.

1.2.1 Short time existence

We focus our attention on curvature flows of this kind: given an embedded hypersurface $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$ as in the previous section, we look for a family of maps $F : \mathcal{M} \times [0,T) \to \mathbb{R}^{n+1}$, with $F_t := F(\cdot,t) : \mathcal{M} \to \mathbb{R}^{n+1}$, solution of

$$\begin{cases} \partial_t F(x,t) = -S(\lambda_1(x,t),\dots,\lambda_n(x,t))\nu(x,t) \\ F(x,0) = F_0(x), \end{cases} \tag{1.1}$$

where F_0 is a smooth embedding of a n-dimensional manifold taken as in the previous section, and ν is chosen pointing outward. S is an homogeneous, symmetric, increasing function of the principal curvatures, which is positive on the positive cone $\Gamma_+ := \{(\lambda_1,\dots,\lambda_n) \ : \ \lambda_1 > 0, \dots \lambda_n > 0\}$. We refer to these flows as *standard contracting flows*.
Since S is symmetric in the principal curvatures, it can be also thought as a $GL(n)$-invariant function of the second fundamental form and the metric. Vice versa, any $GL(n)$-invariant function of the second fundamental form and the metric can be written as a symmetric function of the principal curvatures (see for instance [14, §2] for more details). Then we will use alternately the same notation without ambiguity:

$$S(\lambda_1,\dots,\lambda_n) = S(h^i_j) = S(h_{ij}, g_{ij}).$$

Short time existence for the system (1.1) can be deduced only if it is parabolic. The linearization of the evolution equation in (1.1) is given by

$$\partial_t G = \frac{\partial S}{\partial h_{ij}} g^{ik} g^{jl} \left(\frac{\partial^2 G}{\partial x^k \partial x^l}, \nu \right) \nu + \text{ lower order terms.} \tag{1.2}$$

This equation is degenerate in tangential directions. It can be checked that this is related to the invariance of the flow via tangential diffeomorfisms. Roughly speaking, this property is due to the very geometric nature

of the flow, since the evolution law does not depend on the choice of the parametrization of the hypersurface, but only on its "shape".
We have the following theorem for the local existence of the solution. For more details on this part, see for example [40].

Theorem 1.2.1. *If $F_0 : \mathcal{M} \to \mathbb{R}^n$ is a smooth, compact hypersurface without boundary such that*

$$\frac{\partial S}{\partial \lambda_i} > 0 \quad \forall i = 1, \ldots, n$$

then (1.1) has a unique smooth solution at least on a short time interval $[0, T)$, $T > 0$.

Moreover, we have the following result, see [9], [14] and [40] for more details.

Theorem 1.2.2. *Assume $S(\lambda_1, \ldots, \lambda_n) = f(s(\lambda_1, \ldots, \lambda_n))$, where f is positive and increasing, and s is symmetric, homogeneous, concave and increasing in each variable. Let $[0, T)$ be the maximal time existence interval for the solution of (1.1). If $T < +\infty$, then either $\liminf_{t \to T} \min_{\mathcal{M}_t} \frac{\partial S}{\partial \lambda_i} = 0$ for some i, or $\limsup_{t \to T} \max_{\mathcal{M}_t} |A|^2 = +\infty$.*

This theorem says that the flow (1.1), with S satisfying certain properties, can stop only if the curvature blows up, or if the parabolicity doesn't hold any more.

1.2.2 A priori estimates: maximum principle and its consequences

A very powerful tool in the study the behaviour of the solutions of parabolic equations is the *maximum principle* that, in particular, allows to estimate a certain quantity evolving in time by some constant that only depends on the initial datum. For more details, see for example [60]. There the maximum principle is given for the Laplace-Beltrami operator, but the same statements hold for any parabolic operator $\Delta_a = a^{ij}\nabla_i\nabla_j$.

Theorem 1.2.3. *Let $t \mapsto (\mathcal{M}, g_{ij}(t))$ a smooth flow of compact manifolds. Let $v, w : \mathcal{M} \times [0, T^*] \to \mathbb{R}$ be C^2 functions such that $v(x, 0) \geqslant w(x, 0)$ for all $x \in \mathcal{M}$. Given a parabolic operator $\Delta_{a(t)}$, suppose*

$$\partial_t v(x, t) \geqslant \Delta_{a(t)} v(x, t) + \nabla_{X(t)} v(x, t) + F(v(x, t), t)$$

$$\partial_t w(x,t) \leqslant \Delta_{a(t)} w(x,t) + \nabla_{X(t)} w(x,t) + F(w(x,t),t)$$

for all $(x,t) \in \mathcal{M} \times [0,T^*]$, *where, for each time* t, $X(t)$ *is a vector field and* $F(t,\cdot) : \mathbb{R} \to \mathbb{R}$ *is a Lipschitz function. Then*

$$v(x,t) \geqslant w(x,t) \quad \forall t \in [0,T^*].$$

Corollary 1.2.4. *Let* $t \mapsto (\mathcal{M}, g_{ij}(t))$ *a smooth flow of compact manifolds. Let* $X(t)$, $\Delta_{a(t)}$ *as in the hypotheses of Theorem 1.2.3, and* $f : \mathcal{M} \times [0,T^*] \to \mathbb{R}$ *be* C^2 *functions such that, on* $\mathcal{M} \times [0,T^*]$,

$$\partial_t f(x,t) \geqslant \Delta_{a(t)} f(x,t) + \nabla_{X(t)} f(x,t)$$

or, respectively,

$$\partial_t f(x,t) \leqslant \Delta_{a(t)} f(x,t) + \nabla_{X(t)} f(x,t).$$

Then, for any $t \in [0,T^*]$,

$$f(x,t) \geqslant \min_{y \in \mathcal{M}} f(y,0)$$

or, respectively,

$$f(x,t) \leqslant \max_{y \in \mathcal{M}} f(y,0).$$

We also give the maximum principle version for tensors, see [33, Theorem 9.1].

Theorem 1.2.5. *Let* $t \mapsto (\mathcal{M}, g_{ij}(t))$ *a smooth flow of compact manifolds,* $\Delta_{a(t)}$ *a parabolic operator. Let* v^k *be a vector field and* M_{ij} *a symmetric tensor on the evolving manifolds, and* $N_{ij} = p(M_{ij}, g_{ij})$ *a polynomial in* M_{ij} *formed by contracting products of* M_{ij} *with itself using the metric. Suppose that, on* $[0,T^*]$,

$$\partial_t M_{ij} = \Delta_a M_{ij} + u^k \nabla_k M_{ij} + N_{ij}.$$

Suppose also that N_{ij} *satisfies the null-eigenvector condition i.e., for all null-eigenvector* w^k *of* M_{ij}, $N_{ij} w^i w^j \geqslant 0$. *If* $M_{ij} \geqslant 0$ *at* $t = 0$, *then it remains so on* $[0,T^*]$.

Two very important applications of the maximum principle are the following results.

Proposition 1.2.6. *Let $F' : \mathcal{M}' \times [0,T) \to \mathbb{R}^{n+1}$ and $F'' : \mathcal{M}'' \times [0,T) \to \mathbb{R}^{n+1}$ be two hypersurfaces moving by (1.1), with $\mathcal{M}'$ compact. Then the distance between them is nondecreasing in time.*

As a consequence, we obtain the following corollary, called *avoidance property.*

Corollary 1.2.7. ***(Avoidance property)*** *Let F', F'' as in the hypotheses of Proposition 1.2.6. If $F'(\mathcal{M}',0)$ and $F''(\mathcal{M}'',0)$ are disjoint, then $F'(\mathcal{M}',t)$ and $F''(\mathcal{M}'',t)$ keep disjoint for all times $t \in [0,T)$.*

Proposition 1.2.8. *If the initial hypersurface is compact and embedded, then it remains embedded during the flow (1.1).*

For the proofs of Propositions 1.2.6 and 1.2.8 see for example [48]. The proofs there are given for the mean curvature flow, but they still hold for a generic flow (1.1) with S as in our hypotheses.

1.3 Constrained flows

In Chapters 2, 3 and 4 we are going to consider flows of the kind

$$\begin{cases} \partial_t F(x,t) = \left[-S(\lambda_1(x,t),\dots,\lambda_n(x,t)) + h(t)\right]\nu(x,t) \\ F(x,0) = F_0(x), \end{cases} \tag{1.3}$$

with S, ν and $F_0(\mathcal{M})$ as in (1.1). The function $h(t)$ only depends on time. Such flows are called *constrained flows*, since typically the term h derives from some constraint on the hypersurfaces, and contrasts the contractive thrust produced by S. In our case, h will be chosen in order to force the area or the volume enclosed by the hypersurface to be constant.
Since the term $h(t)$ only depend on time, its presence does not interfere with the parabolicity of the flow. Then Theorems 1.2.1 and 1.2.2 are still valid for (1.3), as well as for the maximum principle. Nevertheless, the avoidance property does not hold any more, since in the evolution equations needed for the proof an extra non local term appears, compromising the applicability of the maximum principle.
Typically, the constrained flows that one analyses present a kind of balancing between the two terms in the speed function. Then the results are about the asymptotic behaviour and convergence to an equilibrium position. This will be our case too.

1.4 Evolution equations

In [40], Huisken and Polden give the evolution equations for the main geometric quantities associated to any flow of the kind

$$\partial_t F(x,t) = V(x,t)\nu(x,t), \tag{1.4}$$

provided that is a nonlinear parabolic equation of the second order. They consider hypersurfaces of any ambient manifolds, but here we restrict our attention to the Euclidean ambient space. We have the following proposition, coming from Lemma 7.4, Lemma 7.5 and Lemma 7.6 in [40].

Proposition 1.4.1. *Under a flow of the kind (1.4), the following equations hold:*

$$\begin{aligned}
\partial_t g_{ij} &= 2V h_{ij} \\
\partial_t g^{ij} &= -2V h^{ij} \\
\partial_t d\mu &= V H d\mu \\
\partial_t \nu &= -\nabla V \\
\partial_t h_{ij} &= -\nabla_i \nabla_j V + V h_{ik} h_j^k \\
\partial_t h_j^i &= -\nabla^i \nabla_j V + V h_k^i h^{kj} \\
\partial_t H &= -\Delta V - |A|^2
\end{aligned}$$

The following identity will be useful in the next chapters in order to rewrite some of the evolution terms in a more convenient way. It can be recovered by the proof of Corollary 3.3 in [40].

Proposition 1.4.2. *If Λ is a smooth symmetric function of the principal curvatures, its Hessian matrix satisfies*

$$\begin{aligned}
\nabla_i \nabla_j \Lambda = {} & \frac{\partial \Lambda}{\partial h_{kl}} \nabla_k \nabla_l h_{ij} + \frac{\partial^2 \Lambda}{\partial h_{kl} \partial h_{pq}} \nabla_i h_{kl} \nabla_j h_{pq} \\
& - \frac{\partial \Lambda}{\partial h_{kl}} (h_{kl} h_i^m h_{mj} - h_k^m h_{il} h_j^m + h_{kj} h_i^m h_l^m - h_k^m h_{ij} h_{ml}).
\end{aligned}$$

CHAPTER 2

Volume preserving flow by scalar curvature in the Euclidean space

2.1 Presentation of the problem

Let $\mathcal{M}$ be an oriented, compact n-dimensional manifold without boundary. Consider the problem (1.3) given by

$$\begin{cases} \partial_t F(x,t) = [-\sigma(x,t) + h(t)]\nu(x,t) \\ F(x,0) = F_0(x), \end{cases} \tag{2.1}$$

where:

- ν denotes the outer unit normal vector of the evolving hypersurface $\mathcal{M}_t := F_t(\mathcal{M})$;
- $\sigma(x,t) = E_k^\alpha(x,t)$ with $\alpha \geqslant \frac{1}{k}$ and E_k the k-th symmetric polynomial in the principal curvatures, i.e.

  $$E_k(x,t) = \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} \lambda_{i_1}(x,t) \dots \lambda_{i_k}(x,t),$$

  with $\lambda_i, \dots \lambda_j$ the principal curvatures of $\mathcal{M}_t$ and $k = 1, \dots, n$;
- The function $h(t)$ is defined as

  $$h(t) := \frac{1}{A(\mathcal{M}_t)} \int_{\mathcal{M}_t} \sigma d\mu, \tag{2.2}$$

where $A(\mathcal{M}_t)$ is the n-dimensional measure of $\mathcal{M}_t$.

Such a definition of $h(t)$ ensures that the volume $Vol(\Omega_t)$ is preserved by the flow, where Ω_t is the $(n+1)$-region bounded by $\mathcal{M}_t$. We will prove the following result.

Theorem 2.1.1. *Let $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$, with $n \geqslant 1$, be a smooth embedding of an oriented, compact n-dimensional manifold without boundary, such that $F_0(\mathcal{M})$ is strictly convex. Then the flow (2.1) has a unique smooth solution, which exists for any time $t \in [0, \infty)$. The solution is strictly convex and converges in the Hausdorff distance, as $t \to \infty$, to a round sphere with the same volume as the initial datum. Furthermore, if $\alpha = 1$ and $k = 2$, the convergence is smooth and exponentially fast.*

Notice that there is no requirement of any pinching condition of the curvatures of the initial datum except for the strict convexity.

Short time existence and evolution equations. By Theorem 1.2.1 in Chapter 1, a flow of the form (2.1) is parabolic at least on a short time interval if

$$\frac{\partial \sigma}{\partial \lambda_i}(x, 0) > 0 \quad i = 1, \ldots, n \tag{2.3}$$

and then admits a unique smooth solution for short times. Condition (2.3) is guaranteed, in our case, by the strict convexity of $\mathcal{M}_0$. Moreover, by Theorem 1.2.2, we have the following result.

Theorem 2.1.2. *Let $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$ be a smooth embedding of an oriented, compact n-dimensional manifold without boundary, such that $F_0(\mathcal{M})$ is strictly convex. Then the flow (2.1) has a unique smooth solution $\mathcal{M}_t$ defined on a maximal time interval $[0, T)$. If $T < +\infty$, then either*

$$\liminf_{t \to T} \min_{\mathcal{M}_t} \frac{\partial \sigma}{\partial \lambda_i} = 0 \;\; \exists i, \quad \textit{or} \quad \limsup_{t \to T} \max_{\mathcal{M}_t} |A|^2 = +\infty.$$

We give now the evolution equations for the main geometric quantities associated to the flow (2.1). They can be obtained from Proposition 1.4.1 taking $V(x, t) = -\sigma(x, t) + h(t)$ in (1.4).

Proposition 2.1.3. *We have the following evolution equations under the flow (2.1):*

$$
\begin{aligned}
\partial_t g_{ij} &= 2(-\sigma + h)h_{ij},\\
\partial_t g^{ij} &= -2(-\sigma + h)h^{ij},\\
\partial_t \nu &= \nabla\sigma,\\
\partial_t d\mu &= H(-\sigma + h)d\mu,\\
\partial_t h^i_j &= \nabla^i\nabla_j\sigma - (h-\sigma)h^i_l h^l_j\\
&= \Delta_{\dot\sigma} h^i_j + \ddot\sigma(\nabla^i A, \nabla_j A) + tr_{\dot\sigma}(h_{lm}h^m_n)h^i_j - (h + (\alpha k - 1)\sigma)h^i_n h^n_j,\\
\partial_t H &= \Delta_{\dot\sigma} H + tr_{g^{-1}}\left[\ddot\sigma(\nabla_i A, \nabla_j A)\right] + H tr_{\dot\sigma}(h_{ml}h^l_r) - (h + (\alpha k - 1)\sigma)|A|^2,\\
\partial_t \sigma &= \Delta_{\dot\sigma}\sigma + (\sigma - h)tr_{\dot\sigma}(h_{ml}h^l_r),\\
\partial_t u &= \Delta_{\dot\sigma} u + tr_{\dot\sigma}(h_{lm}h^m_n)u - (\alpha k + 1)\sigma + h.
\end{aligned}
$$

Proof. The first four evolution equations follow directly from Proposition 1.4.1 setting $V = -\sigma + h$. By Proposition 1.4.1 we also have

$$\partial_t h_{ij} = \nabla_i\nabla_j\sigma - (\sigma - h)h_{ik}h^k_j. \tag{2.4}$$

By Proposition 1.4.2 with $\Lambda = \sigma$, we have

$$
\begin{aligned}
\nabla_i\nabla_j\sigma &= \Delta_{\dot\sigma} h_{ij} + \ddot\sigma(\nabla_i A, \nabla_j A)\\
&\quad \dot\sigma^{lm}(h_{lm}h^n_i h_{nj} - h^n_l h_{im}h^n_j + h_{lj}h^n_i h^n_m - h^n_l h_{ij}h_{nm})\\
&= \Delta_{\dot\sigma} h_{ij} + \ddot\sigma(\nabla_i A, \nabla_j A) + tr_{\dot\sigma}(h_{ln}h^n_m)h_{ij}\\
&\quad - \alpha k h_{in}h^n_j + \dot\sigma^{kl}(h^n_l h_{im}h^n_j - h_{lj}h^n_i h^n_m)
\end{aligned}
\tag{2.5}
$$

where in the last equality we used Lemma 2.1.4. Furthermore,

$$\dot\sigma^{kl}(h^n_l h_{im}h^n_j - h_{lj}h^n_i h^n_m) = 0$$

since all the other addenda in the right side of the last equality in (2.5) are symmetric in i and j, as well as $\nabla_i\nabla_j\sigma$.
From the evolution of h_{ij} and g^{ij}, the evolutions of h^i_j and H can be easily computed. Once we have the evolution of h^i_j, we can compute

$$
\begin{aligned}
\partial_t\sigma &= \frac{\partial\sigma}{\partial h^i_j}\partial_t h^i_j = \frac{\partial\sigma}{\partial h^i_j}\nabla^i\nabla_j\sigma - \frac{\partial\sigma}{\partial h^i_j}(h-\sigma)h^i_l h^l_j\\
&= \Delta_{\dot\sigma}\sigma + (\sigma - h)tr_{\dot\sigma}(h_{il}h^l_j).
\end{aligned}
$$

Finally, for the evolution of the support function see [24, Lemma 3.5], formula (3.4).

2.1.1 Symmetric polynomials and convex sets

It is convenient to define the symmetric polynomials also for $k = 0, n+1$ setting $E_0 \equiv 1$ and $E_{n+1} \equiv 0$. To simplify some formulas, it is useful to introduce the normalized symmetric polynomials

$$\tilde{E}_k := \binom{n}{k}^{-1} E_k, \qquad k = 0, \ldots, n, \tag{2.6}$$

which satisfy $\tilde{E}_k(1, \ldots, 1) = 1$. For the purposes of this thesis, these functions will only be evaluated in the positive cone Γ_+.

The polynomials E_k and $\tilde{E}_k$ can be also regarded as a function of the Weingarten operator of $\mathcal{M}$. We will use the same symbol in the two cases, since the meaning will be clear from the context. We also recall some well known properties, see e.g. Theorem 2.3 in [42], Lemma 2.1 in [24] and the references therein, and [51].

Lemma 2.1.4. *The following relations hold, for any $k = 1, \ldots, n$ and $(\lambda_1, \ldots, \lambda_n) \in \Gamma_+$.*

(i) $\frac{\partial E_k}{\partial \lambda_i} \lambda_i^2 = HE_k - (k+1)E_{k+1} \geqslant \frac{k}{n} HE_k$.

(ii) $\tilde{E}_{k+1}^{\frac{1}{k+1}} \leqslant \tilde{E}_k^{\frac{1}{k}}$*, with equality if and only if* $\lambda_1 = \cdots = \lambda_n$ *and* $k < n$.

(iii) As a function on $\mathcal{M}_t$, $\nabla^i \frac{\partial E_k}{\partial h^i_j} = 0$ *for any* $j = 1, \ldots, n$.

(iv) If $\sigma = E_k^\alpha$*, then* $\frac{\partial \sigma}{\partial \lambda_i} \lambda_i = \alpha k \sigma$.

During this chapter, we will see that flows of the kind (2.1) are related to mixed volumes. *Mixed volumes* are a classical notion in convex analysis, see e.g. [15, 21, 55]. We recall here the definitions and properties required for our analysis.

Given a compact convex set $\Omega \subset \mathbb{R}^{n+1}$ and $t > 0$, consider the set

$$\Omega + tB := \{x + ty \ : \ x \in \Omega, |y| \leqslant 1\}.$$

It can be proved, see [21, §19.3.6] that the volume of this set is a polynomial of degree $n+1$ in t and can be therefore written as

$$\mathrm{Vol}(\Omega + tB) = \sum_{i=0}^{n+1} \binom{n+1}{i} \alpha_i t^i,$$

for suitable coefficients α_i depending on Ω. We then define the *k-th mixed volume* of Ω as $V_i(\Omega) = \alpha_{n+1-i}$, for $i = 0, \ldots, n+1$. It can be proved that, for any Ω,

$$V_{n+1}(\Omega) = \mathrm{Vol}\,(\Omega), \qquad V_n(\Omega) = A(\partial\Omega), \qquad V_0 = \alpha_n,$$

where α_n is the volume of the unit sphere in $\mathbb{R}^{n+1}$. Thus, mixed volumes can be regarded as a generalization of volume and area. They are known also as *cross sectional measures* or *quermassintegrals*.

Mixed volumes depend continuously on the set: if $\{\Omega_l\}$ is a sequence of convex sets converging to Ω in the Hausdorff topology, then

$$V_i(\Omega_l) \to V_i(\Omega), \qquad i = 1, \ldots, n+1.$$

If the convex set Ω has a smooth boundary, mixed volumes admit an equivalent characterization as boundary integrals of the elementary symmetric functions of the curvatures. In fact, it can be proved that

$$V_{n-k}(\Omega) = \begin{cases} Vol(\Omega) & \text{if } k = -1 \\ (n+1)^{-1} \int_{\mathcal{M}_t} \tilde{E}_k d\mu & \text{if } k = 0, 1, \ldots, n-1. \end{cases}$$

An important result related to the mixed volumes are the so-called Minkowski identities, which say the following. On any closed convex hypersurface $\mathcal{M}$ and for any $l = 1, \ldots, n$, we have

$$\int_{\mathcal{M}} \tilde{E}_l d\mu = \int_{\mathcal{M}} u\, \tilde{E}_{l+1}\, d\mu, \tag{2.7}$$

where $u = u_{p_0}$ is the support function centred at any point and $\tilde{E}_l, \tilde{E}_{l+1}$ are defined as in (2.6) in Chapter 2. These properties were originally proved by Minkowski and Kubota. It was later proved by Hsiung [34] that they also hold without the convexity assumption.

