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PhD's Thesis in Physics XXXI Cycle

A CRITICAL STUDY OF TWO PATI-SALAM MODELS

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February 27, 2019

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Chapter 1

Introduction

One of the open questions in the Standard Model (SM) is what the origin for the mass hierarchy and mixing of fermions is. This is the so called "flavor problem", it arises because in the lagrangian of the Standard Model the masses and mixing angles of fermions are completely arbitrary, their values are explained by ad hoc Yukawa couplings to fit the experimental data without giving a theoretical motivation that makes us able to understand such numbers.

A possible way to solve this problem is to use flavour symmetries and/or symmetries like GUT and/or of partial unification, like Pati-Salam models [1, 2, 3, 4, 5, 6], in order to decrease the number of free parameters in the models. For example, in the minimal $SU(5)$ the relation between the electron and the down quark masses $M_e = M_d^T$ is valid, in this way it is possible to determine completely a Yukawa matrix from the other; therefore the number of independent parameters of the theory is reduced.

In the model proposed by J.C. Pati and A. Salam in 1974 [1], the authors assumed the unification of quarks and leptons by means of the introduction of a fourth color carried by quarks, in addition to the three of $SU(3)_C$, and related to the leptonic number. In this way the unification of quarks and leptons comes extending the group $SU(3)_C$ to $SU(4)$. Because in 1974 the third fermionic family of quarks and leptons had not yet been discovered, the model describe only the first two fermionic families of quarks and leptons. The gauge group proposed by the authors is $SU(4) \otimes [SU(2)_L^{1+2}] \otimes [SU(2)_R^{1+2}]$, this just because, using the quantum numbers related to the isospin ($I_3 = \pm 1/2$), the strangeness ($S = 0, 1$) and the charm ($C = 0, 1$), the quarks can be collected in quartets. Under this group the known matter fields can be embeded in the multiplets:

$$\Psi_{L,R} = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 = \nu_e \\ d_1 & d_2 & d_3 & d_4 = e^- \\ s_1 & s_2 & s_3 & s_4 = \mu^- \\ c_1 & c_2 & c_3 & c_4 = \nu_\mu \end{pmatrix}_{L,R} \quad . \tag{1.1}$$

In the so-called "basic model", the fermionic fields obtain mass through Yukawa interaction with the scalar multiplet $A = (1, 4, \bar{4})$:

$$\mathcal{L}_{mass} = \lambda[\bar{\Psi}_L \langle A \rangle \Psi_R + h.c.] \qquad \text{with} \qquad \langle A \rangle = \mathbf{diag}[a_1, a_2, a_3, a_4]\,, \tag{1.2}$$

which generates the tree level mass relations:

$$m_d = m_e\,, \quad m_u = m_{\nu_e}\,, \quad m_s = m_\mu \quad \text{and} \quad m_c = m_{\nu_\mu}\,. \tag{1.3}$$

Although a distinction between quark and lepton masses can be reached introducing new scalar multiplets, the different strong interactions of quarks and leptons will produce however radiative corrections of the order:

$$\frac{m_q^{phys} - m_l^{phys}}{m^{tree}} \approx \frac{3g_4^2}{(4\pi)^2} \ln\left(\frac{M_X^2}{M_V^2}\right)\,, \tag{1.4}$$

with g_4 the coupling constant of $SU(4)$, V_i the gauge mesons of $SU(3)$ and $X_i^\pm$ the leptoquarks, new vector bosons which connect quark and leptons. For the neutrino sector instead, the authors suggest the introduction of singlet fermions, $s_0^e = (1,1,1)$ and $s_0^\mu = (1,1,1)$, in order to obtain massless neutrinos.

Interesting scenarios related to Pati-Salam have been investigated, among others, in [7, 8, 9] where family simmetries [10] have been employed to construct the lepton and quark mass matrices. In particular, the authors of [7] investigated the possibility to combine a Pati-Salam model with the discrete flavour symmetry based on the discrete group S_4 that gives rise to the quark-lepton complementarity relation $\theta_{12} + \theta_C \sim \pi/4$ [11, 13] between the neutrino solar angle θ_{12} and the the Cabibbo angle θ_C and, in addition, provides a good description of the fermion masses and mixings, both in the quark and in the lepton sectors. In [8], instead, the flavor group is based on the $A_4 \times Z_5$ family symmetry and the construction makes use of the see-saw mechanism with very hierarchical right-handed neutrinos to predict the entire PMNS mixing matrix as well as a Cabibbo angle $\theta_C \sim 1/4$. In the latter paper, a Pati-Salam extension of the standard model was proposed, incorporating a flavor symmetry based on the $\Delta(27)$ group. The theory realizes a Froggatt-Nielsen picture of quark mixing [14] and a predictive pattern of neutrino oscillations in which the CP symmetry is violated. Differently from our approach, none of the previous models attempted a complete numerical fit of the model parameters to the fermion observables, so that their agreement to the low energy observables is mainly based on a selection of higher-dimensional operators contributing to the Yukawa sectors.

The thesis work will focus on models of partial unification, in particular two Pati-Salam models [2, 3] based on the gauge group $SU(4) \otimes SU(2)_L \otimes SU(2)_R$, developed in the 90's and then abandoned too early in favor of the more exotic SUSY theories. These two models differ from each other because of the choice of the scalar multiplets, therefore also the scalar potentials and the Yukawa interactions in the Lagrangians will differ. As first consequence of this we will have that: while in the first model the only existing tree level fermion masses are those for some Beyond SM neutrinos, the second model provides, a tree level mass for each fermion. Beside this difference, in order to compare in a fair way these two models, we demand, for both, the existence of an explicit mass for the singlet fermions and the equality, at the partial-unification scale, of the gauge couplings of the groups $SU(2)_L$ and $SU(2)_R$. However, the common features of the two models is the presence of sterile neutrinos and the fact that the hierarchy between the fermion masses will come from loop corrections [15, 16, 17]. Therefore we will study how these corrections come out and in which form, trying to understand, by means of computer analysis, which is the best model to describe the current experimental data [18].