A remarkable property of mixed volumes is the Alexandrov-Fenchel inequality, see e.g. [21, §20]. Its statement is somehow technical and will not be needed here in its general form. We recall instead some special inequalities that can be recovered from Alexandrov-Fenchel's one. For instance, for any $0 < m < l \leqslant n+1$, there exists a constant $C(l, m, n) > 0$ such that, for any compact convex set $\Omega \subset \mathbb{R}^{n+1}$ with non empty interior, we have

$$V_l^m(\Omega) \leqslant C(l, m, n) V_m^l(\Omega), \tag{2.8}$$

and the equality occurs only for spheres. This result can be viewed as a generalization of the isoperimetric inequality, taking $l = n+1$ and $m = n$. Then we can define the $k-th$ *generalized isoperimetric ratio* as

$$\mathcal{I}_k(\Omega) = \frac{V_k^{n+1}(\Omega)}{Vol^k(\Omega)}.$$

$\mathcal{I}_k(\Omega)$ reaches its minimum only for spheres. The standard isoperimetric ratio $\mathcal{I}_n(\Omega)$ will be denoted just by $\mathcal{I}(\Omega)$.
The next result, called Favard inequalities, can also be deduced from Alexandrov-Fenchel's inequality, see [21, §20]. Given a compact convex set $\Omega \subset \mathbb{R}^{n+1}$ with nonempty interior and $i = 1, \ldots, n-1$, we have

$$V_i^2(\Omega) \geqslant V_{i+1}(\Omega)V_{i-1}(\Omega), \tag{2.9}$$

and the inequality is strict unless Ω is a sphere. This can be easily generalized as follows: for any $l = 1, \ldots, n-1$, we have

$$V_{n-l}^{l+1}(\Omega) \geqslant V_{n-l-1}^{l}(\Omega)V_n(\Omega). \tag{2.10}$$

Again, the inequality is strict unless Ω is a sphere. To see why (2.10) holds, observe that the case $l = 1$ is immediate from (2.9). The case of a general l is obtained by induction. Suppose in fact that the assertion is true for $l-1$, that is,

$$V_{n-l+1}^{l}(\Omega) \geqslant V_{n-l}^{l-1}(\Omega)V_n(\Omega).$$

On the other hand, a direct application of (2.9) gives

$$V_{n-l}^{2l}(\Omega) \geqslant V_{n-l-1}^{l}(\Omega)V_{n-l+1}^{l}(\Omega).$$

Multiplying the two inequalities, we obtain (2.10).

We recall that the *inner* [resp. *outer*] *radius* of Ω is the radius of the biggest $(n+1)$-dimensional sphere contained in Ω [resp. the smallest $(n+1)$-dimensional sphere that contains Ω]. We indicate inner and outer radii respectively by $R_-(\Omega)$ and $R^+(\Omega)$. We call *inball* [resp. *extball*] a ball contained in Ω of radius $R_-(\Omega)$ [resp. that contains Ω of radius $R^+(\Omega)$]. We will need the following property, that allows to control the "shape" of a convex domain in function of its isoperimetric ratio. It can be found in [8, Proposition 5.1] or [43, Lemma 4.4].

Proposition 2.1.5. *For any $n \geqslant 1$ and $c_1 > 0$ there exist $c_2 = c(c_1, n)$ with the following property. Let $\Omega \subset \mathbb{R}^n$ be a compact, convex set with non empty interior such that $\mathcal{I}(\Omega) \leqslant c_1$. Then Ω satisfies*

$$\frac{R^+(\Omega)}{R_-(\Omega)} \leqslant c_2.$$

□

2.2 Long time existence

2.2.1 Preserving of convexity

We want to show the that strict convexity is a property preserved by the flow. We will follow the strategy used in [14], where the authors consider flows driven by general homogeneous speeds in the standard non volume-preserving case. In our case we have some additional terms in the computations due to the presence of $h(t)$, but we will see easily that these terms do not interfere with the success of the proof.

Let introduce some preliminaries and notations. Since $\mathcal{M}_0$ is a convex hypersurface, we can use the Gauss map parametrization given by

$$X : \mathbb{S}^n \longrightarrow \mathcal{M}_0 \subset \mathbb{R}^{n+1}$$

$$z \longmapsto u(z)z + \bar{\nabla} u(z)$$

which takes z in the unique point in $\mathcal{M}_0$ with outward normal direction z. Here u is the support function $u(z) = \sup_{q \in \Omega_0}(q, z) = (X(z), z)$, and $\bar{\nabla}$ is the gradient on the sphere $\mathbb{S}^n$ with respect to the standard metric $\bar{g}_{ij}$. If we set

$$\tau_{ij} = \overline{\nabla}_i \overline{\nabla}_j u + \bar{g}_{ij} u$$

then it can be checked that the eigenvalues of τ_{ij} with respect to $\bar{g}$ are the principal radii of curvature $r_1, \ldots, r_n$, with $r_i = \lambda_i^{-1}$. To describe the flow in this setting, it is convenient to define

$$\Phi(r_1, \ldots, r_n) = \left(\sigma\left(\frac{1}{r_1}, \ldots, \frac{1}{r_n}\right)\right)^{-1/\alpha k}.$$

It is well known that Φ is a concave function (see for instance [46]). As Φ is concave, the function $\sigma^{1/\alpha k} = E_k^{1/k}$ is said to be *inverse concave*. Inverse

concavity plays an important role in the study of geometric flows, see for instance [11].
We can also regard Φ as functions of τ_{ij} and we can write the flow equation as

$$\partial_t u(z,t) = -\left(\Phi(\tau_{ij}(z,t))\right)^{-1} + h(t).$$

Denote the derivatives of Φ with respect to τ_{ij} as

$$\dot{\Phi}^{lm} = \frac{\partial \Phi}{\partial \tau_{lm}} \qquad\qquad \ddot{\Phi}^{lm,pq} = \frac{\partial^2 \Phi}{\partial \tau_{lm} \partial \tau_{pq}}.$$

Then τ_{ij} satisfies the following equation.

Lemma 2.2.1.

$$\begin{aligned}\partial_t \tau_{ij} = \alpha k \Phi^{-\alpha k-1}[\dot{\Phi}^{lm}\overline{\nabla}_l\overline{\nabla}_m\tau_{ij} + \ddot{\Phi}^{lm,pq}\overline{\nabla}_i\tau_{pq}\overline{\nabla}_j\tau_{lm} - (\alpha k+1)\Phi^{-1}\overline{\nabla}_i\Phi\overline{\nabla}_j\Phi] \\ - \alpha k \Phi^{-\alpha k-1}\dot{\Phi}^{lm}\bar{g}_{lm}\tau_{ij} + (\alpha k-1)\Phi^{-\alpha k}\bar{g}_{ij} + h(t)\bar{g}_{ij}.\end{aligned} \tag{2.11}$$

Proof. Follows from [14, Lemma 10], also noticing that the additional term involving $h(t)$ comes out from

$$\begin{aligned}\partial_t \tau_{ij} &= \partial_t(\bar{\nabla}_i\bar{\nabla}_j u + \bar{g}_{ij}u) = \bar{\nabla}_i\bar{\nabla}_j\partial_t u + \bar{g}_{ij}\partial_t u \\ &= -\bar{\nabla}_i\bar{\nabla}_j\Phi^{-1} - \bar{g}_{ij}\Phi^{-1} + h(t)\bar{g}_{ij}.\end{aligned}$$

□

Proposition 2.2.2. *Let $\mathcal{M}_t$ be a convex solution of (2.1) on a time interval $[0,T_0)$ and suppose that $h(t) \leqslant h^*$ for every $t \in [0,T_0)$ for a suitable $h^* > 0$. If we set $\lambda_{\min}(t) = \min_{x\in\mathcal{M}_t}\lambda_1(x,t)$, then we have*

$$\lambda_{\min}(t) \geqslant \frac{1}{\lambda_{\min}(0)^{-1} + h^* t}.$$

Proof. On $[0,T_0)$ we use the Gauss map parametrization and we have the evolution equation of τ_{ij} given by Lemma 2.2.1. This is a parabolic equation where the first order terms give a negative contribution, due to the concavity of Φ. Furthermore, by Lemma 4 in [14],

$$\dot{\Phi}^{lm}\bar{g}_{lm} = \sum_{i=1}^{n}\frac{\partial\Phi}{\partial r_i} \geqslant \binom{n}{k}^{-1/\alpha k}.$$

Also, observe that

$$\Phi \leqslant \binom{n}{k}^{-1/\alpha k} r_1.$$

Then, at a maximum eigenvalue of τ_{ij}, the reaction terms in (2.11) give

$$\begin{aligned}
-\alpha k \Phi^{-\alpha k-1} \dot{\Phi}^{lm} \bar{g}_{lm} \tau_{ij} + (\alpha k - 1)\Phi^{-\alpha k} \bar{g}_{ij} + h(t)\bar{g}_{ij} \\
\leqslant -\binom{n}{k}^{-1/\alpha k} \Phi^{-\alpha k-1} r_1 \bar{g}_{ij} + h^* \bar{g}_{ij} \\
\leqslant h^* \bar{g}_{ij}.
\end{aligned}$$

Then, by the maximum principle for tensors recalled in Theorem 1.2.5, the radii can increase, but only by an amount which is bounded as long as $h(t)$ is bounded. More precisely, if $r_1(0)$ denotes the largest radius at time 0, the maximum principle for tensors implies that the matrix $\tau_{ij} - (r_1(0) + h^* t)\bar{g}_{ij}$ remains negative definite for all times, that is, the principal radii on $\mathcal{M}_t$ are bounded from above by $r_1(0) + h^* t$. The assertion follows. □

Corollary 2.2.3. *Let $[0, T)$ be the maximal interval of existence of the solution of (2.1). Then $\mathcal{M}_t$ is convex for all $t \in [0, T)$. In addition, if $T < +\infty$, then the curvature of $\mathcal{M}_t$ becomes unbounded as $t \to T$.*

Proof. As $h(t)$ is bounded on any compact subinterval of $[0, T)$, the convexity of $\mathcal{M}_t$ follows from the previous proposition. If $T < +\infty$ and the curvature is bounded, then we also have a bound on $h(t)$ for $t \in [0, T)$, and the previous proposition shows that $\mathcal{M}_t$ remains uniformly convex as $t \to T$. This shows that the flow is uniformly parabolic and has bounded curvature on $[0, T)$. Well known regularity results, see e.g. [49, 24, 13], give uniform bounds on all derivatives of the solution and imply that $\mathcal{M}_t$ converges to a smooth strictly convex limit as $t \to T$. Then we can restart the flow, in contradiction with the maximality of T. □

2.2.2 A monotone quantity

An important feature of the flow (2.1) is the monotonicity of a suitable isoperimetric ratio. This property was observed by M. Gage [31] for the area preserving mean curvature flow.
First of all notice, with the following Lemma, that the definition of $h(t)$ given by (2.2) keeps the volume constant during the flow. Denote by Ω_t the $(n+1)$-dimensional region enclosed by $\mathcal{M}_t$.

Lemma 2.2.4. *The volume $Vol(\Omega_t)$ is preserved by the flow (2.1).*

Proof.

$$\begin{aligned}\partial_t Vol(\Omega_t) &= \int_{\mathcal{M}_t} (-\sigma + h) d\mu \\ &= -\int_{\mathcal{M}_t} \sigma d\mu + \frac{1}{A(\mathcal{M}_t)} \int_{\mathcal{M}_t} \sigma d\mu \int_{\mathcal{M}_t} d\mu = 0\end{aligned}$$

□

We observe that there is a particular mixed volume, related to k, that exhibits a property of monotonicity.

Lemma 2.2.5. *Along the flow (2.1),with $\sigma = E_k^\alpha$ for a given $k = 1, 2, \ldots n$, we have*

$$\frac{d}{dt} \int_{\mathcal{M}_t} E_{k-1} d\mu \leqslant 0,$$

and the inequality is strict unless $\mathcal{M}_t$ is a round sphere.

Proof. By Proposition 2.1.3 and Lemma 2.1.4 and integrating by parts, we have

$$\begin{aligned}\frac{d}{dt} \int_{\mathcal{M}_t} E_{k-1} d\mu &= \int_{\mathcal{M}_t} \frac{\partial E_{k-1}}{\partial h^i_j} \left(\nabla^i \nabla_j E_k + (\sigma - h) h^i_m h^m_j \right) d\mu \\ &\quad + \int_{\mathcal{M}_t} E_{k-1} H(-\sigma + h) d\mu \\ &= \int_{\mathcal{M}_t} \left\{ (\sigma - h) \left(H E_{k-1} - k E_k \right) + E_{k-1} H(-\sigma + h) \right\} d\mu \\ &= k \int_{\mathcal{M}_t} E_k (-\sigma + h) d\mu = -k \int_{\mathcal{M}_t} (\sigma - h)(E_k - h^{1/\alpha}) d\mu.\end{aligned}$$

which is a negative quantity, since the function $q \mapsto q^\alpha$ is increasing. Moreover, this quantity is zero only if E_k is constant on the hypersurface, and this can only happen for round spheres (see [51]). □

Using (2.8) and Lemma 2.2.5 we obtain the following corollary.

Corollary 2.2.6. *There exist constants $\underline{V}$, $\overline{V} > 0$ depending only on $\mathcal{M}_0$ and k, n such that, along the flow (2.1),*

$$\underline{V} \leqslant V_{n-k+1}(\Omega_t) \leqslant \overline{V}.$$

Proof. From Lemma 2.2.5 and (2.8) with $l = n+1$ and $m = n-1$ it follows

$$V_{n-k+1}(\Omega_0) \geqslant V_{n-k+1}(\Omega_t) \geqslant \tilde{C}\ Vol(\Omega_t)^{\frac{n-k+1}{n+1}} = \tilde{C}\ Vol(\Omega_0)^{\frac{n-k+1}{n+1}},$$

for a suitable $\tilde{C} = \tilde{C}(n,k) > 0$. $\square$

It is now natural consider the generalized isoperimetric ratio involving the $(n-k+1)$-th mixed volume:

$$\mathcal{I}_{n-k+1}(\Omega_t) = \frac{V^{n+1}_{n-k+1}(\Omega_t)}{Vol^{n-k+1}(\Omega_t)}.$$

By Lemma 2.2.5, $\mathcal{I}_{n-k+1}(\Omega_t)$ is decreasing along the flow and, in particular, bounded from above.

Proposition 2.2.7. *For any $n \geqslant 1$, $1 \leqslant k \leqslant n$ and $c_1 > 0$ there exist $c_2 = c(c_1, n)$ with the following property. Let $\Omega \subset \mathbb{R}^n$ be a compact, convex set with non empty interior such that $\mathcal{I}_{n-k+1}(\Omega) \leqslant c_1$. Then Ω satisfies*

$$\frac{R^+(\Omega)}{R_-(\Omega)} \leqslant c_2,$$

with $R^+(\Omega)$ and $R_-(\Omega)$ respectively the outer and inner radius associated to Ω.

Proof. We observe that a bound on $\mathcal{I}_{n-k+1}$ implies a bound on the standard isoperimetric ratio involving the area. In fact, we have

$$\frac{A(\partial\Omega)^{(n+1)}}{Vol(\Omega)^n} = \frac{V_n^{(n+1)}(\Omega)}{V^n_{n+1}(\Omega)} \leqslant \frac{[V_{n-k+1}(\Omega)]^{\frac{n(n+1)}{n-k+1}}}{V^n_{n+1}(\Omega)^n} = [\mathcal{I}_{n-k+1}(\Omega)]^{\frac{n}{n-k+1}}.$$

The assertion then follows from Proposition 2.1.5. $\square$

Let us set $R_-(t) = R_-(\Omega_t)$ and $R^+(t) = R^+(\Omega_t)$. By Proposition 2.2.2 we know that the solution of (2.1) stays strictly convex along the flow. Then we can use Proposition 2.2.7 to get the following corollary.

Corollary 2.2.8. *There exist constants $R^+, R_- > 0$ such that along the flow*

$$R_- < R_-(t) \leqslant R^+(t) < R^+$$

Proof. By virtue of the boundedness of the isoperimetric ratio, we can use Proposition 2.2.7 to say that $\frac{R^+(t)}{R_-(t)}$ is uniformly bounded by a constant c_2 depending only on n, $A(\mathcal{M}_0)$ and $Vol(\Omega_0)$. Then, comparing $Vol(\Omega_t)$ with the volume of a ball and using Corollary 2.2.6, we find

$$V_1 \leqslant Vol(\Omega_t) \leqslant \omega_n \frac{(R^+(t))^{n+1}}{n+1} \leqslant \omega_n \frac{(c_2 R_-(t))^{n+1}}{n+1} \leqslant c_2^{n+1} Vol(\Omega_t) \leqslant c_2^{n+1} V_2,$$

where $\omega_n = A(\mathbb{S}^n)$. Then we obtain bounds from both sides on $R_-(t)$ and $R^+(t)$. $\square$

2.2.3 Upper bound on the velocity

Thanks to Corollary 2.2.8 and Proposition 2.2.3, we are now able to control uniformly the velocity of the flow, and obtain curvature bounds which imply the long time existence for the solution. To do this, we follow a method first introduced by Tso [61] and adapted by Andrews and by McCoy [8, **?**] to the volume preserving setting.

Lemma 2.2.9. *Given $\bar{t} \in [0, T)$, let $\bar{q} \in \Omega_{\bar{t}}$ be such that $B_{R_-(\bar{t})}(\bar{q}) \subset \Omega_{\bar{t}}$. Then*

$$B_{R_-(\bar{t})/2}(\bar{q}) \subset \Omega_t \qquad \forall t \in [\bar{t}, \min\{\bar{t} + \tau, T\})$$

for some constant $\tau > 0$ that only depends on $n, A(\mathcal{M}_0)$ and $Vol(\Omega_0)$.

Proof. Define $r(x,t) := |F(x,t) - \bar{q}|$ and consider $u(x,t) = u_{\bar{q}}(x,t)$ the support function with respect to the point $\bar{q}$. Then

$$\partial_t r = \frac{1}{2r}\partial_t r^2 = (h - \sigma)\frac{u}{r} > -\sigma\frac{u}{r} > -\sigma. \tag{2.12}$$

Let $r_B(t)$ be the radius of the ball centred in $\bar{q}$ such that

$$\begin{cases} r_B'(t) = -\binom{n}{k}^{\alpha} \dfrac{1}{r_B^{\alpha k}(t)} \\ r_B(\bar{t}) = R_- \end{cases} \tag{2.13}$$

Define $f(x,t) := r(x,t) - r_B(t)$. Using (2.12), we obtain

$$\partial_t f > -\sigma + \binom{n}{k}^{\alpha} \frac{1}{r_B^{\alpha k}}.$$

At time $\bar{t}$, $f(\cdot, \bar{t}) > 0$. Suppose that there exists a first time $t^* > \bar{t}$ such that $f(x^*, t^*) = 0$ at some point x^*. Then $\partial_t f(x^*, t^*) \leqslant 0$. In addition, the ball with radius $r_B(t^*)$ touches $\mathcal{M}_{t^*}$ from the inside at the point $F(x^*, t^*)$, which implies

$$\sigma(x^*, t^*) \leqslant \binom{n}{k}^{\alpha} \frac{1}{r_B^{\alpha k}(t^*)} \implies \partial_t f(x^*, t^*) > 0.$$

From this contradiction, it follows that for every time t where the flow (2.13) is defined, we have $r(x, t) \geqslant r_B(t)$. By the explicit expression for the solution of (2.13) we have

$$r_B(t) = \left(R_-(\bar{t})^3 - \frac{6}{n(n-1)}(t - \bar{t}) \right) \geqslant \frac{R_-(\bar{t})}{2} \Longleftrightarrow t - \bar{t} \leqslant \frac{7}{12n(n-1)} R_-(\bar{t}),$$

then we can choose

$$\tau = \frac{7}{12n(n-1)} R_-.$$

□

Proposition 2.2.10. *There exists a positive constant C_1, only depending on $\mathcal{M}_0$, such that*

$$\sigma(x, t) < C_1$$

for every $(x, t) \in \mathcal{M} \times [0, T)$.

Proof. By Lemma 2.2.9, for every $\bar{t} \in [0, T)$, exists $\bar{q}$ such that

$$B_{R_-/2}(\bar{q}) \subset \Omega_t \qquad \forall t \in [\bar{t}, \min\{T, \bar{t} + \tau\}).$$

Let us set $u(x, t) = u_{\bar{q}}(x, t)$. Choosing $c := \frac{R_-}{4}$ we obtain, by the convexity of $\mathcal{M}_t$,

$$c \leqslant u - c \leqslant d, \qquad \forall\, t \in [\bar{t}, \min\{T, \bar{t} + \tau\}), \tag{2.14}$$

where $d = \sup_{[0,T)}(\operatorname{diam} \mathcal{M}_t)$ is finite by Corollary 2.2.8. Then, the function

$$W(x, t) := \frac{\sigma(x, t)}{u(x, t) - c}$$

is well defined on $[\bar{t}, \min\{T, \bar{t}+\tau\})$. Standard computations show that

$$\begin{aligned}\partial_t W &= \frac{(u-c)\partial_t \sigma - \sigma \partial_t u}{(u-c)^2} \\ &= \frac{(u-c)\Delta_{\dot\sigma}\sigma - \sigma \Delta_{\dot\sigma}(u-c)}{(u-c)^2} + (\alpha k+1)W^2 \\ &\quad - \frac{cW}{u-c} tr_{\dot\sigma}(h_{il}h^l_j) - h\frac{W + tr_{\dot\sigma}(h_{il}h^l_j)}{u-c}\end{aligned}$$

and

$$\Delta_{\dot\sigma} W = \frac{(u-c)\Delta_{\dot\sigma}\sigma - \sigma \Delta_{\dot\sigma}(u-c)}{(u-c)^2} - \frac{2}{u-c}\langle \nabla u, \nabla W\rangle_{\dot\sigma}$$

Now, define

$$\overline{W}(t) := \sup_{\mathcal{M}_t} W(x,t) \qquad\qquad X(t) := \{x \in \mathcal{M} | W(x,t) = \overline{W}(t)\}$$

where by "$\sup_{\mathcal{M}_t}$" we mean the supremum taken on $\mathcal{M} \times \{t\}$.

Then, using Lemma 2.1.4, we find that the upper Dini derivative $D_+\overline{W}$ satisfies

$$\begin{aligned}D_+\overline{W} &\leqslant \sup_{X(t)}\left\{(\alpha k+1)W^2 - \frac{cW}{u-c} tr_{\dot\sigma}(h_{il}h^l_j)\right\} \\ &\leqslant \frac{\overline{W}}{u-c}\sup_{X(t)}\left\{(\alpha k+1)\sigma - \alpha k c\frac{H}{n}\sigma\right\} \\ &= \overline{W}^2 \sup_{X(t)}\left\{\alpha k+1 - \alpha k c\frac{H}{n}\right\} \\ &\leqslant \overline{W}^2 \sup_{X(t)}\left\{\alpha k+1 - \alpha k c\binom{n}{k}^{-1/k} E_k^{1/k}\right\} \\ &\leqslant \overline{W}^2 \left(\alpha k+1 - \alpha k c^{1+1/\alpha k}\binom{n}{k}^{-1/k} \overline{W}^{1/\alpha k}\right)\end{aligned}$$

where, for the last inequality, we used (2.14).

Now, choose $\tilde{C} > 0$ large enough such that

$$\begin{cases} \tilde{C} \geqslant c^{-\alpha k - 1} \left(\dfrac{\alpha k + 2}{\alpha k} \right)^{\alpha k} \dbinom{n}{k}^{\alpha} \\ \frac{1}{\tilde{C}} < \tau \end{cases} \tag{2.15}$$

so that, if $\overline{W} > \tilde{C}$, then

$$D_+ \overline{W} \leqslant -\overline{W}^2.$$

Then, by a comparison argument,

$$W \leqslant \max \left\{ \max_{\mathcal{M}_0} W, \tilde{C} \right\} \qquad \text{on } [0, \min\{\tau, T\}) \tag{2.16}$$

in the case $\bar{t} = 0$, and

$$W \leqslant \max \left\{ \frac{1}{t - \bar{t}}, \tilde{C} \right\} \qquad \text{on } [\bar{t}, \min\{\bar{t} + \tau, T\})$$

for a general $\bar{t}$. Then we also have

$$W \leqslant \tilde{C} \quad \text{on } \left[\bar{t} + \frac{1}{\tilde{C}}, \min\{\bar{t} + \tau, T\} \right). \tag{2.17}$$

Since $\bar{t}$ is arbitrary, combining (2.16) and (2.17) and using the second condition of (2.15), we obtain

$$W \leqslant \max \left\{ \max_{\mathcal{M}_0} W, \tilde{C} \right\} \qquad \text{on } [0, T),$$

which implies the assertion, since $\sigma \leqslant dW$ by (2.14). □

If $k > 1$, the bound on σ provided by the above theorem does not imply that the curvature is bounded. In fact, there remains the possibility that some principal curvatures become unbounded while others tend to zero. However, we can already exclude this behaviour on any finite time interval, and obtain that the solution exists for all times. We begin by estimating the mixed volumes together with the volume preserving term.

Corollary 2.2.11. *All mixed volumes $V_i(\Omega_t)$ are bounded from above and below by positive constants uniformly for $t \in [0, T)$. Similarly, there are two constants $\beta, \gamma > 0$, only depending on $\mathcal{M}_0$ such that, on $[0, T)$*

$$\beta \leqslant h(t) \leqslant \gamma.$$

Proof. The bound from below follows from (2.8) and the volume preserving property

$$V_i(\Omega_t) \geqslant C\,\mathrm{Vol}(\Omega_t)^{\frac{n-i}{n+1}} = C\,\mathrm{Vol}(\Omega_0)^{\frac{n-i}{n+1}}.$$

Here we denote by C all constants depending on i, n but not on t. Inequalities (2.8) also give a bound from above for $n-k+1 \leqslant i \leqslant n$, thanks to Corollary 2.2.6. In the case $1 \leqslant i \leqslant n-k$, we can use Lemma 2.1.4 and Proposition 2.2.10 to obtain

$$V_i(\Omega_t) = C\int_{\mathcal{M}_t} E_{n-i} d\mu \leqslant C\int_{\mathcal{M}_t} E_k^{\frac{n-i}{k}} d\mu \leqslant CA(\mathcal{M}_t) = CV_n(\Omega_t) \leqslant C.$$

The boundedness from above of $h(t)$ follows from Proposition 2.2.10. Since the mixed volume are uniformly bounded from both sides, a bound from below on $h(t)$ is equivalent to a bound on $\int_{\mathcal{M}_t} \sigma d\mu$. Let $\eta > 0$, and set $\tilde{\mathcal{M}}_t = \{x \in \mathcal{M} \mid E_k(x,t) \geqslant \eta\}$. Then,

$$\begin{aligned} C \leqslant V_{n-k}(\Omega_t) = C\int_{\mathcal{M}_t} E_k d\mu = C\int_{\tilde{\mathcal{M}}_t} E_k d\mu + C\int_{\mathcal{M}_t\smallsetminus\tilde{\mathcal{M}}_t} E_k d\mu \\ CA(\tilde{\mathcal{M}}_t) + C\eta A(\mathcal{M}_t) \leqslant CA(\tilde{\mathcal{M}}_t) + C, \end{aligned}$$

then

$$A(\tilde{\mathcal{M}}_t) \geqslant C$$

and we can conclude

$$\int_{\mathcal{M}_t} \sigma d\mu \geqslant \int_{\tilde{\mathcal{M}}_t} \sigma d\mu \geqslant \eta^{\alpha} A(\tilde{\mathcal{M}}_t) \geqslant C.$$

□

We can now prove that the solution to (2.1) exists for all times.

Theorem 2.2.12. *The solution $\mathcal{M}_t$ of the flow 2.1 exists for $t \in [0, +\infty)$.*

Proof. Suppose that the maximal time T is finite. By Proposition 2.2.2 and Corollary 2.2.11, we obtain that the principal curvatures are bounded from below for all $t \in [0, T)$ by some constant λ_0. It follows, using Proposition 2.2.10,

$$\lambda_n = \frac{\lambda_{n-k+1}\cdots\lambda_n}{\lambda_{n-k+1}\cdots\lambda_{n-1}} \leqslant \frac{E_k}{\lambda_0^{k-1}} \leqslant \frac{C_1^{\frac{1}{\alpha}}}{\lambda_0^{k-1}},$$

which shows that the curvatures are also bounded from above on $[0, T)$. This contradicts Corollary 2.2.3 and shows that T is infinite. □

2.3 Convergence to a sphere

2.3.1 Hausdorff convergence

A crucial property of the flow (2.1) is that, as t goes to infinity, E_k tends to its mean in an integral sense.

Theorem 2.3.1. *As $t \to +\infty$ we have $\int_{\mathcal{M}_t} |\sigma - h(t)|^2 \, d\mu \to 0$*

Proof. Let us estimate the derivative of our integral, which can be rewritten as

$$\int_{\mathcal{M}_t} |\sigma - h(t)|^2 d\mu = \int_{\mathcal{M}_t} \sigma^2 d\mu - \frac{1}{|\mathcal{M}_t|} \left(\int_{\mathcal{M}_t} \sigma d\mu \right)^2 .$$

We find, using Proposition 2.1.3 and 2.1.4,

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{M}_t} \sigma d\mu &= \int_{\mathcal{M}_t} (\sigma - h)(tr_{\dot{\sigma}}(h_{ik} h_j^k) - H\sigma) d\mu \\ &= \int_{\mathcal{M}_t} (\sigma - h)((\alpha - 1)H\sigma - \alpha(k+1)E_k^{\alpha-1} E_{k+1}) d\mu \end{aligned}$$

Since h, E_k, E_{k+1} are all uniformly bounded, as well as the area of $\mathcal{M}_t$, then

$$\left| \frac{d}{dt} \int_{\mathcal{M}_t} \sigma d\mu \right| \leqslant C \int_{\mathcal{M}_t} H d\mu + C$$

which is also uniformly bounded, since the integral of H is equal to $V_{n-1}(\Omega_t)$ up to a constant factor. In addition, we have

$$\frac{d}{dt} |\mathcal{M}_t| = - \int_{\mathcal{M}_t} H(\sigma - h) d\mu.$$

Therefore

$$\left| \frac{d}{dt} |\mathcal{M}_t| \right| \leqslant C \int_{\mathcal{M}_t} H d\mu,$$

which is uniformly bounded. Finally we compute

$$\frac{d}{dt} \int_{\mathcal{M}_t} \sigma^2 d\mu = \int_{\mathcal{M}_t} \left(-2|\nabla \sigma|^2_{\dot{\sigma}} + \sigma(\sigma - h) tr_{\dot{\sigma}}(h_{ik} h_j^k) - \sigma H(\sigma - h) \right) d\mu$$

where $|\nabla E_k|^2_{\dot\sigma} = \dot\sigma^{ij}\nabla_i E_k \nabla_j E_k$. The gradient term gives a negative contribution, while all the remaining terms have a bounded integral by similar arguments as before. It follows that we can find an upper bound

$$\frac{d}{dt}\int_{\mathcal{M}_t} |\sigma - h|^2 d\mu \leqslant C, \tag{2.18}$$

where C does not depend on t. On the other hand, since from Lemma 2.2.5 V_{n-k} is decreasing, then

$$\int_0^\infty \left(\int_{\mathcal{M}_t} |\sigma - h||E_k - h^{1/\alpha}| d\mu \right) dt < +\infty.$$

If $0 \leqslant \alpha \leqslant 1$, it can be easily checked that

$$|\sigma - h| \leqslant \frac{|E_k - h^{1/\alpha}|}{h^{(1-\alpha)/\alpha}} \leqslant \beta^{(\alpha-1)/\alpha}|E_k - h^{1/\alpha}|,$$

where the last inequality comes from Corollary 2.2.11.
If $\alpha \geqslant 1$, then from Proposition 2.2.10 and Corollary 2.2.11 it follows that

$$|\sigma - h| \leqslant \alpha(\max\{E_k, h^{1/\alpha}\})^{\alpha-1}|E_k - h^{1/\alpha}| \leqslant C|E_k - h^{1/\alpha}|$$

for some constant $C > 0$. Then, in any case, there exists a constant $C' > 0$ such that

$$\int_0^\infty \left(\int_{\mathcal{M}_t} |\sigma - h|^2 d\mu \right) dt \leqslant C' \int_0^\infty \left(\int_{\mathcal{M}_t} |\sigma - h||E_k - h^{1/\alpha}| d\mu \right) dt < +\infty. \tag{2.19}$$

Let us set $l := \limsup_{t\to+\infty} \int_{\mathcal{M}_t} |\sigma - h|^2 d\mu$. If $l > 0$, then (2.19) implies that $\int_{\mathcal{M}_t} |\sigma - h|^2 d\mu$ oscillates infinitely many times between 0 and l with an arbitrarily large speed as $t \to \infty$. However, the one-sided bound (2.18) is enough to exclude that $\int_{\mathcal{M}_t} |\sigma - h|^2 d\mu$ has arbitrarily fast oscillations. Therefore the integral must tend to zero.

□

Lemma 2.3.2. *For any $p > 0$, we have*

$$\lim_{t\to\infty} \int_{\mathcal{M}_t} \sigma^p d\mu - |\mathcal{M}_t| h(t)^p = \lim_{t\to\infty} \int_{\mathcal{M}_t} |\sigma^p - h(t)^p|\, d\mu = 0.$$

Proof. Thanks to our bounds on σ and h, we easily check as for the proof of the previous Theorem that, if $0 < p < 1$, then

$$|\sigma^p - h^p| \leqslant \frac{|\sigma - h|}{h^{1-p}} \leqslant C|\sigma - h|$$

while, if $p \geqslant 1$,

$$|\sigma^p - h^p| \leqslant p(\max\{\sigma, h\})^{p-1}|\sigma - h| \leqslant C|\sigma - h|.$$

Thus, for any $p > 0$, we find

$$\begin{aligned} \left| \int_{\mathcal{M}_t} \sigma^p d\mu - |\mathcal{M}_t| h(t)^p \right| &\leqslant \int_{\mathcal{M}_t} |\sigma^p - h(t)^p| d\mu \leqslant C \int_{\mathcal{M}_t} |\sigma - h(t)| d\mu \\ &\leqslant C|\mathcal{M}_t|^{1/2} \left(\int_{\mathcal{M}_t} |\sigma - h(t)|^2 d\mu \right)^{1/2}, \end{aligned}$$

which tends to zero as $t \to +\infty$, by the previous theorem. □

The next lemma is inspired by the proof in [34] that a convex hypersurface with constant E_k is a sphere.

Lemma 2.3.3. *We have*

$$\lim_{t\to\infty} \int_{\mathcal{M}_t} \left(\tilde{E}_{k-1} - \tilde{E}_k^{\frac{k-1}{k}} \right) d\mu = 0.$$

Proof. By Lemma 2.1.4, the integral at the right-hand side is nonnegative. Therefore, we only need to show that its lim sup is nonnpositive. Let us set

$$\tilde{h}(t) = \binom{n}{k}^{-\alpha} h(t).$$

We have, using (2.7) and Lemma 2.1.4,

$$\begin{aligned} \int_{\mathcal{M}_t} \tilde{E}_{k-1}\, d\mu &= \int_{\mathcal{M}_t} \tilde{E}_k(F,\nu)\, d\mu \\ &= \tilde{h}^{\frac{k-1}{\alpha k}} \int_{\mathcal{M}_t} \tilde{E}_k^{\frac{1}{k}}(F,\nu)\, d\mu + \int_{\mathcal{M}_t} \left(\tilde{E}_k^{\frac{k-1}{k}} - \tilde{h}^{\frac{k-1}{\alpha k}} \right) \tilde{E}_k^{\frac{1}{k}}(F,\nu)\, d\mu \\ &\leqslant \tilde{h}^{\frac{k-1}{\alpha k}} \int_{\mathcal{M}_t} \tilde{E}_1(F,\nu)\, d\mu + \int_{\mathcal{M}_t} \left(\tilde{E}_k^{\frac{k-1}{k}} - \tilde{h}^{\frac{k-1}{\alpha k}} \right) \tilde{E}_k^{\frac{1}{k}}(F,\nu)\, d\mu \\ &= \tilde{h}^{\frac{k-1}{\alpha k}} A(t) + \int_{\mathcal{M}_t} \left(\tilde{E}_k^{\frac{k-1}{k}} - \tilde{h}^{\frac{k-1}{\alpha k}} \right) \tilde{E}_k^{\frac{1}{k}}(F,\nu)\, d\mu. \end{aligned}$$

Up to a translation, we can assume that $\max |(F,\nu)| \leqslant R_+(t) \leqslant C$. Therefore, taking into account the boundedness of E_k and Lemma 2.3.2 with $p = \frac{k-1}{\alpha k}$, we have

$$\left| \int_{\mathcal{M}_t} \left(\tilde{E}_k^{\frac{k-1}{k}} - \tilde{h}^{\frac{k-1}{\alpha k}} \right) \tilde{E}_k^{\frac{1}{k}} (F,\nu)\, d\mu \right| \leqslant C \int_{\mathcal{M}_t} \left| \tilde{E}_k^{\frac{k-1}{k}} - \tilde{h}^{\frac{k-1}{\alpha k}} \right| d\mu \to 0 \text{ as } t \to \infty.$$

We then deduce, using Lemma 2.3.2,

$$\limsup_{t\to\infty} \int_{\mathcal{M}_t} \left(\tilde{E}_{k-1} - \tilde{E}_k^{\frac{k-1}{k}} \right) d\mu \quad = \quad \limsup_{t\to\infty} \left(\tilde{h}^{\frac{k-1}{\alpha k}} A(t) - \tilde{h}^{\frac{k-1}{\alpha k}} A(t) \right) = 0.$$

which concludes our proof.

∎

Lemma 2.3.4. *Set* $\bar{E}_k(t) = \frac{1}{A(t)} \int_{\mathcal{M}_t} E_k d\mu$. *Then*

$$\lim_{t\to\infty} |h(t)^{\frac{1}{\alpha}} - \bar{E}_k(t)| = 0.$$

Proof. Follows from Lemma 2.3.2 taking $p = \frac{1}{\alpha}$. ∎

Theorem 2.3.5. *As $t \to +\infty$, the hypersurfaces $\mathcal{M}_t$, up to translations, converge in the Hausdorff metric to a round sphere with the same volume as $\mathcal{M}_0$.*

Proof. By Blaschke's theorem, see e.g. [52], the convex sets Ω_t's, possibly up to translations, are compact with respect to the Hausdorff metric. As recalled in the preliminaries, the mixed volumes are continuous with respect to the Hausdorff convergence. In particular, any limit has the same ordinary volume as $\mathcal{M}_0$. If the conclusion of our theorem does not hold, there exists a sequence Ω_{t_i} converging to a limit Ω_∞ which is not a round sphere. We observe that $\bar{E}_k(t) = V_{n-k}(\Omega_t)/V_n(\Omega_t)$, and we deduce

$$\bar{E}_k(t_i) \to \frac{V_{n-k}(\Omega_\infty)}{V_n(\Omega_\infty)}.$$

By Lemma 2.3.3, Lemma 2.3.2 and Lemma 2.3.4, we deduce that

$$\begin{aligned} V_{n-k+1}(\Omega_\infty) &= \frac{1}{n+1} \lim_{i\to\infty} \int_{\mathcal{M}_{t_i}} \tilde{E}_{k-1}\, d\mu = \frac{1}{n+1} \lim_{i\to\infty} \int_{\mathcal{M}_{t_i}} \tilde{E}_k^{\frac{k-1}{k}}\, d\mu \\ &= \frac{1}{n+1} \lim_{i\to\infty} |M_{t_i}| \tilde{h}(t_i)^{\frac{k-1}{\alpha k}} = \frac{1}{n+1} \lim_{i\to\infty} |M_{t_i}| \bar{E}_k^{\frac{k-1}{k}} \\ &= V_n^{\frac{1}{k}}(\Omega_\infty) V_{n-k}^{\frac{k-1}{k}}(\Omega_\infty). \end{aligned}$$

It follows that

$$V^k_{n-k+1}(\Omega_\infty) = V^{k-1}_{n-k}(\Omega_\infty)V_n(\Omega_\infty).$$

Therefore, the set Ω_∞ satisfies the equality case in (2.10) and hence is a sphere, in contradiction with our assumption. $\square$

2.3.2 Smooth convergence for scalar curvature flow

In the case $k = 2$ and $\alpha = 1$, where the speed is given by the scalar curvature, we are able to show that all principal curvatures of our hypersurfcae remain bounded as time goes to infinity.

Proposition 2.3.6. *There exist a constant $C_2 > 0$ such that on* $[0, \infty)$

$$\lambda_i \leqslant C_2 \quad \forall i = 1, \dots, n.$$

Proof. We can rewrite the evolution of H as in Corollary 4.2 of [2] :

$$\partial_t H = \Delta_{\dot\sigma} H + |\nabla H|^2 - |\nabla A|^2 - E_2|A|^2 + (H|A|^2 - C)H - h|A|^2 \tag{2.20}$$

where $C = \sum_{i=1}^n \lambda_i^3$. At a local maximum point for H, the terms containing derivatives are non positive. Let us analyze the reaction terms. Since $E_2 \leqslant C_1$, we can write

$$\begin{aligned} H|A|^2 - C = |A|^2 \sum_{i=1}^{n-1} \lambda_i + \sum_{i=1}^{n-1} (\lambda_n - \lambda_i)\lambda_i^2 \leqslant |A|^2 \sum_{i=1}^{n-1} \lambda_i + \lambda_n \sum_{i=1}^{n-1} \lambda_i^2 \\ \leqslant |A|^2 \sum_{i=1}^{n-1} \lambda_i + (n-1)\lambda_n\lambda_{n-1}^2 \leqslant |A|^2 \sum_{i=1}^{n-1} \lambda_i + (n-1)C_1\lambda_{n-1}. \end{aligned}$$

Then we can estimate

$$\begin{aligned} -E_2|A|^2 + (H|A|^2 - C)H &\leqslant -\lambda_n|A|^2 \sum_{i=1}^{n-1} \lambda_i + H|A|^2 \sum_{i=1}^{n-1} \lambda_i + (n-1)nC_1\lambda_{n-1}\lambda_n \\ &\leqslant |A|^2(H - \lambda_n) \sum_{i=1}^{n-1} \lambda_i + (n-1)nC_1\lambda_{n-1}\lambda_n \\ &\leqslant (n-1)n \left\{(n-1)(\lambda_n\lambda_{n-1})^2 + C_1\lambda_n\lambda_{n-1}\right\} \\ &\leqslant (n-1)n^2C_1^2, \end{aligned}$$

We conclude from equation (2.20) that, at any local maximum of H,

$$\partial_t H \leqslant (n-1)n^2 C_1^2 - \frac{\beta}{n} H^2$$

with α as in Corollary 2.2.11. The maximum principle implies

$$H(x,t) \leqslant \max\left\{\max_{\mathcal{M}_0} H, nC_1\sqrt{\frac{(n-1)n}{\beta}}\right\}.$$

at any time $t \in [0,\infty)$. Since $\mathcal{M}_t$ is convex, the same bound holds for any principal curvature. $\square$

Once we have the boundedness of all principal curvatures, the last step is to show that the flow is uniformly parabolic as $t \to \infty$. To do this, we obtain a bound from below on the speed.

Proposition 2.3.7. *There exists a positive constant C_3, only depending on n and $\mathcal{M}_0$, such that*

$$E_2(x,t) > C_3$$

for every $(x,t) \in \mathcal{M} \times [0,\infty)$.

Proof. We already know that $\mathcal{M}_t$ converge to a round sphere in the Hausdorff metric, up to a translation. Therefore, for any $\varepsilon > 0$, there exists T_ε such that, for any $t_0 \geqslant T_\varepsilon$, there exists a point $q = q(t_0)$ such that

$$B_{R-\varepsilon}(q) \subset \Omega_{t_0} \subset B_{R+\varepsilon}(q).$$

Since the speed is bounded, there exists $\tau = \tau(\varepsilon)$ such that

$$B_{R-2\varepsilon}(q) \subset \Omega_t \subset B_{R+2\varepsilon}(q), \qquad t \in [t_0, t_0+\delta].$$

If we now consider the support function $u = (F-q,\nu)$ and we set $c = R-3\varepsilon$, we have

$$\varepsilon \leqslant u - c \leqslant 5\varepsilon$$

on $\mathcal{M}_t$, for every $t \in [t_0, t_0+\tau]$. On this time interval, we consider the function

$$W(x,t) = \frac{E_2(x,t)}{c - u(x,t)}.$$

Some computations show that

$$
\begin{aligned}
(\partial_t - \Delta_{\dot\sigma})W &= \frac{2}{c-u}\langle \nabla u, \nabla W\rangle_{\dot\sigma} - 3W^2 - \frac{cW}{c-u}(HE_2 - 3E_3) \\
&\quad + \frac{h}{c-u}W - \frac{h}{c-u}(HE_2 - 3E_3) \\
&\geqslant \frac{2}{c-u}\langle \nabla u, \nabla W\rangle_{\dot\sigma} - W^2(3 + cH) + Wh\left(\frac{1}{c-u} - H\right)
\end{aligned}
$$

Let $\bar{H}$ denote the supremum of H along the flow, and let us choose $\varepsilon = (10\bar{H})^{-1}$, so that

$$
\frac{1}{c-u} - H \geqslant \frac{1}{5\varepsilon} - \bar{H} = \bar{H}.
$$

Then, at any point where the minimum of W on $\mathcal{M}_t$ is attained, we have

$$
\partial_t W \geqslant -W^2(3 + c\bar{H}) + Wh\bar{H} \geqslant W(\alpha\bar{H} - W(3 + R\bar{H})).
$$

This shows that W cannot attain a new minimum smaller than $\frac{\alpha\bar{H}}{3+R\bar{H}}$ at a time $t \geqslant T_\varepsilon$, and implies that E_2 is bounded from below by a positive constant for all times. $\square$

From Proposition 2.3.7, it follows that at least two principal curvatures are uniformly bounded from below, i.e. there exists $\lambda > 0$ such that

$$
\lambda_{n-1}(x,t), \lambda_n(x,t) > \lambda \qquad \text{for all } (x,t) \in \mathcal{M} \times [0,\infty).
$$

Then the operator $\dot\sigma^{ij}$ is uniformly parabolic on $[0,\infty)$ since, taken $\omega = (\omega_1 \ldots, \omega_n) \in \mathbb{R}^n$,

$$
\dot\sigma^{ij}\omega_i\omega_j = \frac{\partial E_2}{\partial \lambda_i}\omega_i^2 = (H - \lambda_i)\omega_i^2 \geqslant (H - \lambda_n)|\omega|^2 \geqslant \lambda_{n-1}|\omega|^2 > \lambda|\omega|^2.
$$

Arguing as in the proof of Theorem 6.4 in [24], we find that all the derivatives of the curvatures are bounded on $[0,\infty)$. Therefore, the Hausdorff convergence of the $\mathcal{M}_t$'s to a sphere is also a convergence in the C^∞ norm.