In Chapter 2 we study the first model where the scalar content is composed by two Higgs multiplets in the same representation as fermions, which under the Pati-Salam group are $(4, 1, 2)$ and $(4, 2, 1)$, so we have a complete left-right symmetry. After discussing the field content and the form of the Lagrangian, for the case without fermion inter-family mixing, in Section 2.1 we continue the analysis of the model computing in particular the tensorial structure of the scalar potential and the conditions for the minimum, obtaining in this way the form of the vacuum expectation values (VEVs) of the Higgs fields responsible for the symmetry breaking of the Pati-Salam and the Standard Model groups. The model analysis goes on, in Section 2.2 and 2.3, evaluating the Higgs and gauge boson mass spectra, two ingredients needed in the one-loop diagrams calculation for the fermion mass radiative generations, since in this model no SM particles get a tree level mass. In Section

2.4 we use the matching conditions between the running gauge coupling constants of the Standard Model and the Pati-Salam model in order to estimate the order of magnitude of the partial-unification scale. In Section 2.5 we focus on neutrinos, in particular on their masses and mixings, because these will be other fundamental ingredients for loop corrections to the fermion mass, being the neutrinos the only fermions propagating in that type of diagrams. In Section 2.6, using the dimensional regularization, we compute the one-loop corrections to the fermion masses coming from the diagrams involving scalars and gauge bosons exchange, also demonstrating that the model can not generate at all masses for the charged leptons. In Section 2.7 we show the first results obtained inserting in a scan code built up in the Mathematica software the fermion mass formulas, found in the previous sections, in order to search the model parameter set able to reproduce the experimental mass values. We will see that the model can generate realistic masses for all the quarks except for the top. In Section 2.8 we implement the model with the fermion inter-family mixing, generalizing the Lagrangian and defining the CKM and the PMNS matrices. We conclude the chapter with the Section 2.9, where we extend the model including two more multiplets, again in the representations $(4, 1, 2)$ and $(4, 2, 1)$ but with zero VEVs, in order to solve the problems of the unrealstic top mass and the no mass generations for the charged leptons. What we find, after repeating numerical analysis with the new two Higgs multiplets, is that the extended model is able to generate good values for the quarks and charged leptons mass spectra, but not for the neutrinos and the CKM and PMNS mixings.

We carry out an analogous study in Chapter 3 for the second model, where the scalar content is given, in this case, by the multiplet in the representations $(4, 1, 2)$, as in the previous model, and a bi-doublet $(1, 2, 2)$ under the group $SU(4) \otimes SU(2)_L \otimes SU(2)_R$. With these scalar multiplets, unlike the previous model, all the SM fermions get a tree level mass, while the correct mass hierarchy should be again generated by loop corrctions. However we will see that this second model does not seem able to generate the correct fermion mass spectra.

In this work we evaluate only the one-loop corrections to the fermion mass values and mixings, at the partial-unification scale. Furthermore, in order to make our numerical analysis more simple, we do not take into account any CP phase, cosidering, for both the models, real parameters.

Chapter 2

Model 1

The first Pati-Salam model we analyze was developed by M. P. Worah [2]. How we will see in the following, the peculiarity of this model is to provide masses to the Standard Model fermions radiatively generated at one loop, but not for a set of Beyond SM neutrinos whose masses are generated at tree level. Compared to the original paper, we will make some changes in the scalar sector analysis that will be justified later. Let us start now describing the fields content of this model. In the group $SU(4) \otimes SU(2)_L \otimes SU(2)_R$ the fermions of Standard Model are assigned to the representations $(4,2,1)$ and $(4,1,2)$. In the matrix form:

$$\Psi^{\alpha i(\mathbf{f})}_{L,R} = \begin{pmatrix} u_1 & d_1 \\ u_2 & d_2 \\ u_3 & d_3 \\ \nu_e & e^- \end{pmatrix}^{(\mathbf{f})}_{L,R} \tag{2.1}$$

where $i = 1, 2$ is the index for $SU(2)_{L,R}$, $\alpha = 1, 2, 3, 4$ for SU(4) and $\mathbf{f} = 1, 2, 3$ corresponds to the fermionic family. Moreover, the model contains also fermionic singlets of the group, $(1, 1, 1)$, an extra sterile neutrino $s_0^{(\mathbf{f})}$ for each family $\mathbf{f}$ of fermions. The gauge bosons of the $SU(2)_{L,R}$ groups, instead, are defined as usual using the Gell-Mann matrices $\sigma^a_{L,R}$:

$$W^\mu_{L,R} = \sum_{a=1}^{3} \frac{\sigma^a_{L,R}}{2} W^{\mu a}_{L,R} = \frac{1}{2} \begin{pmatrix} W^{\mu 3} & W^{\mu 1}_{L,R} - iW^{\mu 2}_{L,R} \\ W^{\mu 1}_{L,R} + iW^{\mu 2}_{L,R} & -W^{\mu 3} \end{pmatrix}_{L,R} ; \tag{2.2}$$

for the gauge bosons related to the group $SU(4)$, instead, we use the matrices λ^a (definited in Appendix A):

$$G^\mu = \sum_{a=1}^{15} \frac{\lambda^a}{2} G^{\mu a} = \\ = \frac{1}{2} \begin{pmatrix} \frac{B^\mu}{\sqrt{6}} + \frac{G^{\mu 8}}{\sqrt{3}} + G^{\mu 3} & G^{\mu 1} - iG^{\mu 2} & G^{\mu 4} - iG^{\mu 5} & G^{\mu 9} - iG^{\mu 10} \\ G^{\mu 1} + iG^{\mu 2} & \frac{B^\mu}{\sqrt{6}} + \frac{G^{\mu 8}}{\sqrt{3}} - G^{\mu 3} & G^{\mu 6} - iG^{\mu 7} & G^{\mu 11} - iG^{\mu 12} \\ G^{\mu 4} + iG^{\mu 5} & G^{\mu 6} + iG^{\mu 7} & \frac{B^\mu}{\sqrt{6}} - \frac{2G^{\mu 8}}{\sqrt{3}} & G^{\mu 13} - iG^{\mu 14} \\ G^{\mu 9} + iG^{\mu 10} & G^{\mu 11} + iG^{\mu 12} & G^{\mu 13} + iG^{\mu 14} & -\sqrt{\frac{3}{2}} B^\mu \end{pmatrix}, \tag{2.3}$$

where $B^\mu = G^{\mu 15}$ is the gauge boson that couples to the hypercharge Y, while the bosons $G^{\mu a}$ with $a = 1, \ldots 8$ represent the gluons of $SU(3)_C$.