Finally, in order to obtain the exponential rate of the convergence we can observe that, after a certain time t^*, the pinching condition (1.6) appearing in [24] holds. Then we can apply Theorem 7.7 of that paper to conclude that the hypersurfaces $\mathcal{M}_t$ converges exponentially to a round sphere, with no need to add space isometries. The proof of Theorem 2.1.1 is complete.

CHAPTER 3

Volume/Area preserving non homogeneous flows: Euclidean case

In this chapter and in the next one we will present curvature flows with velocities that are not homogeneous functions of the principal curvatures. The study of these flows is not in complete generality, but the key ingredient is the fact that the velocity is an increasing function of the mean curvature (which is homogeneous of degree one in the principal curvatures). In addition to monotonicity, we have to require some other hypotheses which are just technical and satisfied by a large class of functions.

3.1 Presentation of the problem

Let $\mathcal{M}$ be an oriented, compact n-dimensional manifold without boundary. Consider the problem (1.3) given by

$$\begin{cases} \partial_t F(x,t) = [-\phi(H(x,t)) + h(t)]\nu(x,t) \\ F(x,0) = F_0(x), \end{cases} \tag{3.1}$$

where:

- $\phi : [0, +\infty) \to \mathbb{R}$ is a continuous function, C^2 differentiable in $(0, +\infty)$ with the following properties:

  i) $\phi(0) = 0, \qquad \lim_{\alpha \to \infty} \phi(\alpha) = \infty;$

ii) $\phi'(\alpha) > 0 \quad \forall \alpha > 0$;

iii) $\lim_{\alpha \to 0} \frac{\phi'(\alpha)\alpha^2}{\phi(\alpha)} = 0, \qquad \lim_{\alpha \to \infty} \frac{\phi'(\alpha)\alpha^2}{\phi(\alpha)} = \infty$;

iv) $\lim_{\alpha \to 0} \phi'(\alpha)\alpha = 0$;

v) $\phi''(\alpha)\alpha + 2\phi'(\alpha) \geqslant 0 \quad \forall \alpha > 0$.

- The function $h(t)$ is either defined as

$$h(t) := \frac{1}{|\mathcal{M}_t|} \int_{\mathcal{M}_t} \phi(H) d\mu \tag{3.2}$$

or as

$$h(t) := \frac{\int_{\mathcal{M}_t} H\phi(H) d\mu}{\int_{\mathcal{M}_t} H d\mu}. \tag{3.3}$$

The choice of the constraining term h is made in order to keep the volume enclosed by $\mathcal{M}_t$ constant in case (3.2), and in order to keep the area of $\mathcal{M}_t$ constant in case (3.3).

We will prove the following result.

Theorem 3.1.1. *Let $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$, with $n \geqslant 1$, be a smooth embedding of an oriented, compact n-dimensional manifold without boundary, such that $F_0(\mathcal{M})$ is strictly convex. Then the flow (3.1) with $h(t)$ given by (3.2) (resp. (3.3)) has a unique smooth solution, which exists for any time $t \in [0, \infty)$. The solution is still strictly convex and converges smoothly, as $t \to \infty$, to a round sphere that encloses the same volume (resp. has the same area) as the initial datum $\mathcal{M}_0$.*

This theorem can be regarded as a generalization of the result in [58], where the case $\phi(\alpha) = \alpha^k$ with $k > 0$ was considered. Here we are able to treat a more general class of speeds depending on the mean curvature, where no assumption of homogeneity or convexity/concavity is made. As we said before, the main assumption is the positivity of ϕ'. The additional requirements we put on ϕ are satisfied in most of the natural examples. For example, linear combinations of powers

$$\phi(\alpha) = \sum_{i=1}^{l} c_i \alpha^{k_i} \quad c_i, k_i > 0$$

satisfy assumptions i)-v). There are also easy examples with non polynomial growth: for example

$$\phi(\alpha) = \log(1+\alpha) \quad \text{or} \quad \phi(\alpha) = e^{\alpha} - 1$$

satisfy our hypotheses.

Short time existence and evolution equations. Even if ϕ is not homogeneous, the dependence on H and the positivity of ϕ' ensure the parabolicity of the problem. In fact, as we have seen in Chapter 1, the linearization of the evolution equation in (3.1) is an equation of the form

$$\partial_t G = \phi' g^{kl} \left(\frac{\partial^2 G}{\partial x^k \partial x^l}, \nu \right) \nu + \text{l.o.t.}$$

Since $H > 0$ on the initial datum, then $\phi' > 0$ at time zero, and then by Theorem 1.2.1 the flow (3.1) has a unique smooth solution $\mathcal{M}_t$ defined on a maximal time interval $[0, T)$. Moreover, by Theorem 1.2.2, T is finite only if the curvature blows up, or if the flow loses parabolicity.
The evolution equations for (3.1) can be recovered by Proposition 1.4.1 setting $V(x,t) = -\phi(H(x,t)) + h(t)$ in (1.4).

Proposition 3.1.2. *We have the following evolution equations for the flow* (3.1)*:*

$$\begin{aligned}
&\partial_t g_{ij} = 2(-\phi + h)h_{ij}, \\
&\partial_t g^{ij} = -2(-\phi + h)h^{ij}, \\
&\partial_t \nu = \nabla \phi, \\
&\partial_t d\mu = H(-\phi + h)d\mu, \\
&\partial_t h^i_j = \phi' \Delta h^i_j + \phi'' \nabla^i H \nabla_j H + \phi' |A|^2 h^i_j + (\phi - h - H\phi') h^i_k h^k_j, \\
&\partial_t H = \phi' \Delta H + \phi'' |\nabla H|^2 + (\phi - h)|A|^2, \\
&\partial_t \phi = \phi' \Delta \phi + \phi'(\phi - h)|A|^2, \\
&\partial_t u = \phi' \Delta u + \phi' |A|^2 u - \phi - \phi' H + h.
\end{aligned}$$

Proof. The first four evolution equations follow directly from Proposition 1.4.1. For the second fundamental form we have

$$\partial_t h_{ij} = \nabla_i \nabla_j \phi - (\phi - h) h_{ik} h^k_j. \tag{3.4}$$

By Proposition 1.4.2 with $\Lambda = \phi$,

$$\begin{aligned}
\nabla_i \nabla_j \phi &= \phi' \Delta h_{ij} + \phi'' \nabla_i H \nabla_j H \\
&\quad - g^{kl} \phi' (h_{kl} h_i^m h_{mj} - h_k^m h_{il} h_j^m + h_{kj} h_i^m h_l^m - h_k^m h_{ij} h_{ml}) \\
&= \phi' \Delta h_{ij} + \phi'' \nabla_i H \nabla_j H - \phi' H h_i^m h_{mj} \\
&\quad + g^{kl} h_k^m \phi' (h_{il} h_{mj} - h_{lj} h_{im}) + \phi' |A|^2 h_{ij}.
\end{aligned}$$

Then, from (3.4) we have

$$\begin{aligned}
\partial_t h_{ij} &= \phi' \Delta h_{ij} + \phi'' \nabla_i H \nabla_j H - \phi' H h_i^m h_{mj} + \phi' |A|^2 h_{ij} \\
&\quad - (\phi - h + \phi' H) h_{ik} h_j^k + g^{kl} h_k^m \phi' (h_{il} h_{mj} - h_{lj} h_{im}).
\end{aligned}$$

The last term is zero because of the symmetry of h_{ij}.
From the evolution of h_{ij} and g^{ij}, the evolutions of h_j^i and H can be easily computed. For the function ϕ,

$$\partial_t \phi = \phi' \partial_t H = \phi' (\phi' \Delta H + \phi'' |\nabla H|^2 + (\phi - h)|A|^2),$$

and the conclusion follows noticing that

$$\Delta \phi = \phi' \Delta H + \phi'' |\nabla H|^2.$$

Finally, the evolution of the support function follows from

$$\partial_t u = (F, \nabla \phi) - \phi + h,$$

using also the equality

$$\phi' \Delta u = (F, \nabla \phi) - \phi' |A|^2 u - \phi' H \tag{3.5}$$

that can be calculated as in [**?**, Lemma 4.2]. □

3.2 Long time existence

3.2.1 Preserving of convexity

We show that, if the initial datum $\mathcal{M}_0$ is strictly convex, then the strict convexity is preserved for all time such that the flow is defined.

Proposition 3.2.1. *If $\mathcal{M}_0$ is strictly convex, then $\mathcal{M}_t$ is strictly convex for all $t \in [0, T)$.*

Proof. It is sufficient to show that, if $\mathcal{M}_t$ is strictly convex on $[0, T^*)$, for an arbitrary $T^* < T$, then $\mathcal{M}_{T^*}$ is strictly convex. On the interval $[0, T^*)$, let $b^i_j := (h^i_j)^{-1}$ the inverse of the Weingarten operator. By standard computations, we get the evolution equation of b^i_j:

$$\begin{aligned}\partial_t b^i_j = \phi' \Delta b^i_j - 2\phi' h^m_n \nabla_l b^n_j \nabla^l b^i_m - \phi''(b^i_m \nabla^m H)(b^n_j \nabla_n H)\\ - \phi' |A|^2 b^i_j + (\phi' H - \phi + h)\delta^i_j.\end{aligned}$$

In order to prove that gradient terms give a negative contribution, we rewrite the gradients of b^i_j in terms of gradients of h^i_j:

$$h^m_n \nabla_l b^n_j \nabla^l b^i_m = b^q_j b^i_p b^r_m \nabla_l h^m_q \nabla^l h^p_r.$$

Then we use the following inequality proved by Schulze (see the second-last formula in the proof of Lemma 2.5 of [55] with $k = 1$)

$$-H b^r_m \nabla_l h^m_q \nabla^l h^p_r \leqslant -\nabla^p H \nabla_q H,$$

which gives

$$\begin{aligned}&-\phi' 2 h^m_n \nabla_l b^n_j \nabla^l b^i_m - \phi''(b^i_m \nabla^m H)(b^n_j \nabla_n H)\\ &\leqslant -\frac{1}{H}(2\phi' + \phi'' H)(b^i_m \nabla^m H)(b^n_j \nabla_n H) \leqslant 0,\end{aligned} \tag{3.6}$$

where for the last inequality we used property v) of ϕ. Furthermore, since $[0, T^*)$ is strictly contained in the existence time interval of the solution, there exists $H^* > 0$ such that $0 < H < H^*$ on $[0, T^*)$. Such a bound on H also implies a bound from above on $\phi' H + h$, thanks also to property iv) of ϕ. Then, using (3.6) we obtain

$$\partial_t b^i_j \leqslant \phi' \Delta b^i_j - \phi' |A|^2 b^i_j + c_0,$$

where c_0 is a constant only depending on $n, \mathcal{M}_0$ and H^*.

So, using the maximum principle, b^i_j is bounded on the finite interval $[0, T^*]$, and then on such interval all principal curvatures stay bounded from below by a positive constant. Then, $\mathcal{M}_{T^*}$ is strictly convex and the conclusion follows. □

3.2.2 A monotone quantity

As for the flow seen in Chapter 2, an important feature of the volume/area-preserving flows we are considering here is that the isoperimetric ratio of the hypersurface is non increasing in time. Now we are considering the standard isoperimetric ratio, i.e. $\mathcal{I}(\Omega_t) = \mathcal{I}_n(\Omega_t)$. By the isoperimetric inequality,

$$\mathcal{I}(\Omega) \geqslant (n+1)^n \omega_n, \tag{3.7}$$

where $\omega_n = A(\mathbb{S}^n)$. As before, denote by Ω_t the region enclosed by $\mathcal{M}_t$.

Lemma 3.2.2. *The flows (3.1) with $h(t)$ given by (3.2) and (3.3) preserve the volume $Vol(\Omega_t)$ and the area $A(\mathcal{M}_t)$ respectively.*

Proof. The proof is analogue to Lemma 2.2.4. □

Lemma 3.2.3. *For the flow (3.1) we have*

$$\frac{d}{dt} A(\mathcal{M}_t) \leqslant 0 \qquad \text{in case of } h \text{ given by (3.2)},$$

$$\frac{d}{dt} Vol(\Omega_t) \geqslant 0 \qquad \text{in case of } h \text{ given by (3.3)}.$$

Proof. We start from the volume preserving case. For any t, let us denote by $\bar{H} = \bar{H}(t)$ the value such that $\phi(\bar{H}) = \frac{1}{A(\mathcal{M}_t)} \int_{\mathcal{M}_t} \phi$, which is uniquely defined by the monotonicity of ϕ. Then $\int_{\mathcal{M}_t} [\phi(\bar{H}) - \phi(H)] = 0$, and so

$$\begin{aligned}
\frac{d}{dt} A(\mathcal{M}_t) &= \int_{\mathcal{M}_t} H(-\phi(H) + h)\, d\mu = \int_{\mathcal{M}_t} [H\phi(\bar{H}) - H\phi(H)]\, d\mu \\
&= \int_{\mathcal{M}_t} [H - \bar{H}][\phi(\bar{H}) - \phi(H)]\, d\mu \\
&= \int_{H \geqslant \bar{H}} [H - \bar{H}][\phi(\bar{H}) - \phi(H)]\, d\mu \\
&\qquad\qquad + \int_{H \leqslant \bar{H}} [H - \bar{H}][\phi(\bar{H}) - \phi(H)]\, d\mu.
\end{aligned}$$

Since both terms on the right side are nonpositive, the assertion follows.

Analogously, for the area preserving flow, let $\bar{H} = \frac{1}{A(\mathcal{M}_t)} \int_{\mathcal{M}_t} H$. We have

$$\begin{aligned}
\frac{d}{dt} Vol(\Omega_t) &= \int_{\mathcal{M}_t} [-\phi(H) + h] \, d\mu \\
&= \frac{A(\mathcal{M}_t)}{\int_{\mathcal{M}_t} H \, d\mu} \left(- \int_{\mathcal{M}_t} \phi(H) \bar{H} \, d\mu + \int_{\mathcal{M}_t} \phi(H) H \, d\mu \right) \\
&= \frac{A(\mathcal{M}_t)}{\int_{\mathcal{M}_t} H \, d\mu} \int_{\mathcal{M}_t} [\phi(H) - \phi(\bar{H})][H - \bar{H}] \, d\mu,
\end{aligned}$$

which is nonnegative by an argument similar to the previous case. $\square$

From Lemma 4.2.2 and the isoperimetric inequality (3.7), we deduce the following.

Corollary 3.2.4. *For the flow (3.1) with h given either by (3.2) or (3.3) there exist constants $M_1, M_2, V_1, V_2 > 0$ depending only on $\mathcal{M}_0, \Omega_0$ and n such that*

$$M_1 \leqslant A(\mathcal{M}_t) \leqslant M_2, \qquad V_1 \leqslant Vol(\Omega_t) \leqslant V_2.$$

Since the convexity of the initial datum is preserved along the flow by Proposition 3.2.1, we can use Proposition... to bound uniformly the inner and outer radii. As before, we set $R_-(t) = R_-(\Omega_t)$ and $R^+(t) = R^+(\Omega_t)$.

Corollary 3.2.5. *For a convex $\mathcal{M}_t$ evolving by (3.1), there are positive constants R_- and R^+ such that*

$$R_- < R_-(t) \leqslant R^+(t) < R^+,$$

where R_- and R^+ depend only on n, $A(\mathcal{M}_0)$ and $Vol(\Omega_0)$.

Proof. By virtue of the boundedness of the isoperimetric ratio, we can use Proposition 2.1.5 to say that $\frac{R^+(t)}{R_-(t)}$ is uniformly bounded by a constant c_2 depending only on n, $A(\mathcal{M}_0)$ and $Vol(\Omega_0)$. Then, comparing $Vol(\Omega_t)$ with the volume of a ball and using Corollary 3.2.4, we find

$$V_1 \leqslant Vol(\Omega_t) \leqslant \omega_n \frac{(R^+(t))^{n+1}}{n+1} \leqslant \omega_n \frac{(c_2 R_-(t))^{n+1}}{n+1} \leqslant c_2^{n+1} Vol(\Omega_t) \leqslant c_2^{n+1} V_2.$$

Then we obtain bounds from both sides on $R_-(t)$ and $R^+(t)$. $\square$

3.2.3 Upper bound on the curvatures

Thanks to the uniform bounds on the inner and outer radii and the preserving of the convexity, we can prove that all the principal curvatures are bounded from above. The technique employed is analogue to the one in Chapter 2 to bound the velocity.

Lemma 3.2.6. *Given $\bar{t} \in [0, T)$, let $\bar{q} \in \Omega_{\bar{t}}$ be such that $B_{R_-}(\bar{q}) \subset \Omega_{\bar{t}}$, where R_- is taken as in Corollary 3.2.5. Then*

$$BR_-/2(\bar{q}) \subset \Omega_t \qquad\qquad \forall t \in [\bar{t}, \min\{\bar{t}+\tau, T\})$$

for some constant $\tau > 0$ that only depends on n, $|\mathcal{M}_0|$ and $|\Omega_0|$.

Proof. Define $r(x,t) := |F(x,t) - \bar{q}|$ and set $u(x,t) := (F(x,t) - \bar{q}, \nu(x,t))$. Then

$$\partial_t r = \frac{1}{2r}\partial_t r^2 = (h - \phi(H))\frac{u}{r} > -\phi(H)\frac{u}{r} > -\phi(H). \tag{3.8}$$

Let $r_B(t)$ be the radius of the ball centered in $\bar{q}$ and contracting by

$$r_B'(t) = -\phi\left(\frac{n}{r_B(t)}\right) \tag{3.9}$$

with initial datum $r_B(\bar{t}) = R_-$. Define $f(x,t) := r(x,t) - r_B(t)$. Using (3.8), we obtain

$$\partial_t f > -\phi(H) + \phi\left(\frac{n}{r_B}\right).$$

At time $\bar{t}$, $f(\cdot, \bar{t}) > 0$. Suppose that there exists a first time $t^* > \bar{t}$ such that $f(x^*, t^*) = 0$ at some point x^*. Then $\partial_t f(x^*, t^*) \leqslant 0$. In addition, the ball with radius $r_B(t^*)$ touches $\mathcal{M}_t$ from the inside at the point $F(x^*, t^*)$, which implies

$$H(x^*, t^*) \leqslant \frac{n}{r_B(t^*)} \implies \phi(H(x^*, t^*)) \leqslant \phi\left(\frac{n}{r_B(t^*)}\right).$$

The contradiction shows that,for every time t where the flow (3.9) is defined, we have $r(x,t) \geqslant r_B(t)$. It now suffices to choose $\tau > 0$ such that $r_B(t) \geqslant \frac{R_-}{2}$ for every $t \in [\bar{t}, \min\{T, \bar{t}+\tau\})$. Notice that τ depends neither on the initial time $\bar{t}$ nor on $\bar{q}$. $\square$

Proposition 3.2.7. *At any time $t \in [0,T)$, we have*

$$\phi(H) \leqslant C_1$$

where C_1 is a positive constant only depending on n and $\mathcal{M}_0$.

Proof. By Lemma 3.2.6, for every $\bar{t} \in [0,T)$, exists $\bar{q}$ such that

$$B_{R_-/2}(\bar{q}) \subset \Omega_t \qquad \forall t \in [\bar{t}, \min\{T, \bar{t}+\tau\}).$$

Let us set

$$u(x,t) := (F(x,t) - \bar{q}, \nu(x,t)).$$

Choosing $c := \frac{R_-}{4}$ we obtain, by the convexity of $\mathcal{M}_t$,

$$c \leqslant u - c \leqslant d, \qquad \forall\, t \in [\bar{t}, \min\{T, \bar{t}+\tau\}), \tag{3.10}$$

where $d = \sup_{[0,T)}(\mathrm{diam}\mathcal{M}_t)$ is finite by Corollary 3.2.5. Then, the function

$$W(x,t) := \frac{\phi(H(x,t))}{u(x,t) - c}$$

is well defined on $[\bar{t}, \min\{T, \bar{t}+\tau\})$. Standard computations show that

$$\begin{aligned}
\partial_t W &= \frac{(u-c)\partial_t \phi - \phi \partial_t u}{(u-c)^2} \\
&= \frac{(u-c)\phi' \Delta \phi - \phi \phi' \Delta(u-c)}{(u-c)^2} \\
&\quad - \frac{\phi'}{u-c} h|A|^2 - \frac{\phi}{(u-c)^2}\{h - (\phi' H + \phi) + c|A|^2 \phi'\}
\end{aligned}$$

and

$$\phi' \Delta W = \frac{(u-c)\phi' \Delta \phi - \phi \phi' \Delta(u-c)}{(u-c)^2} - \frac{2\phi'}{u-c}\langle \nabla W, \nabla u \rangle.$$

Now, define

$$\overline{W}(t) := \sup_{\mathcal{M}_t} W(x,t) \qquad\qquad X(t) := \{x \in \mathcal{M} | W(x,t) = \overline{W}(t)\}.$$

Then, discarding the negative h terms, we find that the upper Dini derivative $D_+\overline{W}$ satisfies

$$\begin{aligned} D_+\overline{W} &\leqslant \sup_{X(t)} \partial_t W \leqslant \sup_{X(t)} (\partial_t - \phi' \Delta) W \\ &\leqslant \overline{W}^2 + \overline{W} \sup_{X(t)} \phi' \Big(\frac{H}{u-c} - \frac{c|A|^2}{u-c} \Big) \\ &\leqslant \overline{W}^2 + \overline{W} \sup_{X(t)} \frac{\phi' H}{u-c} \Big(1 - \frac{c^2 H}{nd} \Big) \end{aligned}$$

where for the last inequality we used convexity of $\mathcal{M}_t$ and (3.10).

Let us choose C large enough to satisfy

$$\begin{cases} C \geqslant \frac{3nd}{c^2} \\ \frac{c}{\phi(C)} < \tau \end{cases} \tag{3.11}$$

so that $H \geqslant C$ implies that $1 - \frac{c^2 H}{nd} \leqslant -\frac{2c^2}{3nd} H$. Now, suppose that $\overline{W}(t^*) \geqslant \phi(C)/c$ for some time t^*. Then, using the bound $u - c \geqslant c$ and the monotonicity of ϕ we have that $H(x^*, t^*) \geqslant C$ for any $x^* \in X(t^*)$. Then, we get at time $t = t^*$

$$D_+\overline{W} \leqslant \overline{W}^2 - \frac{2c^2}{3nd} \overline{W} \sup_{X(t^*)} \frac{\phi' H^2}{u-c} = \overline{W}^2 \sup_{X(t^*)} \left\{ 1 - \frac{2c^2 \phi' H^2}{3nd\phi} \right\}.$$

Also, by property $iii)$ of ϕ, we can choose C sufficiently big such that $H \geqslant C$ implies

$$1 - \frac{2c^2 \phi' H^2}{3nd\phi} < -1.$$

Then

$$D_+\overline{W} \leqslant -\overline{W}^2,$$

and so a standard comparison argument implies

$$W \leqslant \max\left\{ \max_{\mathcal{M}_0} W, \frac{\phi(C)}{c} \right\} \qquad \text{on } [0, \min\{\tau, T\}) \tag{3.12}$$

in the case $\bar{t} = 0$, and

$$W \leqslant \max\left\{ \frac{1}{t - \bar{t}}, \frac{\phi(C)}{c} \right\} \qquad \text{on } [\bar{t}, \min\{\bar{t} + \tau, T\})$$

for a general $\bar{t}$. Then we also have

$$W \leqslant \frac{\phi(C)}{c} \quad \text{on} \quad \left[\bar{t} + \frac{c}{\phi(C)}, \min\{\bar{t} + \tau, T\}\right). \tag{3.13}$$

Since $\bar{t}$ is arbitrary, combining (3.12) and (3.13) and using the second condition of (3.11), we obtain

$$W \leqslant \max\left\{\max_{\mathcal{M}_0} W, \frac{\phi(C)}{c}\right\} \qquad \text{on } t \in [0, T),$$

which implies the assertion, since $\phi \leqslant dW$ by (3.10). □

Corollary 3.2.8. *H and h are uniformly bounded on $[0, T)$. In particular, all the principal curvatures are uniformly bounded.*

Proof. The boundedness of H follows from Proposition 3.2.7 and property $i)$ of ϕ, while the boundedness of h follows from the boundedness of ϕ. The last assertion follows from the convexity. □

Theorem 3.2.9. *The solution $\mathcal{M}_t$ of (3.1) exists for any time.*

Proof. Since H and h are bounded, we can retrace the proof of Proposition 3.2.1 taking H^* and h^* independent of T^*. Then, if $T < \infty$, all the principal curvatures are bounded from below by a constant depending on T, hence the flow is uniformly parabolic. By Theorem 1.2.2 in Chapter 1, since T is the maximal time, the curvature blows up as $t \to \infty$. This leads a contradiction with Corollary 3.2.8. Then $T = \infty$. □

3.3 Convergence to a sphere

3.3.1 Lower bound on the mean curvature

In order to prove the convergence of the solution to a sphere, we need to prove the uniform parabolicity on $[0, +\infty)$. To do this, it is essential to have a positive lower bound on H, since Proposition 3.2.1 implies uniform convexity only on finite time intervals. Let us first give a preliminary result.

Lemma 3.3.1. *Given $\bar{t} \in [0, \infty)$, let $\bar{q} \in \Omega_{\bar{t}}$ be such that $\Omega_{\bar{t}} \subset B_{R^+}(\bar{q})$, where R^+ is taken as in Corollary 3.2.5. Then*

$$\Omega_t \subset B_{2R^+}(\bar{q}) \qquad\qquad \forall t \in [\bar{t}, \bar{t} + \sigma]$$

where $\sigma > 0$ is a constant that only depends on n, $|\mathcal{M}_0|$, $|\Omega_0|$ and $\sup_t h(t)$.

Proof. Let us compare $\mathcal{M}_t$ with the sphere centered in $\bar{q}$ whose radius $R(t)$ increases linearly according to

$$R(t) = \tilde{h}(t - \bar{t}) + R^+$$

where $\tilde{h} = \sup_t h(t)$. $R(t)$ grows to $2R^+$ at time $t = \bar{t} + \frac{R^+}{\tilde{h}}$. Denote $\sigma := \frac{R^+}{\tilde{h}}$.

Similarly as in Lemma 3.2.6, set

$$r(x,t) := |F(x,t) - \bar{q}|, \qquad u(x,t) := (F(x,t) - \bar{q}, \nu(x,t)).$$

Then, the function $f(x,t) := R(t) - r(x,t)$ satisfies

$$\partial_t f = \tilde{h} - \frac{hu}{r} + \frac{\phi u}{r} \geqslant \frac{\phi u}{r} \geqslant 0.$$

So $f(x,t) \geqslant 0$ for every time, and $r(x,t) \leqslant R(\bar{t} + \sigma) = 2R^+$ for $t \in [\bar{t}, \bar{t} + \sigma]$. $\square$

Lemma 3.3.2. *There exists $b > 0$ such that*

$$h(t) \geqslant b \quad \forall t \in [0, \infty).$$

Proof. Let us first prove a bound from below on $\frac{1}{A(\mathcal{M}_t)} \int_{\mathcal{M}_t} \phi(H)\, d\mu$. By the Alexandrov-Fenchel inequalities is that there exists a constant C_n only depending by n, such that

$$\int_{\mathcal{M}_t} H\, d\mu \geqslant C_n Vol(\Omega_t)^{\frac{n-1}{n+1}}$$

so, by Corollary 3.2.4 we get

$$\int_{\mathcal{M}_t} H\, d\mu \geqslant C_0,$$

where $C_0 > 0$ is a constant depending by n and the initial datum. By Corollary 3.2.8, there exists some value H^* such that $H \leqslant H^*$ on $\mathcal{M}_t$, for all t. Let $k = \frac{C_0}{2A(M_0)}$ and $\tilde{\mathcal{M}}_t = \{x \in \mathcal{M} | H(x,t) \geqslant k\}$. Then we have

$$\begin{aligned} C_0 &\leqslant \int_{\mathcal{M}_t} H\, d\mu = \int_{\tilde{\mathcal{M}}_t} H\, d\mu + \int_{\mathcal{M}_t \setminus \tilde{\mathcal{M}}_t} H\, d\mu \leqslant H^* A(\tilde{\mathcal{M}}_t) + kA(\mathcal{M}_t) \\ &\leqslant H^* A(\tilde{\mathcal{M}}_t) + \frac{C_0}{2} \end{aligned}$$

where for the last inequality we used the fact that $A(\mathcal{M}_t) \leqslant A(\mathcal{M}_0)$ for all t. Thus

$$A(\tilde{\mathcal{M}}_t) \geqslant \frac{C_0}{2H^*}. \tag{3.14}$$

Let us set $m = \min_{k \leqslant H \leqslant H^*} \frac{\phi(H)}{H}$. Using Corollary 3.2.4, we conclude

$$\begin{aligned}\frac{1}{A(\mathcal{M}_t)} \int_{\mathcal{M}_t} \phi(H)\, d\mu \geqslant \frac{1}{M_2} \int_{\mathcal{M}_t} \phi(H)\, d\mu \geqslant \frac{1}{M_2} \int_{\tilde{\mathcal{M}}_t} \phi(H)\, d\mu \\ \geqslant \frac{m}{M_2} \int_{\tilde{\mathcal{M}}_t} H\, d\mu \geqslant \frac{m}{M_2} A(\tilde{\mathcal{M}}_t) k \geqslant \frac{mC_0 k}{2H^* M_2} > 0,\end{aligned}$$

which gives a uniform bound from below on $h(t)$ in the volume-preserving case. In the area preserving case, the above computations also imply an estimate on $h(t)$ using the inequality

$$\int_{\mathcal{M}_t} \phi(H) H\, d\mu \geqslant \frac{1}{A(\mathcal{M}_t)} \int_{\mathcal{M}_t} H\, d\mu \int_{\mathcal{M}_t} \phi(H)\, d\mu,$$

which was proved in the second part of the proof of Lemma 4.2.2. □

To obtain a lower bound on H, we now use a technique analogous to Proposition 2.2.10, but we reverse the sign of the test function by considering a ball which encloses $\mathcal{M}_t$ instead of an enclosed one. A similar argument was used in [54] for an expanding flow. In contrast to the upper bound in Proposition 2.2.10, the proof of the next result depends crucially on the presence of the nonlocal term $h(t)$.

Proposition 3.3.3. *The mean curvature H is uniformly bounded from below by a positive constant.*

Proof. Given any $\bar{t} \geqslant 0$, let $\bar{q}$ be chosen so that the conclusion of Lemma 3.3.1 holds. We define

$$W(x,t) := \frac{\phi(H)}{c - u(x,t)} \qquad c := 4R^+,$$

which is well defined on $[\bar{t}, \bar{t} + \sigma]$, because on such interval we have

$$\frac{c}{2} \leqslant c - u \leqslant c$$

where $u(x,t) := (F(x,t) - \bar{q}, \nu(x,t))$ as usual. Standard computations show that

$$\begin{aligned}\partial_t W = \phi' \Delta W + \frac{2\phi'}{c-u}(\nabla W, \nabla(c-u)) \\ - \frac{\phi'}{c-u} h|A|^2 - \frac{\phi}{(c-u)^2}\{-h + \phi' H + \phi - c|A|^2 \phi'\}.\end{aligned}$$

Now, define

$$\underline{W}(t) := \inf_{\mathcal{M}_t} W(x,t) \qquad\qquad Y(t) := \{x \in \mathcal{M} | W(x,t) = \underline{W}(t)\}.$$

Then, after disregarding the last positive term, we obtain

$$\begin{aligned} D_-\underline{W} &\geqslant \inf_{Y(t)} \left\{ -\frac{\phi' h H^2}{c-u} + \frac{h}{c-u} W - \frac{\phi' H}{c-u} W - W^2 \right\} \\ &\geqslant \underline{W} \inf_{Y(t)} \left\{ -\frac{\phi' h H^2}{\phi} + \frac{h}{c} - \frac{2\phi' H}{c} - \frac{2\phi}{c} \right\}. \end{aligned} \tag{3.15}$$

Using properties $i)$, $iii)$ and $iv)$ of ϕ, we can fix $\beta > 0$ such that, if $H \in (0, \beta)$, we have

$$\frac{\phi' H^2}{\phi} < \frac{1}{2c}, \qquad \phi + \phi' H < \frac{b}{8} \tag{3.16}$$

where $b > 0$ is the lower bound on $h(t)$ given by Lemma 3.3.2. Suppose now that $\underline{W}(t) < \phi(\beta)/c$ at some time t. Then $\phi(H) \leqslant \beta$ on $Y(t)$ and therefore

$$D_-\underline{W} \geqslant \underline{W} \left\{ -\frac{h}{2c} + \frac{h}{c} - \frac{b}{4c} \right\} \geqslant \frac{b}{4c} \underline{W} > 0. \tag{3.17}$$

This shows that W cannot attain a new minimum smaller than $\phi(\beta)/c$, thus

$$W(x,t) \geqslant \min \left\{ \underline{W}(0), \frac{\phi(\beta)}{c} \right\} \qquad \text{on } [0, \infty).$$

From this we deduce that ϕ, and so H, is bounded from below for all times by a positive constant. $\square$

3.3.2 Smooth convergence to a sphere

Proposition 3.3.3, together with Corollary 3.2.8, implies that H takes values in a fixed compact subset of $(0, +\infty)$ for all times. Therefore $\phi'(H)$ is bounded from above and below by positive constants for all $t \in [0, +\infty)$ and the flow is uniformly parabolic. By classical results, see for instance Theorem 6 in [9], we obtain that all derivatives of the curvatures are bounded for $t \in [0, \infty)$. So, by compactness, the hypersurfaces $\mathcal{M}_t$ converge, up to time subsequences, to a smooth limit $\mathcal{M}_\infty$. To prove that this limit has to be a sphere, we show that ϕ tends to its mean value. Define $\bar{\phi}(t) = \frac{1}{A(\mathcal{M}_t)} \int_{\mathcal{M}_t} \phi d\mu$.

Proposition 3.3.4.

$$\lim_{t\to\infty} \max_{\mathcal{M}_t} |\phi(H(\cdot,t)) - h(t)| = \lim_{t\to\infty} \max_{\mathcal{M}_t} |\phi(H(\cdot,t)) - \bar{\phi}(t)| = 0$$

Proof. We start with the volume preserving flow. Of course in this case $\bar{\phi}(t) = h(t)$. For any t, let $\tilde{H}(t)$ such that $\phi(\tilde{H}(t)) = h(t)$. Then we compute

$$\begin{aligned} \frac{d}{dt} A(\mathcal{M}_t) &= \int_{\mathcal{M}_t} Hh\, d\mu - \int_{\mathcal{M}_t} H\phi(H)\, d\mu \\ &= \int_{\mathcal{M}_t} (H - \tilde{H})(\phi(\tilde{H}) - \phi(H))\, d\mu \\ &= -\int_{\mathcal{M}_t} |H - \tilde{H}||\phi(H) - \phi(\tilde{H})|\, d\mu. \end{aligned}$$

Now, using the bound on ϕ' we obtain

$$\begin{aligned} \frac{d}{dt} A(\mathcal{M}_t) &\leqslant -\frac{1}{\sup \phi'} \int_{\mathcal{M}_t} |\phi(H) - \phi(\tilde{H})|^2\, d\mu. \\ &= -\frac{1}{\sup \phi'} \int_{\mathcal{M}_t} |\phi(H) - h|^2\, d\mu. \end{aligned}$$

Suppose that $|\phi(H) - h| = a$ for some $a > 0$ at some point $(\bar{x}, \bar{t})$. The derivative bounds on the curvature imply that H is uniformly Lipschitz continuous, and then there exists a radius $r(a)$, not depending by $(\bar{x}, \bar{t})$, such that

$$|\phi(H) - h| > \frac{a}{2} \qquad \text{on } B_{r(a)}((\bar{x}, \bar{t}))$$

where $B_{r(a)}((\bar{x}, \bar{t}))$ is the parabolic neighbourhood centered at $(\bar{x}, \bar{t})$ of radius $r(a)$. Then

$$\frac{d}{dt} A(\mathcal{M}_t) < -\eta(a) \qquad \forall t \in [\bar{t} - r(a), \bar{t} + r(a)] \tag{3.18}$$

for some $\eta > 0$ only depending on a.

By Lemma 3.2.3, $A(\mathcal{M}_t)$ is positive and decreasing in time, and so property (3.18) can occur only on a finite number of time intervals, for any given $a > 0$. This shows that $|\phi(H) - h|$ tends to zero uniformly.
For the area preserving flow, define $\bar{H} = \frac{1}{A(\mathcal{M}_t)} \int_{\mathcal{M}_t} H d\mu$. Similarly as before, we compute

$$\begin{aligned}\partial_t Vol(\Omega_t) &= \frac{1}{\bar{H}} \int_{\mathcal{M}_t} (\phi(H) - \phi(\bar{H}))(H - \bar{H}) d\mu \\ &\geqslant C \int_{\mathcal{M}_t} (\phi(H) - \phi(\bar{H}))(H - \bar{H}) d\mu \\ &\geqslant C \inf \phi' \int_{\mathcal{M}_t} |H - \bar{H}|^2 d\mu.\end{aligned}$$

With the same argument as before, we can say that H tends to $\bar{H}$ uniformly as t tends to infinity. Then

$$\begin{aligned}\lim_{t\to\infty} \max_{\mathcal{M}_t} |\phi(H) - h| &= \lim_{t\to\infty} \max_{\mathcal{M}_t} |\phi(H) - \phi(\bar{H}) + \phi(\bar{H}) - h| \\ &= \lim_{t\to\infty} \max_{\mathcal{M}_t} |\phi(H) - \phi(\bar{H})| + \lim_{t\to\infty} \max_{\mathcal{M}_t} |\phi(\bar{H}) - h| \\ &= \lim_{t\to\infty} \max_{\mathcal{M}_t} \left| \phi(\bar{H}) - \frac{\int_{\mathcal{M}_t} H\phi(H) d\mu}{\int_{\mathcal{M}_t} H d\mu} \right| \\ &= \lim_{t\to\infty} \max_{\mathcal{M}_t} |\phi(\bar{H}) - \phi(\bar{H})| = 0,\end{aligned}$$

which concludes the proof. □

Proposition 4.3.1 implies that any possible limit of subsequences of $\mathcal{M}_t$ has constant mean curvature, and so is a sphere. Then, we can conclude that the whole family $\mathcal{M}_t$ converges smoothly to a sphere.

3.3.3 Exponential rate

In order to prove the exponential rate, we will follow a similar method to [56], but the procedure will be simpler because we already have the smooth convergence to a sphere. We define

$$Q = \frac{K}{H^n},$$

where K is the Gaussian curvature $K = E_n$. The first author who used this ratio in order to give a pinching condition was Chow in [26].
$\mathcal{M}_t$ is strictly convex, then Q is well defined and strictly positive. Notice also that $Q \leqslant \frac{1}{n^n}$, and the equality holds if and only if the hypersurface is totally umbilic. Then we know that Q converges smoothly to the constant value $\frac{1}{n^n}$. Our goal is to show that this convergence is exponential.

Proposition 3.3.5. *The quantities K, Q evolve according to*

$$\begin{aligned}
\partial_t K &= \phi'\Delta K - \frac{(n-1)}{n}\frac{\phi'|\nabla K|^2}{K} + \frac{K}{H^2}\phi'|H\nabla h^i_j - h^i_j \nabla H|^2_{g,b} \\
&\quad -\frac{H^{2n}}{nK}\phi'\left|\nabla \frac{K}{H^n}\right|^2 + K\phi'' b^i_j \nabla_i H \nabla^j H + KH(\phi - \phi' H - h) \\
&\quad + nK\phi'|A|^2; \\
\partial_t Q &= \phi'\Delta Q + \frac{(n+1)}{nH^n}\phi'\langle \nabla Q, \nabla H^n\rangle - \frac{(n-1)}{nK}\phi'\langle \nabla Q, \nabla K\rangle \\
&\quad -\frac{Q^{-1}}{n}\phi'|\nabla Q|^2 + \frac{Q}{H^2}\phi'|H\nabla h^i_j - h^i_j\nabla H|^2_{g,b} + Q\phi''|\nabla H|^2_{b-\frac{n}{H}g} \\
&\quad + \frac{Q}{H}(\phi' H - \phi + h)(n|A|^2 - H^2).
\end{aligned}$$

where

$$\begin{aligned}
|H\nabla h^i_j - h^i_j\nabla H|^2_{g,b} &= b^n_m b^s_r (H\nabla_i h^m_n - h^m_n \nabla_i H)(H\nabla^i h^r_s - h^r_s \nabla^i H) \\
|\nabla H|^2_{b-\frac{n}{H}g} &= \left(b^{ij} - \frac{n}{H}g^{ij}\right)\nabla_i H \nabla_j
\end{aligned}$$

Proof. By Proposition 3.1.2, we get

$$\partial_t H^n = \phi'\Delta H^n + nH^{n-2}(\phi'' H - (n-1)\phi')|\nabla H|^2 + nH^{n-1}(\phi - h)|A|^2.$$

Using twice the derivative law for the determinant,

$$\begin{aligned}
\Delta K &= \nabla_r\left(K b^i_j \nabla^r h^j_i\right) \\
&= K b^i_j \Delta h^j_i + \frac{|\nabla K|^2}{K} + K\nabla_r b^i_j \nabla^r h^j_i.
\end{aligned}$$

We can use this equation to compute the evolution of K:

$$\begin{aligned}
\partial_t K &= K b^i_j \partial_t h^j_i \\
&= \phi'\Delta K - \phi'\frac{|\nabla K|^2}{K} - K\phi'\nabla_r b^i_j \nabla^r h^j_i + K\phi'' b^i_j \nabla_i H \nabla^j H \\
&\quad + HK(\phi - h - \phi' H) + nK\phi'|A|^2
\end{aligned}$$

As in the proof of Lemma 2.1 of [56], we have

$$K\phi'\nabla_r b^i_j \nabla^r h^j_i = -\frac{1}{n}\frac{\phi'|\nabla K|^2}{K} - \frac{K}{H^2}\phi'|H\nabla h^i_j - h^i_j\nabla H|^2_{g,b} + \frac{H^{2n}}{nK}\phi'\left|\nabla\frac{K}{H^n}\right|^2$$

and then

$$\begin{aligned}-\phi'\frac{|\nabla K|^2}{K} - K\phi'\nabla_r b^i_j\nabla^r h^j_i &= -\frac{(n-1)}{n}\frac{\phi'|\nabla K|^2}{K}\\ &\quad+\frac{K}{H^2}\phi'|H\nabla h^i_j - h^i_j\nabla H|^2_{g,b} - \frac{H^{2n}}{nK}\phi'\left|\nabla\frac{K}{H^n}\right|^2.\end{aligned}$$

From the last equality, we get the evolution of K.
By definition of Q, we get:

$$\Delta Q = \frac{\Delta K}{H^n} - \frac{K\Delta H^n}{H^{2n}} - \frac{2}{H^{2n}}\langle\nabla K,\nabla H^n\rangle + 2\frac{Q}{H^{2n}}|\nabla H^n|^2.$$

Then we have

$$\begin{aligned}\partial_t Q &= \frac{1}{H^n}\partial_t K - \frac{K}{H^{2n}}\partial_t H^n\\ &= \phi'\Delta Q + \frac{2}{H^{2n}}\phi'\langle\nabla K,\nabla H^n\rangle - 2\frac{Q}{H^{2n}}\phi'|\nabla H^n|^2\\ &\quad+n(n-1)\frac{Q}{H^2}\phi'|\nabla H|^2 - \frac{n-1}{n}\phi'\frac{|\nabla K|^2}{KH^n}\\ &\quad-\frac{Q^{-1}}{n}\phi'|\nabla Q|^2 + \frac{Q}{H^2}\phi'|H\nabla h^i_j - h^i_j\nabla H|^2_{g,b} + Q\phi''|\nabla H|^2_{b-\frac{n}{H}g}\\ &\quad+\frac{Q}{H}(\phi' H - \phi + h)(n|A|^2 - H^2).\end{aligned}$$

The conclusion follows observing that

$$\begin{aligned}\langle\nabla Q,\nabla H^n\rangle &= \frac{1}{H^n}\langle\nabla K,\nabla H^n\rangle - \frac{Q}{H^n}|\nabla H^n|^2,\\ \langle\nabla Q,\nabla K\rangle &= \frac{1}{H^n}|\nabla K|^2 - \frac{Q}{H^n}\langle\nabla H^n,\nabla K\rangle.\end{aligned}$$

□

Similarly as in [56] we consider the function $f = \frac{1}{n^n} - Q$. By the results of the previous section, we already know that $\mathcal{M}_t$ converges to a sphere and so f converges smoothly to zero. Now we want to prove that this convergence is exponentially fast. The following Lemma collect two known results, the first one by Huisken in [35], and the second one by Schulze in [56]

Lemma 3.3.6. *If there exists* $\varepsilon \in (0, \frac{1}{n}]$ *such that* $\lambda_i \geqslant \varepsilon H$ *for all* $i = 1, \dots, n$, *then*

(i) $|H\nabla h^i_j - h^i_j \nabla H|^2 \geqslant \frac{1}{2}\varepsilon^2 H^2 |\nabla H|^2$.