We can see that we have nine vector fields more than in the SM: three are the gauge bosons for the right sector $SU(2)_R$ defined, similarly to the W_L^μ bosons of the SM, as:

$$W_R^{\mu\pm} = \frac{W_R^{\mu 1} \mp iW_R^{\mu 2}}{\sqrt{2}} \quad \text{and} \quad W_R^{\mu 0} = W_R^{\mu 3}\,; \tag{2.4}$$

then we have six new gauge bosons, the leptoquarks, that link quarks and leptons:

$$X_1^{\mu\pm} = \frac{G^{\mu 9} \mp iG^{\mu 10}}{\sqrt{2}}, \quad X_2^{\mu\pm} = \frac{G^{\mu 11} \mp iG^{\mu 12}}{\sqrt{2}} \quad \text{and} \quad X_3^{\mu\pm} = \frac{G^{\mu 13} \mp iG^{\mu 14}}{\sqrt{2}}\,. \tag{2.5}$$

All these gauge bosons interact with fermions and scalars as usual by means of the covariant derivative:

$$D^\mu \equiv \partial^\mu + ig_L W_L^\mu + ig_R W_R^\mu + ig_4 G^\mu\,. \tag{2.6}$$

Regarding the scalar part, Worah introduces two Higgs bosons: the right-handed ones, in the representation $(4, 1, 2)$:

$$R^{\alpha i} = \begin{pmatrix} R_{u1} & R_{d1} \\ R_{u2} & R_{d2} \\ R_{u3} & R_{d3} \\ R_\nu & R_e \end{pmatrix} \qquad \text{with} \qquad R_{\alpha i} = (R^{\alpha i})^*\,, \tag{2.7}$$

that will be used for the symmetry breaking of the Pati-Salam group to the SM:

$$SU(4) \otimes SU(2)_L \otimes SU(2)_R \to SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \,, \tag{2.8}$$

and the left-handed Higgs bosons $L^{\alpha i} \sim (4, 2, 1)$, with $L_{\alpha i} = (L^{\alpha i})^*$, that will be responsible for the symmetry breaking of the electro-weak group to the electromagnetism:

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \to SU(3)_C \otimes U(1)_Q \,. \tag{2.9}$$

From the fermions and Higgs representations discussed above, the resulting Yukawa interaction for each fermion family **f**, i.e. considering for the moment the case without inter-family mixing, takes the form:

$$\mathcal{L}^{(\mathbf{f})}_{Yukawa} = -k^{(\mathbf{f})}_L \left[\bar{s}^{(\mathbf{f})}_0 L_{\alpha i} \Psi^{\alpha i (\mathbf{f})}_L + h.c. \right] - k^{(\mathbf{f})}_R \left[\bar{s}^{c\,(\mathbf{f})}_0 R_{\alpha i} \Psi^{\alpha i (\mathbf{f})}_R + h.c. \right] , \tag{2.10}$$

where $k^{(\mathbf{f})}_{L,R}$ are real numbers. So the total Lagrangian of the model has the form:

$$\begin{aligned} \mathcal{L} = \sum_{\mathbf{f}=1}^{3} \Bigg\{ & i\bar{\Psi}^{(\mathbf{f})}_L \gamma^\mu D_\mu \Psi^{(\mathbf{f})}_L + i\bar{\Psi}^{(\mathbf{f})}_R \gamma^\mu D_\mu \Psi^{(\mathbf{f})}_R + i\bar{s}^{(\mathbf{f})}_0 \gamma^\mu \partial_\mu s^{(\mathbf{f})}_0 + i\bar{s}^{c\,(\mathbf{f})}_0 \gamma^\mu \partial_\mu s^{c\,(\mathbf{f})}_0 - \\ & - \left[\bar{s}^{c\,(\mathbf{f})}_0 m^{(\mathbf{f})}_0 s^{(\mathbf{f})}_0 + h.c. \right] + \mathcal{L}^{(\mathbf{f})}_{Yukawa} \Bigg\} - V(L,R) + \mathcal{L}_{gauge\,fixing} + \\ & + D_\mu L_{\alpha i} D^\mu L^{\alpha i} + D_\mu R_{\alpha i} D^\mu R^{\alpha i} - \frac{1}{2} \mathbf{Tr} \left[\mathbb{G}_{\mu\nu} \mathbb{G}^{\mu\nu} + \mathbb{W}_{L\mu\nu} \mathbb{W}^{\mu\nu}_L + \mathbb{W}_{R\mu\nu} \mathbb{W}^{\mu\nu}_R \right] , \end{aligned} \tag{2.11}$$

where

$$\mathbb{G}^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu + i g_4 [G^\mu, G^\nu] \,, \tag{2.12a}$$

$$\mathbb{W}^{\mu\nu}_L = \partial^\mu W^\nu_L - \partial^\nu W^\mu_L + i g_L [W^\mu_L, W^\nu_L] \,, \tag{2.12b}$$

$$\mathbb{W}^{\mu\nu}_R = \partial^\mu W^\nu_R - \partial^\nu W^\mu_R + i g_R [W^\mu_R, W^\nu_R] \,, \tag{2.12c}$$

with g_4, g_L and g_R the gauge coupling constants for the group $SU(4)$, $SU(2)_L$ and $SU(2)_R$, respectively; while the gauge-fixing part is given by:

$$\begin{aligned} \mathcal{L}_{gauge\,fixing} = & -\frac{1}{2\xi_4} (\partial \cdot G)^2 - \frac{1}{2\xi_{2R}} (\partial \cdot W_R)^2 - \frac{1}{2\xi_{2L}} (\partial \cdot W_L)^2 + \\ & + \bar{C}^a_4 \partial \cdot D C^a_4 + \bar{C}^a_{2R} \partial \cdot D C^a_{2R} + \bar{C}^a_{2L} \partial \cdot D C^a_{2L} \,, \end{aligned} \tag{2.13}$$

where ξ_N, $\bar{C}^a_N$ and C^a_N are the gauge parameters, the anti-ghosts and the ghosts of the group $SU(N)$ (with $N = 4, 2R, 2L$) respectively. As we can see in (2.11), there are no explicit mass terms for the fermions of Standard Model, respectively.

Looking at the representations of the fermions we can define the charge as:

$$Q = \frac{\sigma_{3L}}{2} + \frac{Y}{2} = \frac{\sigma_{3L}}{2} + \left(\frac{\sigma_{3R}}{2} + \frac{B-L}{2}\right) = \frac{\sigma_{3L}}{2} + \left(\frac{\sigma_{3R}}{2} + \sqrt{\frac{2}{3}}\frac{\lambda_{15}}{2}\right), \tag{2.14}$$

where Y represents the hypercharge, B the barionic number and L the leptonic number, while λ_{15} and σ_3 are respectively the diagonal generators for SU(4) and SU(2).