(ii) There exists $\delta > 0$ *such that* $\quad \dfrac{n|A|^2 - H^2}{H^2} \geqslant \delta f$

Proposition 3.3.7. *There are a time* $\bar{t} > 0$ *and two constants* c, $\delta > 0$ *such that for every time* $t \geqslant \bar{t}$ *we have :*

$$f \leqslant c e^{-\delta t}.$$

Proof. By Proposition 3.3.5 we can compute the evolution equation for f:

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \phi' \Delta f - \frac{(n+1)}{nH^n}\phi' \langle \nabla f, \nabla H^n \rangle + \frac{(n-1)}{nK}\phi' \langle \nabla f, \nabla K \rangle \\
&\quad + \frac{Q^{-1}}{n}\phi' |\nabla f|^2 - \frac{Q}{H^2}\phi' |H\nabla h^i_j - h^i_j \nabla H|^2_{g,b} - Q\phi'' |\nabla H|^2_{b - \frac{n}{H}g} \\
&\quad - \frac{Q}{H}\left(\phi' H - \phi + h\right)\left(n|A|^2 - H^2\right).
\end{aligned} \tag{3.19}$$

First, we prove that the gradient terms give a negative contribution for large times. Since $\mathcal{M}_t$ converges smoothly to a round sphere, then for all $\varepsilon > 0$ there exists $\bar{t}_1 > 0$ such that, for all $t \geqslant \bar{t}_1$,

$$\lambda_i \geqslant \frac{1}{n(1+\varepsilon)} H \qquad\qquad \forall i = 1, \dots, n \tag{3.20}$$

which implies

$$\left| b - \frac{n}{H} g \right| < \frac{\varepsilon}{H}.$$

Moreover, by Lemma 3.3.6, part (i), (3.20) also implies

$$|H\nabla h^i_j - h^i_j \nabla H|^2 \geqslant \frac{H^2}{2n^2(1+\varepsilon)^2} |\nabla H|^2.$$

Now, we have

$$\begin{aligned}
|H\nabla h^i_j - h^i_j \nabla H|^2_{g,b} &= \frac{1}{\lambda_i \lambda_j}(H\nabla_m h^i_j - h^i_j \nabla_m H)^2 \\
&\geqslant \frac{1}{H^2}(H\nabla_m h^i_j - h^i_j \nabla_m H)^2 \\
&= \frac{1}{H^2}|H\nabla h^i_j - h^i_j \nabla H|^2.
\end{aligned}$$

Collecting all these estimates, we find

$$\begin{aligned}
&-\frac{Q}{H^2}\phi'|H\nabla h^i_j - h^i_j\nabla H|^2_{g,b} - Q\phi''|\nabla H|^2_{b-\frac{n}{H}g} \\
&\qquad \leqslant -\frac{Q}{H^4}\phi'|H\nabla h^i_j - h^i_j\nabla H|^2 + \left|b-\frac{n}{n}g\right|^2 Q\phi''|\nabla H|^2 \\
&\qquad \leqslant -\frac{Q}{H}\left(\frac{\phi'}{2n^2(1+\varepsilon)H} - \varepsilon\phi''\right)^2 |\nabla H|^2.
\end{aligned}$$

Since ε is arbitrary, we can found $\bar{t}_1$ such that the last quantity is non positive. Now we want to show that for large enough times, the reaction terms (the last line terms in the evolution of f) leads to a negative multiple of f. In fact, by Proposition 4.3.1, for every positive η, there exists a time $t_2 > 0$ such that for every $t \geqslant t_2$ the following holds:

$$|\phi - h| \leqslant \eta.$$

Then, if we choose η small enough, we can find a positive constant δ_2 such that

$$\phi' H - \phi + h \geqslant \delta_2,$$

when $t \geqslant t_2$. Also, by (3.20) and Lemma 3.3.6, part (ii), there exists a positive constant δ_3 such that

$$\frac{n|A|^2 - H^2}{H^2} \geqslant \delta_3 f.$$

Finally, since all the curvatures are uniformly bounded from above and below, there exists $\delta_4 > 0$ such that

$$QH \geqslant \delta_4.$$

Let $\bar{t} = \max\{t_1, t_2\}$ and $\delta = \delta_2\delta_3\delta_4$, then for $t \geqslant \bar{t}$ we have

$$\frac{\partial f}{\partial t} \leqslant \phi'\Delta f - \frac{(n+1)}{nH^n}\phi'\langle\nabla f, \nabla H^n\rangle + \frac{(n-1)}{nK}\phi'\langle\nabla f, \nabla K\rangle + \frac{Q^{-1}}{n}\phi'|\nabla f|^2 - \delta f,$$

and the thesis follow by the maximum principle. □

Arguing as in Theorem 3.5 of [56] we obtain:

Corollary 3.3.8. *The second fundamental form A converges exponentially in C^∞ to the one of a round sphere. In particular, there exist positive constants c', δ' such that*

$$|\phi(H) - h| \leqslant c' e^{-\delta' t}$$

From the previous corollary it follows that the limit hypersurface exists with no need to add isometries. In fact, for any $0 \leqslant t_1 < t_2$

$$\begin{aligned}\max_{\mathcal{M}} |F(x,t_1) - F(x,t_2)| &\leqslant \max_{\mathcal{M}} \int_{t_1}^{t_2} |\partial_t F(x,t)| dt \\ &= \max_{\mathcal{M}} \int_{t_1}^{t_2} |\phi - h| dt \leqslant \frac{c'}{\delta'}(e^{-\delta' t_1} - e^{-\delta' t_2})\end{aligned}$$

then the whole family $F(\cdot, t)$ tends to a limit hypersurface for t that goes to infinity. Finally, the smooth convergence of the second fundamental form implies the smooth convergence of the metric and of the embeddings, by standard arguments used for example in [56]. This complete the proof of Theorem 3.1.1.

CHAPTER 4

Volume/Area preserving non homogeneous flows: hyperbolic case

In this chapter we study the hyperbolic version of the flow analysed in the previous chapter. In addition to the monotonicity of the speed function, the other hypotheses are slightly weaker since, as we will see, we don't need to require a specific behaviour in zero.

4.1 Presentation of the problem

Let $\mathbb{H}^{n+1}_a$ be the hyperbolic space of constant sectional curvature $-a^2 < 0$ and let us take a smooth oriented, compact and without boundary hypersurface $F_0 : \mathcal{M} \to \mathbb{H}^{n+1}_a$. We consider a family of maps $F : \mathcal{M} \times [0, T) \to \mathbb{H}^{n+1}_a$, evolving according the law:

$$\begin{cases} \partial_t F(x,t) = [-\phi(H(x,t)) + h(t)]\nu(x,t) \\ F(x,0) = F_0(x), \end{cases} \tag{4.1}$$

where:

- $\phi : [0, +\infty) \to \mathbb{R}$ is a continuous function, C^2 differentiable in $(0, +\infty)$ such that

  $i)$ $\phi(\alpha) > 0, \quad \phi'(\alpha) > 0 \quad \forall \alpha > 0;$

ii) $\lim_{\alpha\to\infty} \phi(\alpha) = \infty$;

iii) $\lim_{\alpha\to\infty} \dfrac{\phi'(\alpha)\alpha^2}{\phi(\alpha)} = \infty$;

iv) $\phi''(\alpha)\alpha \geqslant -2\phi'(\alpha) \quad \forall\alpha > 0$.

- The function $h(t)$ is either defined as (3.2) (volume preserving) or (3.3) (area preserving)

Remark 4.1.1. If ϕ is a convex function, properties $ii), iii)$ and $iv)$ on ϕ just follows from the convexity: $ii)$ and $iv)$ are trivial, and for $iii)$ we can write $\frac{\phi(\alpha)}{\alpha} = \frac{\phi(\alpha)-\phi(0)}{\alpha} + \frac{\phi(0)}{\alpha}$. By the convexity of ϕ, the first addendum of the right side is an increasing function, then $\left(\frac{\phi(\alpha)}{\alpha}\right)' \geqslant -\frac{\phi(0)}{\alpha^2}$, which implies $\phi'(\alpha)\alpha \geqslant \phi(\alpha) - \phi(0)$. Finally,

$$\lim_{\alpha\to\infty} \frac{\phi'(\alpha)\alpha^2}{\phi(\alpha)} \geqslant \lim_{\alpha\to\infty} \frac{\phi(\alpha)-\phi(0)}{\phi(\alpha)}\alpha = \infty$$

Using also property $ii)$ on ϕ.

We will consider initial data which are convex by horospheres. We say that a hypersurface is *convex by horospheres* (*$\hbar$-convex* for short) if it bounds a domain Ω such that at every point $p \in \mathcal{M} = \partial\Omega$ there exists a horosphere of $\mathbb{H}^{n+1}_a$ passing through p such that Ω is contained in the region bounded by the horosphere. Such a definition is the natural analogue of convexity in Euclidean space, since the horospheres touching the boundary from inside take the place of tangent hyperplanes. In [17] was proved that $\mathcal{M}$ is $\hbar$-convex if and only if at any point $\lambda_i \geqslant a$ for all i. Then we see that this condition is stronger than convexity.
The result we are going to prove is the following.

Theorem 4.1.1. *Let $F_0 : \mathcal{M} \to \mathbb{H}^{n+1}_a$, with $n \geqslant 1$, be a smooth embedding of an oriented, compact n-dimensional manifold without boundary, such that $F_0(\mathcal{M})$ is $\hbar$-convex. Then the flow (4.1) with $h(t)$ given by (3.2) (resp. (3.3)) has a unique smooth solution, which exists for any time $t \in [0, \infty)$. The solution is $\hbar$-convex for any time and converges smoothly and exponentially, as $t \to \infty$, to a geodesic sphere that encloses the same volume (resp. has the same area) as the initial datum $\mathcal{M}_0$.*

Short time existence and evolution equations. The short time existence and the uniqueness of the solution follow in the same way as for the flow (3.1), since the linearization of (4.1) is analogue to (3.1). Then there exists a unique solution to (4.1) on a maximal time interval $[0, T)$.
In Chapter 1, Proposition 1.4.1 we recalled the evolution equations for curvature flows of Euclidean hypersurfaces. Indeed, Huisken and Polden in [40] gave very general evolution equations for hypersurfaces of any Riemannian manifolds. In general, these equations are slightly complicated because of the presence of additional terms depending on the geometry of the ambient manifold. However, in the case of hyperbolic space, the Riemann tensor has a simple form, depending only on the sectional curvature and the metric. We recall it in the following proposition.

Proposition 4.1.2. *If N_K is a space form of constant sectional curvature K, then the Riemann tensor is written as*

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc}).$$

In particular, $\nabla^{N_K} R_{abcd} = 0$, where ∇^{N_K} is the Levi-Civita connection of N_K.

Using this fact, the evolution equations for (4.1) can be easily computed from [40].

Proposition 4.1.3. *We have the following evolution equations for the flow (4.1):*

$$\begin{aligned}
\partial_t g_{ij} &= 2(-\phi + h)h_{ij},\\
\partial_t g^{ij} &= -2(-\phi + h)h^{ij},\\
\partial_t \nu &= \nabla\phi,\\
\partial_t d\mu &= H(-\phi + h)d\mu,\\
\partial_t h_{ij} &= \phi'\Delta h_{ij} - (\phi' H + \phi - h)h_{il}h^l_j + \phi'|A|^2 h_{ij} + \phi''\nabla_i H \nabla_j H\\
&\qquad - a^2(\phi' H + \phi - h)g_{ij} + na^2\phi' h_{ij},\\
\partial_t H &= \phi'\Delta H + \phi''|\nabla H|^2 + (\phi - h)|A|^2 - na^2(\phi - h),\\
\partial_t \phi &= \phi'\Delta\phi + \phi'(\phi - h)|A|^2 - na^2\phi'(\phi - h),
\end{aligned}$$

In the next section we are going to give some notation and results that are specific of the hyperbolic geometry. For this reason, we have decided non to include them in Chapter 1, dedicated to general preliminaries.

4.1.1 Some notation and properties of $\hbar$-convex sets

For every constant $a > 0$, we denote with $\mathbb{H}_a^{n+1}$ the hyperbolic space of dimension $n+1$ and constant sectional curvature $-a^2$, and let $(\cdot,\cdot)$ be its standard Riemannian metric. We use for the hyperbolic metric the same notation used for the Euclidean one in the previous chapters, as well as for the other geometric objects (second fundamental form, mean curvature, normal vector, measure, etc). We denote by $d_{\mathbb{H}}$ the hyperbolic distance between points induced by $(\cdot,\cdot)$, and by $\langle\cdot,\cdot\rangle$ the induced metric on the hypersurface. Moreover, we put a bar over any geometrical quantity whenever it is referred to the ambient space $\mathbb{H}_a^{n+1}$, e.g. for the Levi-Civita connection $\bar{\nabla}$.
We will use the following notations for the hyperbolic functions: for any $a > 0$

$$\mathrm{s}_a(t) = \tfrac{\sinh(at)}{a}, \quad \mathrm{c}_a(t) = \cosh(at),$$

$$\mathrm{ta}_a(t) = \tfrac{\mathrm{s}_a(t)}{\mathrm{c}_a(t)} \qquad \mathrm{co}_a(t) = \tfrac{\mathrm{c}_a(t)}{\mathrm{s}_a(t)}$$

Given a point $q \in \mathbb{H}_a^{n+1}$ we set, $\forall p \in \mathbb{H}_a^{n+1}$,

$$r_q(p) = d_{\mathbb{H}}(p,q)$$

$$\partial_{r_p} = \bar{\nabla} r_p.$$

The following theorem is due to [16, 17, 18, 19].

Theorem 4.1.4. *Let Ω be a compact $\hbar$-convex domain of $\mathbb{H}_a^{n+1}$, and let $q \in \Omega$ the center of a inball of Ω. If R_- is the inradius of Ω, then*

1. *the maximal distance $\max d_{\mathbb{H}}(q,\partial\Omega)$ between q and the point in Ω satisfies the inequality*

$$\max d_{\mathbb{H}}(q,\partial\Omega) \leqslant R_- + a\ln\frac{(1+\sqrt{\mathrm{ta}_a\frac{R_-}{2}})^2}{1+\mathrm{ta}_a\frac{R_-}{2}} < R_- + a\ln 2$$

2. *For any interior point p of Ω and any boundary point $q \in \partial\Omega$,*

$$(\nu(q), \partial_{r_p}) \geqslant a\,\mathrm{ta}_a(d_{\mathbb{H}}(p,\partial\Omega)),$$

where $\nu(q)$ is the outer normal vector to $\partial\Omega$ at q.

4.2 Long time existence

4.2.1 Preserving of $\hbar$-convexity and its consequences

Lemma 4.2.1. *The flows (3.1) with $h(t)$ given by (3.2) and (3.3) preserve the volume $Vol(\Omega_t)$ and the area $A(\mathcal{M}_t)$ respectively.*

Proof. The proof is analogue to Lemma 2.2.4. $\square$

Lemma 4.2.2. *Along the flow (4.1) we have*

$$\begin{array}{lll} 1) & \frac{d}{dt}A(\mathcal{M}_t) \leqslant 0, & \frac{d}{dt}Vol(\Omega_t) \geqslant 0. \\ 2) & a_1 \leqslant A(\mathcal{M}_t) \leqslant a_2, & v_1 \leqslant Vol(\Omega_t) \leqslant v_2, \end{array}$$

for some positive constants a_1, a_2, v_1 and v_2.

Proof. 1) The proof is the same of Lemma 3.2.3.
2) It follows from part 1) and the isoperimetric inequality in $\mathbb{H}^{n+1}_a$ (see [62]). $\square$

Proposition 4.2.3. *Let $\mathcal{M}_0$ be an $\hbar$-convex hypersurface of $\mathbb{H}^{n+1}_a$, then $\mathcal{M}_t$ is $\hbar$-convex for any time the flow (4.1) is defined.*

Proof. Since $\mathcal{M}_0$ is $\hbar$-convex, we can consider a time interval $[0, T^*)$, with $T^* < T$, such that $\mathcal{M}_t$ is strictly convex for any time $t \in [0, T^*)$. Then we can define b_i^j, be the inverse matrix of h_i^j. Let us define the tensor $S_{ij} = b_{ij} - \frac{1}{a}g_{ij}$. We have that $\hbar$-convexity is equivalent to the fact that $S_{ij} \leqslant 0$. The first step is to compute the evolution equation of S_{ij}. By Proposition 4.1.3 we get:

$$\begin{aligned} \partial_t h_r^s &= \phi'\Delta h_r^s + \phi''\nabla_r H\nabla^s H - (\phi' H - \phi + h)\, h_r^l h_l^s \\ &\quad -a^2(\phi' H + \phi - h)\,\delta_r^s + na^2\phi' h_r^s. \end{aligned}$$

Since $b_i^k h_k^j = \delta_i^j$ we can compute:

$$\Delta b_i^j = -b_s^j b_i^r \Delta h_r^s - 2b_s^j \nabla_l b_i^r \nabla^l h_r^s.$$

Therefore

$$\begin{aligned} \partial_t b_i^j &= -b_s^j b_i^r \partial_t h_r^s \\ &= \phi'\Delta b_i^j + 2\phi' b_s^j \nabla_l b_i^r \nabla^l h_r^s - b_s^j b_i^r \phi''\nabla_r H\nabla^s H \\ &\quad - (\phi' H - \phi + h)\,\delta_i^j - \phi'|A|^2 b_i^j - a^2(\phi' H + \phi - h)\, b_i^r b_r^j - na^2\phi' b_i^j \end{aligned}$$

Finally, by Proposition 4.1.3, we have:

$$\begin{aligned}\partial_t S_{ij} &= \phi' \Delta S_{ij} + 2\phi' b_{sj} \nabla_l b_i^r \nabla^l h_r^s - \phi'' b_i^r b_{sj} \nabla_r H \nabla^s H \\ &\quad + (\phi' H - \phi + h)\, g_{ij} - \phi' |A|^2 b_{ij} + a^2 (\phi' H + \phi - h)\, b_i^r b_{rj} \\ &\quad -na^2 \phi' b_{ij} - 2(\phi - h)\, g_{ij} + \frac{2}{a} (\phi - h)\, h_{ij}.\end{aligned} \tag{4.2}$$

Analogously to the proof of Lemma 2.5 in [55], we can use Codazzi equation and the fact that $\mathcal{M}_t$ is strictly convex in order to estimate the gradient terms in (4.2):

$$\begin{aligned}&2\phi' b_{sj} \nabla_l b_i^r \nabla^l h_r^s - \phi'' b_i^r b_{sj} \nabla_r H \nabla^s H \\ &\quad = -2\phi' b_j^s b_k^r b_i^c \nabla_l b_c^k \nabla^l h_{rs} - \phi'' b_i^r b_{sj} \nabla_r H \nabla^s H \\ &\quad \leqslant -\frac{1}{H} (2\phi' + \phi'' H) (b_i^r \nabla_r H) (b_{sj} \nabla^s H).\end{aligned}$$

Let V be a null eigenvector of S with unit norm. We apply the reaction terms in the equation (4.2) to V. What we get can be estimate as follows on $[0, T^*)$:

$$\begin{aligned}&-\frac{1}{a^2 H} (2\phi' + \phi'' H) (\nabla_i H V^i)^2 + \phi' \left(2H - \frac{|A|^2}{a^2} - na \right) \\ &\quad \leqslant -\frac{\phi'}{na} (H - na)^2 \leqslant 0.\end{aligned}$$

In the last line, we used the hypothesis $iv)$ on the function ϕ and the fact that $|A|^2 \geqslant \frac{H^2}{n}$ for any n-dimensional submanifold. Then $S_{ij} \leqslant 0$ by the maximum principle for symmetric tensors that can be found in Theorem 9.1 of [33]. Thus we have that, until $\mathcal{M}_t$ is strictly convex, the solution is $\hbar$-convex. If, by contradiction, it exists a first time $\bar{T}$ where the solution is not strictly convex, we can apply the previous argument on the interval $[0, \bar{T})$ to conclude that the solution is $\hbar$-convex on $[0, \bar{T}]$. In particular, in $t = \bar{T}$ $h_{ij} \geqslant a g_{ij}$ holds. Then we find a contradiction. $\square$

Remark 4.2.1. An immediate consequence of Proposition 4.2.3 is that, along the flow, $H \geqslant na > 0$ at any space-time point. Then the mean curvature stays away from zero for all times. This is the reason why we don't need to require any particular behaviour at zero for the function $\phi(H)$.

By Proposition 4.2.3, we are able to deduce some geometrical properties. We begin proving that the inradius is uniformly bounded along the flow.

Lemma 4.2.4. *Let $R_-(t)$ be the inradius of the evolving domain Ω_t at time t. Then there are two positive constants c_1 and c_2 such that*

$$c_1 \leqslant R_-(t) \leqslant c_2.$$

Proof. The proof is the same of Lemma 4.1 of [22] with minor modifications in case of the area preserving flow. Let ψ be the inverse function of $s \mapsto vol(\mathbb{S}^n)\int_0^s \mathrm{s}_a(l)dl$ and let ξ be the inverse function of $s \mapsto s + a\ln\frac{(1+\mathrm{ta}_a(s/2))^2}{1+\mathrm{ta}_a(s/2)}$. Note that they are positive increasing functions. Proceeding like in Lemma 4.1 of [22] we get

$$\xi(\psi(V_t)) \leqslant R_-(t) \leqslant \psi(V_t).$$

By Lemma 4.2.2, the thesis follows with $c_1 = \xi(\psi(v_1))$ and $c_2 = \psi(v_2)$. $\square$

As immediate corollary we obtain the following.

Corollary 4.2.5. *For any $t \in [0,T)$, and $p, q \in \Omega_t$, we have*

$$d_{\mathbb{H}}(p,q) < 2(c_2 + a\ln 2).$$

Proof. By Theorem 4.1.4, using also the triangular inequality,

$$\max_{\Omega_t} d_{\mathbb{H}}(p,q) \leqslant 2(R_-(t) + a\ln 2),$$

which is also bounded by the previous lemma. $\square$

4.2.2 Upper bound on the curvatures

From Lemma 4.2.4 we have a positive lower bound on the inner radii $R_-(t)$, so we can take $R_- = c_1$, with c_1 given in Lemma 4.2.4.

Lemma 4.2.6. *There exists $\tau = \tau(a, n, \mathcal{M}_0) > 0$ with the following property: for all $(\bar{q}, \bar{t}) \in \Omega_{\bar{t}} \times [0,T)$ such that $B_{R_-}(\bar{q}) \subset \Omega_{\bar{t}}$, then*

$$B_{R_-/2}(\bar{q}) \subset \Omega_t \qquad \forall t \in [\bar{t}, \min\{\bar{t}+\tau, T\})$$

Proof. Let $\bar{t}, \bar{q}$ be like in the hypotheses. We consider the geodesic sphere centered at $\bar{q}$ that evolves by the standard flow with initial datum R_-, i.e the radius $r_B(t)$ satisfies

$$\begin{cases} r'_B(t) = -\phi(n\mathrm{co}_a(r_B)) \\ r_B(\bar{t}) = R_- \end{cases}$$

We define τ as the time taken by the geodesic sphere of initial radius R_- to contract its radius to $R_-/2$, i.e.

$$\tau = \int_{R_-/2}^{R_-} \frac{ds}{\phi(n\mathrm{co}_a(s))}$$

Notice that τ does not depend neither on $\bar{q}$ nor from $\bar{t}$, but only on R_-. Let $r(x,t) = r_{\bar{q}}(F_t(x))$. Then

$$\partial_t r = (h - \phi(H))(\nu, \partial_r).$$

Then we define the function $g(x,t) = r(x,t) - r_B(t)$ for $t \in [\bar{t}, \bar{t} + \tau]$, and compute the evolution

$$\partial_t g(x,t) = (h - \phi(H))(\nu, \partial_r) + \phi(n\mathrm{co}_a(r_B)), \tag{4.3}$$

where $\phi(H) = \phi(H(x,t))$. Suppose that there exists a first time t^* such that for some $x^* \in \mathcal{M}$ the ball touches the hypersurface $\mathcal{M}_{t^*}$ at the point $F(x^*, t^*)$. Then we have

$$n\mathrm{co}_a(r_B(t^*)) \geqslant H(x^*, t^*) \qquad \partial_t g(x^*, t^*) \leqslant 0$$

Thus, by the monotonicity of ϕ and the fact that $(\nu, \partial_r) \leqslant 1$, we obtain from (4.3)

$$\partial_t g(x^*, t^*) \geqslant h(\nu, \partial_r) > 0$$

From this contradiction we get the result. □

Define the following function, in analogy with [22]:

$$u_q(x,t) = \mathrm{s}_a(r_q)(\nu, \partial_{r_q})$$

with q a given point in Ω_t. We use the same notation used for the support function in the previous chapter just because here u_q will play the same role in the proof of the boundedness of the velocity. Following analogous calculations as in [22], we get the evolution of u:

$$\partial_t u = \phi' \Delta u + \phi' u |A|^2 + (h - \phi - \phi' H)\mathrm{c}_a(r_q)$$

Lemma 4.2.7. *Given any $\bar{t} \in [0, T)$ we have that, in $[\bar{t}, \min\{\bar{t} + \tau, T\})$:*

1. *there exist constants* $C, D > 0$ *such that*

$$C \leqslant r_{\bar{q}} \leqslant D;$$

2. *taken* $c := \frac{a\mathrm{s}_a(C)\mathrm{ta}_a(C)}{2}$,

$$u_{\bar{q}} - c \geqslant c.$$

where $\bar{q}$ *is taken as as in Lemma 4.2.6*

Proof.

1. As a consequence of Lemma 4.2.4 and Lemma 4.2.6, on the time interval $[\bar{t}, \min\{\bar{t} + \tau, T\})$ we have $\frac{c_1}{2} \leqslant \frac{R_-(t)}{2} \leqslant r_{\bar{q}}$. On the other side, by Corollary 4.2.5, $r_{\bar{q}} \leqslant 2(c_2 + a \ln 2)$.

2. It follows by Theorem 2.2 part 2) and some trivial computations.

□

Proposition 4.2.8. *There exists a positive constant* $c_3 = c_3(\mathcal{M}_0, n, a)$ *such that*

$$\phi(H) \leqslant c_3 \qquad \text{on } [0, T).$$

Proof. On any time interval $[\bar{t}, \min\{\bar{t} + \tau, T\})$, we consider

$$W(x,t) := \frac{\phi(H(x,t))}{u_{\bar{q}}(x,t) - c}$$

with $\bar{q}$ as in Lemma 4.2.6 and c as in Lemma 4.2.7. Standard computations show that

$$\begin{aligned}(\partial_t - \phi'\Delta)W &= \frac{2\phi'}{u-c}\langle \nabla W, \nabla u\rangle - h\frac{\phi'}{u-c}(|A|^2 - na^2) - W\frac{h}{u-c}\mathrm{c}_a(r_{\bar{p}}) \\ &\quad + \left(1 + \frac{\phi' H}{\phi}\right) W^2 \mathrm{c}_a(r_{\bar{p}}) - \frac{c}{u-c}\phi'|A|^2 W - na^2\phi' W\end{aligned}$$

By virtue of the $\hbar$-convexity we have $|A|^2 - na^2 \geqslant 0$, then

$$(\partial_t - \phi'\Delta)W \leqslant \frac{2\phi'}{u-c}\langle \nabla W, \nabla u\rangle + \left(1 + \frac{\phi' H}{\phi}\right) W^2 \mathrm{c}_a(D) - \frac{c\phi' H^2}{n(u-c)} W$$

where we also used the fact that $r_{\bar{p}} \leqslant D$ and $|A|^2 \geqslant \frac{H^2}{n}$. We define

$$\overline{W}(t) := \sup_{\mathcal{M}_t} W(x,t) \qquad\qquad X(t) := \{x \in \mathcal{M} | W(x,t) = \overline{W}(t)\}$$

then

$$D_+ W \leqslant \mathrm{c}_a(D)\overline{W}^2 + \overline{W} \sup_{X(t)} \frac{\phi' H}{u-c}\left\{\mathrm{c}_a(D) - \frac{cH}{n}\right\}$$

Let $\tilde{C} > 0$ big enough such that

$$\begin{cases} \tilde{C} \geqslant \frac{3n}{c}\mathrm{c}_a(D) \\ \frac{c}{\phi(\tilde{C})} \leqslant \tau. \end{cases} \tag{4.4}$$

Suppose that there exists a time t^* such that $\overline{W}(t^*) \geqslant \phi(\tilde{C})/c$. Then, using the bound $u-c \geqslant c$ and the monotonicity of ϕ we have that $H(x^*, t^*) \geqslant \tilde{C}$ for any $x^* \in X(t^*)$. Notice that the first condition of (4.4) implies that if $H \geqslant \tilde{C}$, then $\mathrm{c}_a(D) - \frac{cH}{n} \leqslant -\frac{2cH}{3n}$. Then, we get at time $t = t^*$

$$\begin{aligned} D_+\overline{W} &\leqslant \mathrm{c}_a(D)\overline{W}^2 - \frac{2c}{3n}\overline{W} \sup_{X(t^*)} \frac{\phi' H^2}{u-c} \\ &= \overline{W}^2 \sup_{X(t^*)} \left\{\mathrm{c}_a(D) - \frac{2c\phi' H^2}{3n\phi}\right\} \end{aligned}$$

By condition $iii)$ on ϕ , we can suppose $\tilde{C}$ sufficiently large such that $H \geqslant \tilde{C}$ implies

$$\mathrm{c}_a(D) - \frac{2c\phi' H^2}{3n\phi} \leqslant -1$$

Thus at $t = t^*$ we have

$$D_+\overline{W} \leqslant -\overline{W}^2$$

Standard comparison argument then implies that

$$W \leqslant \max\left\{\max_{\mathcal{M}_0} W, \frac{\phi(\tilde{C})}{c}\right\} \qquad \text{on } [0, \min\{\tau, T\}) \tag{4.5}$$

in the case $\bar{t} = 0$, and

$$W \leqslant \max\left\{\frac{1}{t-\bar{t}}, \frac{\phi(\tilde{C})}{c}\right\} \qquad \text{on } [\bar{t}, \min\{\bar{t}+\tau, T\})$$

for a general $\bar{t}$. Then we also have

$$W \leqslant \frac{\phi(\tilde{C})}{c} \quad \text{on } \left[\bar{t} + \frac{c}{\phi(\tilde{C})}, \min\{\bar{t}+\tau, T\}\right). \tag{4.6}$$

Since $\bar{t}$ is arbitrary, combining (4.5) and (4.6) and using the second condition of (4.4) that allows to cover the entire time interval, we obtain

$$W \leqslant \max\left\{\max_{\mathcal{M}_0} W, \frac{\phi(\tilde{C})}{c}\right\} \qquad \text{on } t \in [0,T),$$

which implies the assertion, since $\phi \leqslant (\mathrm{s}_a(D) - c)W$. □

Corollary 4.2.9. *The quantities H, h and $|A|$ are uniformly bounded along the flow.*

Proof. By property *ii)* on ϕ , an upper bound on ϕ implies a bound on H. Then, by Proposition 4.2.8, H is uniformly bounded. The boundedness of h also follows from the boundedness of ϕ. Thanks to $\hbar$-convexity, $|A| \leqslant H$, and so $|A|$ is bounded too. □

From the bounds on H it follows that also ϕ' is uniformly bounded from both sides, and then the flow is uniformly parabolic. Then, by Theorem 1.2.2 we obtain the long time existence of the flow together with the existence of a limit.

Theorem 4.2.10. *The solution $\mathcal{M}_t$ of the flow 4.1 exists for any time. Moreover, up to time subsequences and space isometries, $\mathcal{M}_t$ converges to a smooth limit $\mathcal{M}_\infty$.*

4.3 Convergence to a geodesic sphere

4.3.1 Smooth convergence to a geodesic sphere

As for the Euclidean version, we will prove that the limit hypersurface $\mathcal{M}_\infty$ has to be a geodesic sphere by showing that the mean curvature tends to a constant value. Then, we will show that the rate of the convergence is exponential. From this, we will deduce in particular that the hypersurfaces converge to a geodesic sphere with no need to add isometries. The proof of the firs part follows identically as for the flow 3.1, then we just write the conclusion.

Proposition 4.3.1. *The velocity $\phi(H)$ tends uniformly to h, i.e.*

$$\lim_{t\to\infty}\max_{\mathcal{M}_t}|\phi(H(x,t))-h(t)|=0$$

Proposition 4.3.1 implies that any possible limit of subsequences of $\mathcal{M}_t$ has constant mean curvature, and so, by a classical result of Alexandrov [1], it is a geodesic sphere. Then standard techniques, see e.g. [5, 22], allow now to conclude that the whole family $\mathcal{M}_t$ converges smoothly to a geodesic sphere up to isometries.

4.3.2 Exponential rate

We will use the same method seen in the previous chapter, adapted for an hyperbolic setting. We follow the paper of Guo, Li and Wu [32]. As for the Euclidian case, our situation is simplified by that fact that we already have the convergence to a sphere. Furthermore, the function ϕ is H-dependent, while in [32] were considered homogeneous one degree functions of the principal curvatures.
In analogy with [32], define the *perturbed* Weingarten operator $\tilde{h}_j^i = h_j^i - a\delta_j^i$. Its trace, norm and determinant will be denoted as $\tilde{H} = \mathrm{Tr}\tilde{h}_j^i$, $|\tilde{A}|^2 = \tilde{h}_i^j\tilde{h}_j^i$ and $\tilde{K} = \det\tilde{h}_j^i$ respectively. We also indicate by $\tilde{b}_j^i$ the inverse matrix of $\tilde{h}_j^i$, and by $\tilde{B}$ its trace. Then, we define

$$\tilde{Q} = \frac{\tilde{K}}{\tilde{H}^n}.$$

Since the hypersurfaces $\mathcal{M}_t$ approach uniformly to a geodesic sphere as t goes to infinity, then $\mathcal{M}_t$ is strictly $\hbar$-convex for t sufficiently large. Then $\tilde{Q}$ is well defined and strictly positive for t sufficiently large. This fact makes things work even if we choose an initial datum not necessary *strictly* $\hbar$-convex, but just $\hbar$-convex. Again, $\tilde{Q} \leqslant \frac{1}{n^n}$, and the equality holds if and only if the hypersurface is totally umbilic. Then we know that $\tilde{Q}$ converges smoothly to the constant value $\frac{1}{n^n}$. As for the Euclidean flow, our goal is to show that this convergence is exponential.

Proposition 4.3.2. *The quantities $\tilde{K}$, $\tilde{Q}$ evolve according to*

$$\begin{aligned}
\partial_t \tilde{K} &= \phi' \Delta \tilde{K} - \frac{(n-1)}{n} \frac{\phi' |\nabla \tilde{K}|^2}{\tilde{K}} + \frac{\tilde{K}}{\tilde{H}^2} \phi' |\tilde{H} \nabla \tilde{h}^i_j - \tilde{h}^i_j \nabla \tilde{H}|^2_{g,\tilde{b}} \\
&\quad - \frac{\tilde{H}^{2n}}{n\tilde{K}} \phi' \left| \nabla \frac{\tilde{K}}{\tilde{H}^n} \right|^2 + \tilde{K} \phi'' \tilde{b}^i_j \nabla_i \tilde{H} \nabla^j \tilde{H} + \tilde{K}\tilde{H}(\phi - \phi' H - h) \\
&\quad + 2an\tilde{K}(\phi - h) + n\tilde{K}\phi' |\tilde{A}|^2 + a\tilde{K}\phi'|\tilde{A}|^2 \tilde{B}; \\
\partial_t \tilde{Q} &= \phi' \Delta \tilde{Q} + \frac{(n+1)}{n\tilde{H}^n} \phi' \langle \nabla \tilde{Q}, \nabla \tilde{H}^n \rangle - \frac{(n-1)}{n\tilde{K}} \phi' \langle \nabla \tilde{Q}, \nabla \tilde{K} \rangle \\
&\quad - \frac{\tilde{Q}^{-1}}{n} \phi' |\nabla \tilde{Q}|^2 + \frac{\tilde{Q}}{\tilde{H}^2} \phi' |\tilde{H} \nabla \tilde{h}^i_j - \tilde{h}^i_j \nabla \tilde{H}|^2_{g,\tilde{b}} + \tilde{Q} \phi'' |\nabla \tilde{H}|^2_{\tilde{b} - \frac{n}{\tilde{H}} g} \\
&\quad + \frac{\tilde{Q}}{\tilde{H}} (\phi' H - \phi + h)(n|\tilde{A}|^2 - \tilde{H}^2) + a\tilde{Q}|\tilde{A}|^2 \left(\tilde{B}^2 - \frac{n^2}{\tilde{H}} \right).
\end{aligned}$$

where

$$\begin{aligned}
|\tilde{H} \nabla \tilde{h}^i_j - \tilde{h}^i_j \nabla \tilde{H}|^2_{g,\tilde{b}} &:= b^n_m b^s_r (\tilde{H} \nabla_i \tilde{h}^m_n - \tilde{h}^m_n \nabla_i \tilde{H})(\tilde{H} \nabla^i \tilde{h}^r_s - \tilde{h}^r_s \nabla^i \tilde{H}) \\
|\nabla \tilde{H}|^2_{\tilde{b} - \frac{n}{\tilde{H}} g} &:= \left(\tilde{b}^{ij} - \frac{n}{\tilde{H}} g^{ij} \right) \nabla_i \tilde{H} \nabla_j \tilde{H}
\end{aligned}$$

Proof. The proof is almost the same as for Lemma 3.3.5, but we write it for completeness. Some easy computations show that $\tilde{H} = H - na$ and $|\tilde{A}|^2 = |A|^2 + na^2 - 2aH$. Hence, by Proposition 4.1.3, we get

$$\begin{aligned}
\partial_t \tilde{h}^j_i &= \phi' \Delta \tilde{h}^j_i + \phi'' \nabla_i H \nabla^j H + (\phi - h - \phi' H) \left(\tilde{h}^l_i \tilde{h}^j_l + 2a\tilde{h}^j_i + a^2 \delta^j_i \right) \\
&\quad + \phi' \left(|\tilde{A}|^2 + 2a\tilde{H} + 2na^2 \right) \left(\tilde{h}^j_i + a\delta^j_i \right) - a^2 \left(\phi' H + \phi - h \right) \delta^j_i, \\
\partial_t \tilde{H}^n &= \phi' \Delta \tilde{H}^n + n\tilde{H}^{n-2} (\phi'' \tilde{H} - (n-1)\phi') |\nabla \tilde{H}|^2 + n\tilde{H}^{n-1} (\phi - h) \left(|\tilde{A}|^2 + 2a\tilde{H} \right).
\end{aligned}$$

Then we can compute the evolution of $\tilde{K}$:

$$\begin{aligned}
\partial_t \tilde{K} &= \tilde{K} \tilde{b}^i_j \frac{\partial \tilde{h}^j_i}{\partial t} \\
&= \phi' \Delta \tilde{K} - \phi' \frac{|\nabla \tilde{K}|^2}{\tilde{K}} - \tilde{K} \phi' \nabla_r \tilde{b}^i_j \nabla^r \tilde{h}^j_i + \tilde{K} \phi'' \tilde{b}^i_j \nabla_i H \nabla^j H \\
&\quad + \tilde{K} \left(\phi - h - \phi' H \right) \left(\tilde{H} + 2an + a^2 \tilde{B} \right) - a^2 \tilde{K} \tilde{B} \left(\phi' H + \phi - h \right) \\
&\quad + \tilde{K} \phi' \left(|\tilde{A}|^2 + 2a\tilde{H} + 2na^2 \right) \left(n + a\tilde{B} \right) \\
&= \phi' \Delta \tilde{K} - \phi' \frac{|\nabla \tilde{K}|^2}{\tilde{K}} - \tilde{K} \phi' \nabla_r \tilde{b}^i_j \nabla^r \tilde{h}^j_i + \tilde{K} \phi'' \tilde{b}^i_j \nabla_i H \nabla^j H \\
&\quad + \tilde{K}\tilde{H}(\phi - \phi' H - h) + 2an\tilde{K}(\phi - h) + n\tilde{K}\phi' |\tilde{A}|^2 + a\tilde{K}\phi' |\tilde{A}|^2 \tilde{B}.
\end{aligned}$$

As for the proof of Proposition 3.3.5,

$$\tilde{K}\phi'\nabla_r\tilde{b}^i_j\nabla^r\tilde{h}^j_i = -\frac{1}{n}\frac{\phi'|\nabla\tilde{K}|^2}{\tilde{K}} - \frac{\tilde{K}}{\tilde{H}^2}\phi'|\tilde{H}\nabla\tilde{h}^i_j - \tilde{h}^i_j\nabla\tilde{H}|^2_{g,\tilde{b}} + \frac{\tilde{H}^{2n}}{n\tilde{K}}\phi'\left|\nabla\frac{\tilde{K}}{\tilde{H}^n}\right|^2$$

and then

$$\begin{aligned}
-\phi'\frac{|\nabla\tilde{K}|^2}{\tilde{K}} - \tilde{K}\phi'\nabla_r\tilde{b}^i_j\nabla^r\tilde{h}^j_i &= -\frac{(n-1)}{n}\frac{\phi'|\nabla\tilde{K}|^2}{\tilde{K}} \\
&\quad+\frac{\tilde{K}}{\tilde{H}^2}\phi'|\tilde{H}\nabla\tilde{h}^i_j - \tilde{h}^i_j\nabla\tilde{H}|^2_{g,\tilde{b}} - \frac{\tilde{H}^{2n}}{n\tilde{K}}\phi'\left|\nabla\frac{\tilde{K}}{\tilde{H}^n}\right|^2.
\end{aligned}$$

From the last equality, we get the evolution of $\tilde{K}$. From

$$\Delta\tilde{Q} = \frac{\Delta\tilde{K}}{\tilde{H}^n} - \frac{\tilde{K}\Delta\tilde{H}^n}{\tilde{H}^{2n}} - \frac{2}{\tilde{H}^{2n}}\nabla_r\tilde{K}\nabla^r\tilde{H}^n + 2\frac{\tilde{Q}}{\tilde{H}^{2n}}|\nabla\tilde{H}^n|^2.$$

we have

$$\begin{aligned}
\partial_t\tilde{Q} &= \frac{1}{\tilde{H}^n}\partial_t\tilde{K} - \frac{\tilde{K}}{\tilde{H}^{2n}}\partial_t\tilde{H}^n \\
&= \phi'\Delta\tilde{Q} + \frac{2}{\tilde{H}^{2n}}\phi'\langle\nabla,\tilde{K}\nabla\tilde{H}^n\rangle - 2\frac{\tilde{Q}}{\tilde{H}^{2n}}\phi'|\nabla\tilde{H}^n|^2 \\
&\quad+n(n-1)\frac{\tilde{Q}}{\tilde{H}^2}\phi'|\nabla\tilde{H}|^2 - \frac{n-1}{n}\phi'\frac{|\nabla\tilde{K}|^2}{\tilde{K}\tilde{H}^n} \\
&\quad-\frac{\tilde{Q}^{-1}}{n}\phi'|\nabla\tilde{Q}|^2 + \frac{\tilde{Q}}{\tilde{H}^2}\phi'|\tilde{H}\nabla\tilde{h}^i_j - \tilde{h}^i_j\nabla\tilde{H}|^2_{g,\tilde{b}} + \tilde{Q}\phi''|\nabla\tilde{H}|^2_{\tilde{b}-\frac{n}{\tilde{H}}g} \\
&\quad+\frac{\tilde{Q}}{\tilde{H}}(\phi'H-\phi+h)(n|\tilde{A}|^2-\tilde{H}^2) + a\tilde{Q}|\tilde{A}|^2\left(\tilde{B}^2-\frac{n^2}{\tilde{H}}\right).
\end{aligned}$$

and the conclusion follows as in the proof of Proposition 3.3.5 noticing

$$\begin{aligned}
\langle\nabla\tilde{Q},\nabla\tilde{H}^n\rangle &= \frac{1}{\tilde{H}^n}\langle\nabla\tilde{K},\nabla\tilde{H}^n\rangle - \frac{\tilde{Q}}{\tilde{H}^n}|\nabla\tilde{H}^n|^2, \\
\langle\nabla\tilde{Q},\nabla\tilde{K}\rangle &= \frac{1}{\tilde{H}^n}|\nabla\tilde{K}|^2 - \frac{\tilde{Q}}{\tilde{H}^n}\langle\nabla\tilde{H}^n,\nabla\tilde{K}\rangle.
\end{aligned}$$

□

Define $\tilde{f} = \frac{1}{n^n} - \tilde{Q}$. We already know that f smoothly converges to zero.

Proposition 4.3.3. *There are a time $\bar{t} > 0$ and two constants $c,\ \delta > 0$ such that for every time $t \geqslant \bar{t}$ we have :*

$$\tilde{f} \leqslant ce^{-\delta t}.$$

Proof. By Proposition 4.3.2 we can compute the evolution equation for f:

$$\begin{aligned}\frac{\partial f}{\partial t} &= \phi'\Delta f - \frac{(n+1)}{n\tilde{H}^n}\phi'\langle\nabla f, \nabla\tilde{H}^n\rangle + \frac{(n-1)}{n\tilde{K}}\phi'\langle\nabla f, \nabla\tilde{K}\rangle \\ &\quad + \frac{\tilde{Q}^{-1}}{n}\phi'|\nabla f|^2 - \frac{\tilde{Q}}{\tilde{H}^2}\phi'|\tilde{H}\nabla\tilde{h}^i_j - \tilde{h}^i_j\nabla\tilde{H}|^2_{g,\tilde{b}} - \tilde{Q}\phi''|\nabla\tilde{H}|^2_{\tilde{b}-\frac{n}{\tilde{H}}g} \\ &\quad - a\phi'\tilde{Q}|\tilde{A}|^2\left(\tilde{B} - \frac{n^2}{\tilde{H}}\right) - \frac{\tilde{Q}}{\tilde{H}}\left(\phi'H - \phi + h\right)\left(n|\tilde{A}|^2 - \tilde{H}^2\right).\end{aligned} \tag{4.7}$$

The gradient terms can be estimated exactly in the same way as in the proof of Proposition 3.3.7 just substituting h^i_j, b^i_j, H, K, Q and f with $\tilde{h}^i_j, \tilde{b}^i_j\tilde{H}, \tilde{K}, \tilde{Q}$ and $\tilde{f}$ respectively: all the passages still hold. Then there exists a time $\bar{t}_1$ such that for all $t \geqslant \bar{t}_1$, the gradient terms are nonpositive.
The reaction term $\frac{\tilde{Q}}{\tilde{H}}\left(\phi'H - \phi + h\right)\left(n|\tilde{A}|^2 - \tilde{H}^2\right)$ is analogue to the one appearing in the proof of Proposition 3.3.7 and can be estimated in the same way. Then, there exists a time $\bar{t}_2$ such that

$$\frac{\tilde{Q}}{\tilde{H}}\left(\phi'H - \phi + h\right)\left(n|\tilde{A}|^2 - \tilde{H}^2\right) \geqslant \delta\tilde{f},$$

for some constant $\delta > 0$. Moreover, by the relation between the harmonic and the arithmetic means of n positive numbers, we have

$$\tilde{B} - \frac{n^2}{\tilde{H}} \geqslant 0.$$

Thus, setting $\bar{t} = \max\{\bar{t}_1, \bar{t}_2\}$, we have

$$\frac{\partial f}{\partial t} \ \leqslant\ \phi'\Delta f - \frac{(n+1)}{n\tilde{H}^n}\phi'\langle\nabla f, \nabla\tilde{H}^n\rangle + \frac{(n-1)}{n\tilde{K}}\phi'\langle\nabla f, \nabla\tilde{K}\rangle + \frac{\tilde{Q}^{-1}}{n}\phi'|\nabla f|^2 - \delta f$$

for all times $t \geqslant \bar{t}$. Then the thesis follows from the maximum principle. □

Arguing as for Corollary 3.3.8, we obtain:

Corollary 4.3.4. *The second fundamental form A converges exponentially in C^{∞} to the one of a geodesic sphere. In particular, there exist positive constants c', δ' such that*

$$|\phi(H) - h| \leqslant c' e^{-\delta' t}$$

Then also in this case, the whole family $F(\cdot, t)$ tends to a unique limit hypersurface for t that goes to infinity. Analogously, the metric and the embedding converge smoothly. This complete the proof of Theorem 4.1.1.