2.1 Higgs potential $V(L, R)$

Let us have a look at how many singlets, under the Pati-Salam group, can be built with at most four Higgs fields, respecting the power counting criterion which demands that the theory, to be renormalizable, contains no parameters of negative dimension in units of mass. We start with the pseudo-real representation $2 \sim \bar{2}$ of $SU(2)$:

$$\begin{aligned} &2 \otimes 2 = 3 \oplus 1 \\ &2 \otimes 2 \otimes 2 = 2 \oplus 4 \oplus 2 \\ &2 \otimes 2 \otimes 2 \otimes 2 = 3 \oplus 1 \oplus 5 \oplus 3 \oplus 3 \oplus 1 \end{aligned} \tag{2.15}$$

(for more details about the tensor products see in Appendix B). Since the Higgs bosons are also in the representation 4 of $SU(4)$, we have to consider the following products as well:

$$\begin{aligned} &4 \otimes 4 = 10 \oplus 6 \\ &\bar{4} \otimes 4 = 1 \oplus 5 \\ &\bar{4} \otimes \bar{4} = 10 \oplus 6 \,, \end{aligned} \tag{2.16}$$

therefore the only gauge invariant quadratic terms we can write have the form:

$$(\bar{4}, 2, 1) \otimes (4, 2, 1) \to L_{i\alpha} L^{i\alpha} \quad \text{and} \quad (\bar{4}, 1, 2) \otimes (4, 1, 2) \to R_{i\alpha} R^{i\alpha}. \tag{2.17}$$

Using three fields we have no $SU(4)$ singlets:

$$
\begin{aligned}
4 \otimes 4 \otimes 4 &= 20 \oplus 20 \oplus 20 \oplus \bar{4} \\
\bar{4} \otimes 4 \otimes 4 &= 36 \oplus 4 \oplus 20 \oplus 4 \\
\bar{4} \otimes \bar{4} \otimes 4 &= \bar{36} \oplus \bar{4} \oplus 20 \oplus 4
\end{aligned}
\tag{2.18}
$$

thus no cubic terms are present in the model. Considering four Higgses, instead, we find:

$$
4 \otimes 4 \otimes 4 \otimes 4 = 35 \oplus 45 \oplus 45 \oplus 20 \oplus 15 \oplus 45 \oplus 20 \oplus 15 \oplus 15 \oplus 1 \tag{2.19}
$$

but this unique singlet assumes a tensorial form of the type: $\epsilon^{\alpha\beta\mu\nu} L_\alpha L_\beta L_\mu L_\nu$, which is equal to zero because the Higgs are bosonic, commutative, fields. The other possibility:

$$
\bar{4} \otimes 4 \otimes 4 \otimes 4 = 70 \oplus 64 \oplus 10 \oplus 10 \oplus 6 \oplus 64 \oplus \bar{10} \oplus 6 \oplus 10 \oplus 6 \tag{2.20}
$$

does not contain any singlets; while the product:

$$
\bar{4} \otimes \bar{4} \otimes 4 \otimes 4 = 84 \oplus \bar{45} \oplus 15 \oplus 1 \oplus 15 \oplus 45 \oplus 20 \oplus 15 \oplus 1 \oplus 15 \tag{2.21}
$$

has two singlets. Combining the third of (2.15) and (2.21) we obtain

$$
(4,2,1) \otimes (\bar{4},2,1) \otimes (4,2,1) \otimes (\bar{4},2,1) = (15 \oplus 1, 3 \oplus 1, 1) \otimes (15 \oplus 1, 3 \oplus 1, 1) \tag{2.22}
$$

which contains the singlets

$$
\begin{aligned}
(1,1,1) \otimes (1,1,1) &\to L_{i\alpha} L^{i\alpha} L_{j\beta} L^{j\beta} \\
(1,3,1) \otimes (1,3,1) &\to L_{i\alpha} L^{j\alpha} L_{j\beta} L^{i\beta} \\
(15,1,1) \otimes (15,1,1) &\to L_{i\alpha} L^{i\beta} L^{j\alpha} L_{j\beta} \\
(15,3,1) \otimes (15,3,1) &\to L_{i\alpha} L^{j\beta} L^{i\alpha} L_{j\beta}\,.
\end{aligned}
\tag{2.23}
$$

Therefore the potential must contain two quartic terms for the scalar multiplet L. The same reasoning followed till now for the multiplet L, can also be used for the Higgs multiplet R.

The next step is to introduce the mixing between the two multiplets L and R, and to do that we have two possible tensorial products which mantain the gauge invariance:

$$(4,2,1)\otimes(\bar{4},2,1)\otimes(4,1,2)\otimes(\bar{4},1,2)=(15\oplus 1,3\oplus 1,1)\otimes(15\oplus 1,1,3\oplus 1)\,, \tag{2.24}$$

which generates the singlets:

$$\begin{aligned}(1,1,1)\otimes(1,1,1)&\rightarrow L_{i\alpha}L^{i\alpha}R_{j\beta}R^{j\beta}\\(15,1,1)\otimes(15,1,1)&\rightarrow L_{i\alpha}L^{i\beta}R^{j\alpha}R_{j\beta}\,,\end{aligned} \tag{2.25}$$

and the product:

$$(4,2,1)\otimes(\bar{4},1,2)\otimes(4,2,1)\otimes(\bar{4},1,2)+h.c.=(15\oplus 1,2,2)\otimes(15\oplus 1,2,2)\times 2\,, \tag{2.26}$$

from which can be obtained the singlets of the form:

$$\begin{aligned}(1,2,2)\otimes(1,2,2)&\rightarrow L_{i\alpha}R^{j\alpha}L^{i\beta}R_{j\beta}+(L_{i\alpha}R^{\alpha}_{j}R^{j\beta}L^{i}_{\beta}+h.c.)\\(15,2,2)\otimes(15,2,2)&\rightarrow L_{i\alpha}R^{j\beta}L^{i\alpha}R_{j\beta}+(L_{i\alpha}R^{\alpha}_{j}R^{j\beta}L^{i}_{\beta}+h.c.)\,.\end{aligned} \tag{2.27}$$

At the end, putting all togheter, we obtain the potential for the Higgs fields of the form:

$$\begin{aligned}V(L,R)=&-2\mu_L^2 L_{i\alpha}L^{i\alpha}+\lambda_{L1}\left(L_{i\alpha}L^{i\alpha}\right)^2+\lambda_{L2}L_{i\alpha}L^{j\alpha}L^{i\beta}L_{j\beta}-\\&-2\mu_R^2 R_{i\alpha}R^{i\alpha}+\lambda_{R1}\left(R_{i\alpha}R^{i\alpha}\right)^2+\lambda_{R2}R_{i\alpha}R^{j\alpha}R^{i\beta}R_{j\beta}+\\&+\lambda_{LR1}L_{i\alpha}L^{i\alpha}R_{j\beta}R^{j\beta}+\lambda_{LR2}L_{i\alpha}R^{j\alpha}L^{i\beta}R_{j\beta}+\\&+\lambda_{LR3}(L_{i\alpha}R^{j\alpha}L^{i}_{\beta}R^{\beta}_{j}+h.c.)\,,\end{aligned} \tag{2.28}$$

where μ_i^2 and λ_i are real parameters.[1]

[1]The potential (2.28) looks different from the one in [2], where an additional term:

$$\lambda_{L3}(L_{i\alpha}L^{j\alpha}L^{\beta}_{j}L^{i}_{\beta}+h.c.) \tag{2.29}$$

appears; however, using the identity $\epsilon^{ij}\epsilon_{kl}=\delta^i_k\delta^j_l-\delta^i_l\delta^j_k$ this can be reparametrized in the following form:

$$(\lambda_{L1}+\lambda_{L3})L_{i\alpha}L^{i\alpha}L_{j\beta}L^{j\beta}+(\lambda_{L2}-\lambda_{L3})L_{i\alpha}L^{j\alpha}L_{j\beta}L^{i\beta}\,. \tag{2.30}$$

2.1.1 Minimum and stability of the Higgs potential

To study the symmetry breaking of the Pati-Salam model and of the Standard Model, we follow Li's method [19], that we rediscuss here in some details. Considering for the moment the case without left-right mixing, the first derivatives of the Higgs potential with respect to the scalar fields assume the following forms:

$$
\begin{aligned}
\frac{\partial V(L)}{\partial L_{k\sigma}} &= -2\mu_L^2 L^{k\sigma} + 2\lambda_{L1} L_{i\alpha} L^{i\alpha} L^{k\sigma} + 2\lambda_{L2} L_{i\alpha} L^{k\alpha} L^{i\sigma} \\
\frac{\partial V(R)}{\partial R_{k\sigma}} &= -2\mu_R^2 R^{k\sigma} + 2\lambda_{R1} R_{i\alpha} R^{i\alpha} R^{k\sigma} + 2\lambda_{R2} R_{i\alpha} R^{k\alpha} L^{i\sigma} \,.
\end{aligned}
\tag{2.31}
$$

Contracting separately the indices of $SU(2)_L$ and $SU(2)_R$, the previous relations can be expressed in terms of the hermitean matrices:

$$
X_\alpha^\sigma = L_{i\alpha} L^{i\sigma} \quad \text{and} \quad Y_\alpha^\sigma = R_{i\alpha} R^{i\sigma} \,, \tag{2.32}
$$

in this way (2.31) becomes:

$$
\begin{aligned}
\frac{\partial V(L)}{\partial L_{k\sigma}} &= -2\mu_L^2 L^{k\sigma} + 2\lambda_{L1} \sum_\alpha X_\alpha^\alpha L^{k\sigma} + 2\lambda_{L_2} X_\alpha^\sigma L^{k\alpha} \\
\frac{\partial V(R)}{\partial R_{k\sigma}} &= -2\mu_R^2 R^{k\sigma} + 2\lambda_{R1} \sum_\alpha Y_\alpha^\alpha R^{k\sigma} + 2\lambda_{R_2} Y_\alpha^\sigma R^{k\alpha} \,.
\end{aligned}
\tag{2.33}
$$

Because $L_{i\alpha}$ and $R_{i\alpha}$ are independent fields, we can diagonalize them simultaneously in the forms:

$$
X_\alpha^\sigma = \delta_\alpha^\sigma x_\sigma \quad \text{and} \quad Y_\alpha^\sigma = \delta_\alpha^\sigma y_\sigma \,. \tag{2.34}
$$

With this definitions, the minimizing equations are given by:

$$
\begin{aligned}
&\Big[-\mu_L^2 + \lambda_{L1} \sum_\alpha x_\alpha + \lambda_{L2} x_\beta\Big] \langle L^{k\beta} \rangle = 0 \\
&\Big[-\mu_R^2 + \lambda_{R1} \sum_\alpha y_\alpha + \lambda_{R2} y_\beta\Big] \langle R^{k\beta} \rangle = 0 \,.
\end{aligned}
\tag{2.35}
$$

Let us focus for the moment only on the left part, for the right one the reasoning will be the same. Using the tensors X_α^β the potential can be written as:

$$\begin{aligned} V(L) &= -2\mu_L^2 X_\alpha^\alpha + \lambda_{L1}(X_\alpha^\alpha)^2 + \lambda_2 X_\alpha^\beta X_\beta^\alpha = \\ &= -2\mu_L^2 \sum_\alpha x_\alpha + \lambda_{L1}\Big(\sum_\alpha x_\alpha\Big)^2 + \lambda_{L2}\sum_\alpha x_\alpha^2 \,. \end{aligned} \tag{2.36}$$

For the first equation in (2.35) we can have two possible solutions: the trivial case $\langle L\rangle = 0$ or $\langle L\rangle \neq 0$. If we consider the general case with a non-zero vacuum expectation value (VEV), the equation (2.35) is reduced to the form:

$$\lambda_{L1}\sum_\alpha x_\alpha + \lambda_{L2}x_\beta = \mu_L^2 \,. \tag{2.37}$$