CHAPTER 5

Some results on entire graphs

5.1 Presentation of the problem

Let $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$ be an embedding of a smooth manifold $\mathcal{M}$ in the Eucledean space $\mathbb{R}^{n+1}$. As usual, denote its image by $\mathcal{M}_0 = F_0(\mathcal{M})$. We assume that $\mathcal{M}_0$ can be written as an entire graph over $\mathbb{R}^n$, i.e. there exists $u_0 : \mathbb{R}^n \to \mathbb{R}$ such that any point $p \in F_0(\mathcal{M})$ can be written as

$$p = (y, u_0(y)), \qquad y \in \mathbb{R}^n.$$

Furthermore we assume that u_0 is a strictly convex function, with $||Du_0||_\infty + ||D^2u_0||_\infty < \infty$.
Then we consider a family of maps $F : \mathcal{M} \times [0,T) \to \mathbb{R}^{n+1}$, with $F_t := F(\cdot, t) : \mathcal{M} \to \mathbb{R}^{n+1}$ satisfying

$$\begin{cases} \partial_t F(x,t) = -\phi(H(x,t))\nu(x,t) \\ F(x,0) = F_0(x), \end{cases} \tag{5.1}$$

where $\nu(\cdot, t)$ denotes the downward unit normal vector of the evolving hypersurface $\mathcal{M}_t := F_t(\mathcal{M})$. The signs of the curvatures are chosen such that $\mathcal{M}_0$ is convex if and only if the function u_0 is convex.
We choose $\phi : [0, +\infty) \to \mathbb{R}$ as a continuous function, C^2 differentiable in $(0, +\infty)$ with the following properties:

i) $\phi(\alpha) > 0 \ \ \forall \alpha > 0, \qquad \phi(0) = 0;$

ii) $\phi'(\alpha) > 0 \quad \forall \alpha > 0$;

iii) $\phi(\alpha) - \phi'(\alpha)\alpha \leqslant 0 \quad \forall \alpha > 0$;

iv) $\phi''(\alpha)\alpha + 2\phi'(\alpha) \geqslant 0 \quad \forall \alpha > 0$;

v) $\lim_{\alpha \to \infty} \phi(\alpha) = \infty$.

Notice that any convex function ϕ satisfies property *iv)*. Furthermore, if ϕ also satisfies $\phi(0) = 0$, and $\phi'_+(0) \geqslant 0$ (where ϕ'_+ means the right side derivative), than automatically $i) - v)$ hold.
We recall that, if a hypersurface is given as a graph of a function $u(y)$, the following expressions holds for the downward unit normal vector, metric tensor and second fundamental form:

$$\begin{aligned} \nu(y) &= \frac{(Du(y), -1)}{\sqrt{1 + |Du(y)|^2}} \\ g_{ij}(y) &= \delta_{ij} + D_i u(y) D_j u(y) \\ h_{ij}(y) &= \frac{D^2_{ij} u(y)}{\sqrt{1 + |Du(y)|^2}}, \end{aligned}$$

where D_i denote the differentiation with respect the coordinate i. Using these expressions, the flow equation (5.1) is equivalent, up to tangential diffeomorphisms, to

$$\begin{cases} \partial_t u = \sqrt{1 + |Du|^2}\phi\left(\frac{1}{\sqrt{1+|Du|^2}}\left(\delta^{ij} - \frac{D^i D^j}{1+|Du|^2}\right) D^2_{ij} u\right) \\ u(y, 0) = u_0(y) \end{cases} \tag{5.2}$$

We will prove the following result.

Theorem 5.1.1. *Let $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$, with $n \geqslant 1$, be a smooth embedding of a n-dimensional manifold, such that F_0 is an entire graph over $\mathbb{R}^n$ of a strictly convex function $u_0 : \mathbb{R}^n \to \mathbb{R}$ satisfying $||Du_0||_\infty + ||D^2 u_0||_\infty < \infty$. Then for any time $t \in [0, \infty)$ there exists a smooth solution $u(\cdot, t)$ of the problem (5.2). The solution is convex and $||Du_0||_\infty + ||D^2 u_0||_\infty$ is bounded by a constant only depending on the initial datum u_0.*

5.2 Compact case

In contrast with the flows studied in the previous chapters, this time we deal with non compact objects. Since our strategy for the proof of the main theorem is based on an approximating procedure by compact hypersurfaces, it is useful in this step to recall some simple facts about the flow (5.1) in the compact case. The evolution equations are given by Proposition 3.1.2 in Chapter 3, setting $h(t) \equiv 0$.

Proposition 5.2.1. *Let consider the flow (5.1) with $\mathcal{M}_0$ a strictly convex, compact hypersurface without boundary. Then the solution $\mathcal{M}_t$ exists unique on a maximal time interval $[0, T)$. Furthermore,*

$$\min_{\mathcal{M}_t} H \geqslant \min_{\mathcal{M}_0} H \qquad \forall t \in [0, T).$$

Proof. Since $H > 0$ on the initial datum, then $\phi' > 0$ on a first time interval. By Theorem 1.2.1, this ensures the existence of a unique solution of (5.1) on a maximal time interval $[0, T)$. Furthermore, by the evolution equation of H, we have

$$\partial_t H \geqslant \phi' \Delta H + \phi'' |\nabla H|^2.$$

Then, by the maximum principle, $\min_{\mathcal{M}_t} H \geqslant \min_{\mathcal{M}_0} H$ on $[0, T)$. □

Proposition 5.2.2. *Let consider the flow (5.1) with $\mathcal{M}_0$ a strictly convex, compact hypersurface without boundary. Then the solution $\mathcal{M}_t$ is strictly convex for any time such that is defined.*

Proof. The proof is the same as for the volume preserving case, given by Proposition 3.2.1. □

Proposition 5.2.3. *Let consider the flow (5.1) with $\mathcal{M}_0$ a strictly convex, compact hypersurface without boundary. The maximal time of the solution is greater than the maximal time of any round sphere contracting by (5.1) with initial datum enclosed by $\mathcal{M}_0$.*

Proof. Similarly as for the proof of Proposition 3.2.7, it can be shown that the curvatures stay bounded as long as the hypersurface encloses a ball of positive radius. On the other hand, the uniform parabolicity of the flow is guaranteed by Proposition 5.2.1. Then, by Theorem 1.2.2 that also holds for velocities of the kind $\phi(H)$, the flow can be continued as long as a ball is contained inside the hypersurface. The thesis follows from the avoidance property 1.2.7. □

5.3 Long time existence

5.3.1 A localized estimate

Since the hypersurface $\mathcal{M}_0$ is unbounded, even if all curvatures are positive at any point, their infimum over $\mathcal{M}_0$ is zero. Then the flow could present some degeneracy when the curvatures approach to zero, and we can't apply the standard parabolic theory to deduce the short time existence of the solution. With this in mind, is fundamental to "localize" the maximum principle on a suitable compact region of the hypersurface that can be enlarged as we like. The following result is analogous to Theorem 2.1 in [29].

Proposition 5.3.1. *Let $F : \mathcal{M} \times [0, \tau] \to \mathbb{R}^{n+1}$, $\tau > 0$, a solution of (5.1), with $\mathcal{M}_t = F(\mathcal{M}, t)$. Suppose that there exist $R > 0$ and $[z_0, z_1] \subset \mathbb{R}$ such that:*

a) $\mathcal{M}_t \cap (B_R(0) \times [z_0, z_1])$ is the graph of a convex function $u(\cdot, t)$, with $u : B_R(0) \times [0, \tau] \to \mathbb{R}$;

b) $\phi' \leqslant \tilde{C}$ on $B_R(0) \times [0, \tau]$, for some constant $\tilde{C} > 0$.

Let $\bar{p} = (0, \bar{y}_{n+1})$, where $0 \in \mathbb{R}^n$, $\bar{y}_{n+1} \in \mathbb{R}$, such that $u(0, t) \leqslant \bar{y}_{n+1}$ for all $t \in [0, \tau]$.
Given any $0 < \tilde{R} \leqslant R$, let $f(x, t) = \tilde{R}^2 - |F(x, t) - \bar{p}|^2 - 2nt\tilde{C}$.
Then, denoting by f_+ the positive part of f, we have the estimate

$$\phi^{-1} f_+(x, t) \leqslant \sup_{\mathcal{M}_0} \phi^{-1} f_+(\cdot, 0) \qquad \forall (x, t) \in supp f_+.$$

Proof. The proof follows the one in [29].
We can assume $u_0 \geqslant 0$. Then, by the choice of the unit normal in (5.1), $u \geqslant 0$ for all times. Given $\tilde{R}$ as in the hypotheses, define $\eta(r) = (\tilde{R}^2 - r)^2$. Notice that, by this choice of $\tilde{R}$, the support of f is strictly contained in $B_R(0) \times [0, \tau]$. Choosing $r(x, t) = |F(x, t) - \bar{p}|^2 + 2nt\tilde{C}$ we have, at any $(x, t) \in B_R(0) \times [0, \tau]$,

$$(\partial_t - \phi'\Delta)\eta = 4\sqrt{\eta}(\phi - \phi' H)(F - \bar{p}, \nu) - 2\phi'|\nabla|F - \bar{p}|^2|^2 + 4n\sqrt{\eta}(\phi' - \tilde{C}), \quad (5.3)$$

while the evolution of ϕ^{-2} is given by

$$(\partial_t - \phi'\Delta)\phi^{-2} = -6\phi'|\nabla\phi^{-1}|^2 - 2\phi'\phi^{-2}|A|^2. \qquad (5.4)$$

(5.3) together with (5.4) gives

$$\begin{aligned}(\partial_t - \phi'\Delta)\phi^{-2}\eta &= 4\phi^{-1}\sqrt{\eta}(\phi - \phi' H)(F - \bar{p}, \nu) - 2\phi^{-2}\phi'|\nabla|F - \bar{p}|^2|^2 \\ &\quad - 6\phi'\eta|\nabla\phi^{-1}|^2 - 2\phi'\eta\phi^{-2}|A|^2 - 2\phi'\langle\nabla\eta, \nabla\phi^{-2}\rangle \\ &\quad + 4n\phi^{-2}\sqrt{\eta}(\phi' - \tilde{C}) \\ &\leqslant -2\phi^{-2}\phi'|\nabla|F - \bar{p}|^2|^2 - 6\phi'\eta|\nabla\phi^{-1}|^2 \\ &\quad - 2\phi'\eta\phi^{-2}|A|^2 - 2\phi'\langle\nabla\eta, \nabla\phi^{-2}\rangle\end{aligned} \tag{5.5}$$

where for the inequality we used property $iii)$ of ϕ, the hypothesis $\phi' \leqslant \tilde{C}$ and the fact that $(F - \bar{p}, \nu) \geqslant 0$ by convexity and the choice of $\bar{p}$. We have the equalities

$$\begin{aligned}-2\phi'\langle\nabla\eta, \nabla\phi^{-2}\rangle &= \phi'\langle\nabla\phi^{-2}, \nabla\eta\rangle - 3\phi'\langle\nabla\phi^{-2}, \nabla\eta\rangle \\ &= \phi'\eta^{-1}\eta\langle\nabla\phi^{-2}, \nabla\eta\rangle - 6\phi'\phi^{-1}\langle\nabla\phi^{-1}, \nabla\eta\rangle \\ &= \phi'\eta^{-1}\langle\nabla(\phi^{-2}\eta), \nabla\eta\rangle - \eta^{-1}\phi'\phi^{-2}|\nabla\eta|^2 - 6\phi'\phi^{-1}\langle\nabla\phi^{-1}, \nabla\eta\rangle\end{aligned}$$

and

$$\eta^{-1}|\nabla\eta|^2\phi^{-2} = \eta^{-1}|2\sqrt{\eta}\nabla|F - \bar{p}|^2|^2\phi^{-2} = 4|\nabla|F - \bar{p}|^2|^2\phi^{-2},$$

then (5.5) becomes

$$\begin{aligned}(\partial_t - \phi'\Delta)\phi^{-2}\eta \leqslant\ & -6\phi'\phi^{-2}|\nabla|F|^2|^2 + \phi'\eta^{-1}\langle\nabla(\phi^{-2}\eta), \nabla\eta\rangle \\ & - 6\phi'\phi^{-1}\langle\nabla\phi^{-1}, \nabla\eta\rangle - 6\phi'\eta|\nabla\phi^{-1}|^2 - 2\phi'\eta\phi^{-2}|A|^2.\end{aligned}$$

Using the Cauchy-Schwarz and the Young inequalities, we can estimate

$$\begin{aligned}-6\phi'\phi^{-1}\langle\nabla\phi^{-1}, \nabla\eta\rangle &\leqslant 6\phi'\phi^{-1}|\langle\nabla\phi^{-1}, \nabla\eta\rangle| \leqslant 6\phi'\phi^{-1}|\nabla\phi^{-1}||\nabla\eta| \\ &\leqslant 6\phi'|\nabla\phi^{-1}|^2\eta + \frac{3}{2}|\nabla\eta|^2\eta^{-1}\phi^{-2} \\ &= 6\phi'|\nabla\phi^{-1}|^2\eta + 6\phi'|\nabla|F - \bar{p}|^2|^2\phi^{-2}\end{aligned}$$

so, finally,

$$(\partial_t - \phi'\Delta)\phi^{-2}\eta \leqslant \phi'\eta^{-1}\langle\nabla(\phi^{-2}\eta), \nabla\eta\rangle.$$

Replacing η with f_+^2 the same computations hold, then we get the result by the maximum principle. $\square$

5.3.2 Costruction of the solution

In order to prove the existence of the solution for short times, we follow an approximating procedure by replacing the initial datum with a compact convex approximating hypersurface, and we use the parabolic theory for compact manifolds to deduce the existence of the solution. Then we enlarge more and more the initial compact datum and we take the limit solution of the approximating flows. We will show that this limit is solution of (5.1). With this technique, the existence for all times, the preserving of convexity together with the property to be an entire graph will follow automatically. We take this procedure from [4], with some modifications due to the fact that in our case we don't have the homogeneity of the velocity. Furthermore, unlike the authors in [4], we don't employ the Harnack-type inequality due to B.Andrews in Theorem 5.21 part (2) of [6].

Let u_0 be as in the hypotheses of Theorem 5.1.1 Without loss of generality, we can assume $u_0(0) = Du_0(0) = 0$.
By Lemma 6 of [4], for any $R > 1$ we can construct a compact convex hypersurface $\mathcal{N}_R$ satisfying the following properties:

1. $\mathcal{N}_R \cap (B_R(0) \times [0, c_0 R]) = \mathcal{M}_0 \cap (B_R(0) \times [0, c_0 R])$ for some $c_0 > 1$ independent of R;

2. $\mathcal{N}_R$ encloses the ball of center $(0, c_0 R)$ and radius R;

3. $\mathcal{N}_R$ has diameter less than $c_1 R$, for some $c_1 > 1$ independent of R;

4. $\max_{\mathcal{N}_R} H \leqslant c_2 \sup_{\mathcal{M}_0} H$ for some $c_2 \geqslant 1$ independent of R.

Notice that H is bounded on $\mathcal{M}_0$ by the hypotheses on u_0.
Let consider the evolution of $\mathcal{N}_R$ by (5.1). By Propositions 5.2.1 and 5.2.2, there exists a unique maximal solution $\mathcal{N}_R(t) = F_R(\cdot, t)$ which is convex, where $F_R : \mathcal{N}_R \times [0, T) \to \mathbb{R}^{n+1}$. Furthermore, by construction and by Proposition 5.2.3, we know that $T \geqslant \tau_R$, where τ_R is the time taken by a sphere of centre $q_0 = (0, c_0 R)$ and initial radius R to contract by (5.1) to a sphere of radius $\frac{R}{2}$. Notice that τ_R is increasing in R, and $\tau_R \to \infty$ as $R \to \infty$.
Denoting $d_R = diam \mathcal{N}_R$ we can consider, on $[0, \tau_R]$, the ratio

$$W(x,t) = \frac{\phi(H(x,t))}{(F_R(x,t) - q_0, \nu) - R/4}$$

which is well defined since, by convexity of $\mathcal{N}_R(t)$ and the Property (3) of $\mathcal{N}_R$,

$$\frac{R}{4} \leqslant (F_R(x,t) - q_0, \nu) - \frac{R}{4} \leqslant (c_1 - \frac{1}{4})R.$$

Following the same procedure of the proof of Proposition 3.2.7 in Chapter 3, we can estimate

$$\phi(H) \leqslant \max\left\{\max_{\mathcal{N}_R} \phi(H), 4\frac{\phi(C_R)}{R} d_R\right\} \quad \text{on } [0, \tau_R] \tag{5.6}$$

where $C_R > 0$ is a constant chosen as in the proof of Proposition 3.2.7 in Chapter 3, and then decreasing with R. Then, by Properties (3), (4) of $\mathcal{N}_R$ and since the diameter of $\mathcal{N}_R(t)$ is decreasing in time and $R > 1$, (5.6) becomes

$$\phi(H) \leqslant \max\left\{\sup_{\mathcal{M}_0} \phi(c_2 H), 4c_1\phi(C_R)\right\} \leqslant C_1^* \quad \text{on } [0, \tau_R] \tag{5.7}$$

where $C_1^* > 0$ does not depend on R. Then, we find

$$|A|^2 \leqslant H^2 \leqslant C_2^* \quad \text{on } [0, \tau_R] \tag{5.8}$$

for some constant $C_2^* > 0$ not depending on R.
Let $p = (y', y'_{n+1})$, $p'' = (y'', y''_{n+1})$ be points in $\mathcal{N}_R(t) \cap \big(B_{R/4}(0) \times [0, c_0 R]\big)$ for some $t \in [0, \tau_R]$. We have

$$0 \leqslant y'_{n+1}, y''_{n+1} < c_0 R, \qquad \frac{|y''_{n+1} - y'_{n+1}|}{|y'' - y'|} \leqslant 4c_0, \tag{5.9}$$

then $\mathcal{N}_R(t) \cap \big(B_{R/4}(0) \times [0, c_0 R]\big)$ is the graph of a convex function with Lipschitz constant smaller or equal to $4c_0$. We denote such a function by $u_R(y,t)$, where $(y,t) \in B_{R/4}(0) \times [0, \tau_R]$.

Let choose a compact set $\tilde{K} \subset \mathbb{R}^n$ and a time $\tilde{t} \in [0, \tau_R)$. If R is sufficiently large, we can assume $\tilde{K} \subset B_{R/4}(0)$. We want to bound ϕ' from below on $\tilde{K} \times [0, \tilde{t}]$ uniformly in R. By (5.8), there exists $\tilde{C} > 0$ such that $\phi' \leqslant \tilde{C}$ on $\tilde{K} \times [0, \tilde{t}]$. Define

$$\bar{z} = \max_{y \in \tilde{K}} u_R(y, 0) = u_0(\bar{y})$$

for some $\bar{y} \in \tilde{K}$. On $\mathcal{N}_R(t) \cap \big(B_{R/4}(0) \times [0, c_0 R]\big)$ with $t \in [0, \tilde{t}]$ we consider, as in Theorem 5.3.1,

$$f(x,t) = \tilde{R}^2 - |F_R(x,t) - \bar{p}|^2 - 2nt\tilde{C},$$

where $\bar{p} = (0, \bar{y} + C_1^* \tilde{t})$, $0 \in \mathbb{R}^n$ and $\tilde{R} > 0$ to decide later. The choice of $\bar{p}$ guarantees that, on $[0, \tilde{t}]$, the graph $(y, u_R(y,t))$ does not intersect $\bar{p}$ at any time. We define continuously f also on $\mathbb{R}^n \times 0$ simply setting $f(x,0) = \tilde{R}^2 - |F(x,0) - \bar{p}|^2$. Then, using (5.2), (5.7) and (5.9) and writing f as a function of (y,t), with $(y,t) \in \tilde{K} \times [0, \tilde{t}]$,

$$\begin{aligned}
f(y,t) &= \tilde{R}^2 - |y|^2 - |u_R(y,t) - u_0(\bar{y}) - C_1^* \tilde{t}|^2 - 2nt\tilde{C} \\
&\geqslant \tilde{R}^2 - |y|^2 - |u_R(y,t) - u_0(\tilde{y})|^2 - (C_1^* \tilde{t})^2 - 2n\tilde{t}\tilde{C} \\
&\geqslant \tilde{R}^2 - |y|^2 - |u_R(y,t) - u_R(\bar{y},t)|^2 - |u_R(\bar{y},t) - u_0(\bar{y})|^2 - (C_1^* \tilde{t})^2 - 2n\tilde{t}\tilde{C} \\
&\geqslant \tilde{R}^2 - |y|^2 - 16c_0^2 |y - \bar{x}|^2 - (2 + 16c_0^2)(C_1^* \tilde{t})^2 - 2n\tilde{t}\tilde{C}.
\end{aligned}$$

Then we can choose $\tilde{R} = \tilde{R}(\tilde{K}, \tilde{t})$ such that $f(y,t) > 1$ on $\tilde{K} \times [0, \tilde{t}]$.
Hence, by Theorem 5.3.1, on $\tilde{K} \times [0, \tilde{t}]$ we have, considering also ϕ as a function of (y,t) on $\tilde{K} \times [0, \tilde{t}]$,

$$\begin{aligned}
\phi(y,t)^{-1} < f_+(y,t)\phi(y,t)^{-1} &\leqslant \max_{\mathcal{N}_R} f_+(\cdot,0)\phi(\cdot,0)^{-1} \\
&= \max_{supp f_+(\cdot,0)} (\tilde{R}^2 - |F(\cdot,0) - \bar{p}|^2)\phi(\cdot,0)^{-1}.
\end{aligned}$$

In the right side the dependency on R has disappeared, then there exists a constant $C = C(\tilde{R}) > 0$ such that $\phi(y,t)^{-1} < C$. Hence, by property $i)$ on ϕ, there exists a constant $\tilde{C}_1 > 0$, independent of R, such that

$$H \geqslant \tilde{C}_1 \quad \text{on } \tilde{K} \times [0, \tilde{t}].$$

Furthermore, by (5.8), there exists $\tilde{C}_2$ also independent of R, such that

$$H \leqslant \tilde{C}_2 \quad \text{on } \tilde{K} \times [0, \tilde{t}].$$

In conclusion, on $\tilde{K} \times [0, \tilde{t}]$, ϕ' is bounded both from above and below by two positive constants that do not depend on R. Then, since $\tilde{K}$ and $\tilde{t}$ can be enlarged as we like, we have shown that on any compact set of $\mathbb{R}^n \times [0, \infty)$, the graphs of u_R are such that ϕ' remains bounded both from above and below by two positive constants independent of R. Then the conclusion follows as in the end of the proof of Theorem 7 in [4].

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