At this point we can distinguish further cases. The first one is when $x_\alpha \neq 0$ for $\alpha = 1,2,3,4$. This request gives us the system:

$$\begin{cases} \lambda_{L1}(x_1+x_2+x_3+x_4) + \lambda_{L2}x_1 = \mu_L^2 \\ \lambda_{L1}(x_1+x_2+x_3+x_4) + \lambda_{L2}x_2 = \mu_L^2 \\ \lambda_{L1}(x_1+x_2+x_3+x_4) + \lambda_{L2}x_3 = \mu_L^2 \\ \lambda_{L1}(x_1+x_2+x_3+x_4) + \lambda_{L2}x_4 = \mu_L^2 \end{cases} \Rightarrow \begin{cases} (4\lambda_{L1}+\lambda_{L2})\sum_{\alpha=1}^4 x_\alpha = 4\mu_L^2 \\ x_1+x_2-x_3-x_4 = 0 \\ x_1-x_2+x_3-x_4 = 0 \\ x_1-x_2-x_3+x_4 = 0 \end{cases} , \tag{2.38}$$

from which we obtain:

$$x_1 = x_2 = x_3 = x_4 = \frac{\mu_L^2}{\lambda_{L1}+\lambda_{L2}} \,, \tag{2.39}$$

that is the solution for the case without symmetry breaking which, inserting (2.39) in (2.36), leads to the minimum:

$$V^{NB} = -\frac{4\mu_L^4}{4\lambda_{L1}+\lambda_{L2}} \,. \tag{2.40}$$

The reason why we consider this as a case without symmetry breaking can be understood looking at the scalar kinetic term in (2.11) responsible, by means of the Higgs mechanism, for the gauge boson masses generation, how we will see in details in the next section. In fact, considering for simplicity only the term involving the gauge bosons G_μ of $SU(4)$:

$$Tr\Big[(D_\mu L)^\dagger (D^\mu L)\Big] = \cdots + g_4^2 Tr\Big[G_\mu L^\dagger L G^\mu\Big] \,, \tag{2.41}$$

and using the notations in (2.32) and (2.34), we obtain, once the Higgs fields get VEV, the mass term for the gauge bosons of the form:

$$g_4^2 Tr\Big[G_\mu A(A^\dagger X A)A^\dagger G^\mu\Big]_{VEV} = g_4^2\Big[\tilde{G}^\alpha_{\mu\beta}(x_\sigma \delta^\sigma_\alpha)\tilde{G}^{\mu\beta}_\sigma\Big]_{VEV} = M_G^2 \tilde{G}_\mu \tilde{G}^\mu\,, \tag{2.42}$$

where A is the unitary matrix which diagonalizes the tensor product $L_{i\alpha}L^{i\sigma}$ and, at the same time, it transforms the interaction basis G_μ into the mass eigenstates $\tilde{G}_\mu$, with M_G^2 their diagonal squared mass matrix. Therefore if the value x_α is the same for $\alpha = 1,2,3,4$ then M_G^2 will be proportional to the identity matrix; this is an analogous case, once rescaled the VEV, to the trivial one $\langle L\rangle = 0$ which mantains all the generators of the group $SU(4)$ exact (not broken), so with all the gauge bosons massless.

The second case, which we are interested to, is when symmetry breaking takes place. However, we have to pay attention to the fact that $SU(4)$ can be broken in two different ways: $SU(4)\to SU(2)+SU(2)$ or $SU(4)\to SU(3)+U(1)$. The breaking pattern of our interest here is the second one, so two possible choices are allowed:

$$\begin{aligned} &x_1 = x_2 = x_3 = 0 \quad \text{and} \quad x_4 = \frac{\mu_L^2}{\lambda_{L1}+\lambda_{L2}} \quad \to \quad V_1^B = -\frac{\mu_L^4}{\lambda_{L1}+\lambda_{L2}}\,; \\ &x_1 = 0 \quad \text{and} \quad x_2 = x_3 = x_4 = \frac{\mu_L^2}{3\lambda_{L1}+\lambda_{L2}} \quad \to \quad V_2^B = -\frac{3\mu_L^4}{3\lambda_{L1}+\lambda_{L2}}\,. \end{aligned} \tag{2.43}$$

Since we need to avoid $SU(3)$ taking a non-vanishing VEV, the right minimum of the potential is V_1^B. Trying to request that V_1^B is an absolute minimum, we can note that exists a function of minima given by:

$$V_{min}(k) = -\frac{k\mu_L^4}{k\lambda_{L1}+\lambda_{L2}} \tag{2.44}$$

with k indicating how many non-zero eigenvalues $x_\alpha \neq 0$ we have for the tensor product $L_{i\alpha}L^{i\beta}$. Deriving $V_{min}(k)$ with respect to k we find:

$$\frac{\partial V_{min}(k)}{\partial k} = -\frac{\lambda_{L2}\mu_L^4}{(k\lambda_{L1}+\lambda_{L2})^2} \tag{2.45}$$

which for $\lambda_{L2} < 0$ is positive, then $V_{min}(k)$ is an increasing function. This means that we have the absolute minima for $k = 1$; in addition, asking that V_1^B is the absolute minimum, we get the additional constraint $\lambda_{L1}+\lambda_{L2} > 0$.

In principle we have the freedom to represent both X^σ_α and Y^σ_α as diagonal 4×4 matrices with only one non-vanishing entry. In view of the breaking of $SU(4) \to SU(3)$, let us fix $\alpha = \sigma = 4$. Then:

$$\langle X \rangle = \frac{\mu_L^2}{\lambda_{L1} + \lambda_{L2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \langle Y \rangle = \frac{\mu_R^2}{\lambda_{R1} + \lambda_{R2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.46}$$

and accordingly:

$$\langle L \rangle = \frac{\mu_L}{\sqrt{\lambda_{L1} + \lambda_{L2}}} \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \end{pmatrix} \qquad \langle R \rangle = \frac{\mu_R}{\sqrt{\lambda_{R1} + \lambda_{R2}}} \begin{pmatrix} 0 & 0 & 0 & a' \\ 0 & 0 & 0 & b' \end{pmatrix}, \tag{2.47}$$

with $| a |^2 + | b |^2 = 1$ and $| a' |^2 + | b' |^2 = 1$. Since both L and R are vectors under $SU(4)$, the choice of giving VEV to the 4^{th} component preserves $SU(3)$; in fact, for every $SU(4)$ generators λ^a, the equalities $\lambda^a \langle X \rangle = 0$ and $\lambda^a \langle Y \rangle = 0$ hold as soon as a run on the subset of λ^a which generate the color group. It remains to specify which $SU(2)$ component of L and R must be vanishing. Given that we want $U(1)_Y$ subgroups to be preserved by $\langle R \rangle$:

$$\begin{aligned} Y\langle R \rangle &= \sigma_{3R}\langle R \rangle + \langle R \rangle \sqrt{\frac{2}{3}} \lambda_{15} = \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & a' \\ 0 & 0 & 0 & b' \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & a' \\ 0 & 0 & 0 & b' \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2b' \end{pmatrix} = 0\,; \end{aligned} \tag{2.48}$$

and $U(1)_Q$ subgroups to be preserved by $\langle L \rangle$:

$$Q\langle L \rangle = \frac{\sigma_{3L}}{2}\langle L \rangle + \frac{Y}{2}\langle L \rangle = \frac{\sigma_{3L}}{2}\langle L \rangle + \langle L \rangle \sqrt{\frac{2}{3}} \frac{\lambda_{15}}{2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b \end{pmatrix} = 0\,, \tag{2.49}$$

the optimal choice is to set $a = a' = 1$ and $b = b' = 0$. Therefore, defining $v_L/\sqrt{2}$ and $v_R/\sqrt{2}$ as the vacuum expectation values taken by the fields L_ν and R_ν respectively , we have:

$$\langle L\rangle = \begin{pmatrix} 0 & 0 & 0 & \frac{v_L}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{with} \qquad v_L = \frac{\sqrt{2}\mu_L}{\sqrt{\lambda_{L1}+\lambda_{L2}}}\,; \tag{2.50}$$

$$\langle R\rangle = \begin{pmatrix} 0 & 0 & 0 & \frac{v_R}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{with} \qquad v_R = \frac{\sqrt{2}\mu_R}{\sqrt{\lambda_{R1}+\lambda_{R2}}}\,. \tag{2.51}$$

It is possible to obtain the same results following an analogus procedure but using, instead of (2.34), the tensors:

$$X_i^j = L_{i\alpha}L^{j\alpha} \to \delta_i^j x_j \quad \text{and} \quad Y_i^j = R_{i\alpha}R^{j\alpha} \to \delta_i^j y_j\,, \tag{2.52}$$

where we contracted the $SU(4)$ indices. In this case, considering for simplicity again the left part only, we can distinguish two cases when $\langle L\rangle \neq 0$:

$$\begin{aligned} & x_1 = x_2 = \frac{\mu_L^2}{2\lambda_{L1}+\lambda_{L2}} \quad \to \quad V^{NB} = -\frac{2\mu_L^4}{2\lambda_{L1}+\lambda_{L2}}\,; \\ & x_2 = 0 \quad \text{and} \quad x_1 = \frac{\mu_L^2}{\lambda_{L1}+\lambda_{L2}} \quad \to \quad V^{B} = -\frac{\mu_L^4}{2\lambda_{L1}+\lambda_{L2}}\,; \end{aligned} \tag{2.53}$$

and demandig that the minimum of symmetry breaking is an absolute minimum:

$$\begin{cases} V^B < 0 \\ V^B < V^{NB} \end{cases} \tag{2.54}$$

we find again the same conditions for the parameters: $\lambda_{L1}+\lambda_{L2} > 0$ and $\lambda_{L2} < 0$.

We have to note that the choice $x_\alpha = x_4$ and $x_i = x_1$ as the components that take VEV, in (2.43) and (2.53) respectively, is needed in order to reproduce the correct symmetry breaking patterns: $SU(4) \to SU(3)$ and $SU(2) \to U(1)$. Furthermore, while v_R will be at an high scale, where a partial-unification of quarks and leptons is achieved, the energy scale of v_L will be of the order of the electro-weak scale, $v_L \approx 246$ GeV. As a note, we can relate the complex scalar fields φ_1 and φ_2, which compose the SM Higgs doublet φ, with the two fields of the multiplet L having the same quantum number in the following way:

$$|\langle\varphi\rangle| = \begin{pmatrix} |\langle\varphi_1\rangle| \\ |\langle\varphi_2\rangle| \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ |v_L| \end{pmatrix} \sim \begin{pmatrix} |\langle L_e^*\rangle| \\ |\langle L_\nu^*\rangle| \end{pmatrix} \tag{2.55}$$

where the VEV is taken by $\varphi_2 \sim (L_\nu)^*$ because we have to produce the electro-weak symmetry breaking $SU(2)_L \otimes U(1)_Y \to U(1)_Q$.

Let us now reintroduce the LR mixing. Eq.(2.31) are then modified as follows:

$$\begin{aligned}\frac{\partial V(L,R)}{\partial L_{k\sigma}} &= -2\mu_L^2 L^{k\sigma} + 2\lambda_{L1} L_{i\alpha} L^{i\alpha} L^{k\sigma} + 2\lambda_{L2} L_{i\alpha} L^{k\alpha} L^{i\sigma} + \lambda_{LR1} L^{k\sigma} R_{i\alpha} R^{i\alpha} + \\ &\quad + \lambda_{LR2} R^{j\sigma} L^{k\beta} R_{j\beta} + 2\lambda_{LR3} L^k_\beta R^\sigma_i R^{i\beta} \\ \frac{\partial V(L,R)}{\partial R_{k\sigma}} &= -2\mu_R^2 R^{k\sigma} + 2\lambda_{R1} R_{i\alpha} R^{i\alpha} R^{k\sigma} + 2\lambda_{R2} R_{i\alpha} R^{k\alpha} L^{i\sigma} + \lambda_{LR1}\, R^{k\sigma} L_{i\alpha} L^{i\alpha} + \\ &\quad + \lambda_{LR2} L_{i\alpha} R^{k\alpha} L^{i\sigma} + 2\lambda_{LR3} L^{i\sigma} L^\beta_i R^k_\beta \quad .\end{aligned} \tag{2.56}$$

The presence of the terms proportional to λ_{LR3} do not allow us to repeat the same procedure done above. In fact, the tensor $X^{\sigma\beta} = L^{i\sigma} L^\beta_i$ is not hermitean and, in addition, does not commute with $X^\alpha_\gamma = L_{i\gamma} L^{i\alpha}$; thus it cannot be put in the diagonal form in the same basis as, for example, X^σ_β. We can only say that $X^\alpha_\gamma X^\gamma_\beta = X^{\alpha\gamma} X_{\gamma\beta}$ which means that the eigenvalues of the two *hermitean* matrices are the same. If we consider the contraction of the $SU(4)$ indices we can rewrite the first equation in (2.56) as:

$$\begin{aligned}\frac{\partial V(L,R)}{\partial L_{k\sigma}} &= \Big[- 2\mu_L^2 + 2\lambda_{L1} X^i_i \Big] L^{k\sigma} + 2\lambda_{L2} X^k_i L^{i\sigma} + \lambda_{LR1} Y^i_i L^{k\sigma} + \\ &\quad + \lambda_{LR2} (Z^k_i)^* R^{i\sigma} + 2\lambda_{LR3} Z^k_i R^{i\sigma}\end{aligned} \tag{2.57}$$

where $Z^i_j = L_{j\alpha} R^{i\alpha}$ and $(Z^i_j)^* = L^{j\alpha} R_{i\alpha}$. At this point, since the diagonalization of the product $L_{k\alpha} L^{l\alpha}$ procedes as:

$$L_{k\alpha} L^{l\alpha} \to (U_L^\dagger)^k_i X^l_k (U_L)^j_l = x_j \delta^j_i = (U_L^\dagger)^k_i L_{k\beta} (V_L)^\beta_\alpha (V_L^\dagger)^\alpha_\rho L^{l\rho} (U_L)^j_l \,, \tag{2.58}$$

and that for the right-handed fields as:

$$R_{k\alpha} R^{l\alpha} \to (U_R^\dagger)^k_i Y^l_k (U_R)^j_l = y_j \delta^j_i = (U_R^\dagger)^k_i R_{k\beta} (V_R)^\beta_\alpha (V_R^\dagger)^\alpha_\rho R^{l\rho} (U_R)^j_l \,, \tag{2.59}$$

with $U_{L,R}$ matrices of $SU(2)_{L,R}$ and $V_{L,R}$ matrices of $SU(4)$, we could diagonalize the tensor Z^l_k as:

$$L_{k\alpha} R^{l\alpha} \to (U_L^\dagger)^k_i L_{k\beta} (V_L)^\beta_\alpha (V_R^\dagger)^\alpha_\beta R^{l\beta} (U_R)^j_l \,, \tag{2.60}$$

but only if we make the restrictive assumption $V_L^\dagger V_R = \delta^i_j$. Instead of that, we proceed in the following way. Having already established that $\langle L_\nu \rangle = v_L/\sqrt{2}$ and $\langle R_\nu \rangle = v_L/\sqrt{2}$

are the good VEVs for our scheme of symmetry breaking, we can enforce them to be solution also for the case with LR mixing. In particular, assuming $\langle L_{14}\rangle = v_L/\sqrt{2}$ and $\langle R_{14}\rangle = v_R/\sqrt{2}$, the VEVs must satisfy the following relations:

$$\begin{aligned} -2\mu_L^2 + (\lambda_{L1}+\lambda_{L2})v_L^2 + \frac{1}{2}(\lambda_{LR1}+\lambda_{LR2})v_R^2 &= 0\,; \\ -2\mu_R^2 + (\lambda_{R1}+\lambda_{R2})v_R^2 + \frac{1}{2}(\lambda_{LR1}+\lambda_{LR2})v_L^2 &= 0\,. \end{aligned} \tag{2.61}$$

Then we can compute V_{min}:

$$\begin{aligned} V_{min} &= -\frac{1}{4(\lambda_{L1}+\lambda_{L2})(\lambda_{R1}+\lambda_{R2})-(\lambda_{LR1}+\lambda_{LR2})^2}\times \\ &\quad \times\left[4(\lambda_{R1}+\lambda_{R2})\mu_L^4 + 4(\lambda_{L1}+\lambda_{L2})\mu_R^4 - 4(\lambda_{LR1}+\lambda_{LR2})\mu_L^2\mu_R^2\right] \equiv \\ &\equiv \alpha\,\mu_L^4 + \beta\,\mu_R^4 + \gamma\,\mu_L^2\mu_R^2\,, \end{aligned} \tag{2.62}$$

and impose that the coefficients α, β and γ, of μ_L^4, μ_R^4 and $\mu_L^2\mu_R^2$ respectively, are simultaneously negative. In this way we "minimize" the minimum. This gives the two possibilities:

$$\left(C<0\,\&\,B>0\,\&\,A>\frac{C^2}{4B}\right) \quad \text{or} \quad \left(C>0\,\&\,B<0\,\&\,\frac{C^2}{4B}<A<0\right), \tag{2.63}$$

where $A=\lambda_{L1}+\lambda_{L2}$, $B=\lambda_{R1}+\lambda_{R2}$ e $C=\lambda_{LR1}+\lambda_{LR2}$. Again, the conditions: $\lambda_{L1}+\lambda_{L2}>0$ and $\lambda_{R1}+\lambda_{R2}>0$, are recovered in the limit of vanishing LR couplings.

2.2 Higgs masses

Let us calculate now the second derivatives of the Higgs potential:

$$
\begin{aligned}
\frac{\partial^2 V(L,R)}{\partial L^{y\nu}\partial L_{x\mu}} &= \Big[-2\mu_L^2 + 2\lambda_{L1}L^2 + \lambda_{LR1}R^2\Big]\delta^x_y\delta^\mu_\nu + 2\lambda_{L2}\Big[\delta^x_y L^{i\mu}L_{i\nu} + \delta^\mu_\nu L^{x\alpha}L_{y\alpha}\Big] + \\
&+ 2\lambda_{L1}L_{y\nu}L^{x\mu} + \lambda_{LR2}R^{i\mu}R_{i\nu}, \\
\frac{\partial^2 V(L,R)}{\partial R^{y\nu}\partial L_{x\mu}} &= \lambda_{LR1}R_{y\nu}L^{x\mu} + \lambda_{LR2}L^{x\alpha}R_{y\alpha}\delta^\mu_\nu + 2\lambda_{LR3}\Big[L^x_\alpha R^\alpha_y\delta^\mu_\nu - R^\mu_y L^x_\nu\Big], \\
\frac{\partial^2 V(L,R)}{\partial R^{y\nu}\partial R_{x\mu}} &= \Big[-2\mu_R^2 + 2\lambda_{R1}R^2 + \lambda_{LR1}L^2\Big]\delta^x_y\delta^\mu_\nu + 2\lambda_{R2}\Big[\delta^x_y R^{i\mu}R_{i\nu} + \delta^\mu_\nu R^{x\alpha}R_{y\alpha}\Big] + \\
&+ 2\lambda_{R1}R_{y\nu}R^{x\mu} + \lambda_{LR2}L^{i\mu}L_{i\nu}, \\
\frac{\partial^2 V(L,R)}{\partial L^{y\nu}\partial R_{x\mu}} &= \lambda_{LR1}L_{y\nu}R^{x\mu} + \lambda_{LR2}R^{x\alpha}L_{y\alpha}\delta^\mu_\nu + 2\lambda_{LR3}\Big[R^x_\alpha L^\alpha_y\delta^\mu_\nu - L^\mu_y R^x_\nu\Big].
\end{aligned}
\tag{2.64}
$$

Imposing VEV structures obtained above, we find the block diagonal 8×8 mass squared matrix:

$$
M^2_{LR} = \begin{pmatrix} M^2_{LR_u} & 0 & 0 & 0 \\ 0 & M^2_{LR_\nu} & 0 & 0 \\ 0 & 0 & M^2_{LR_d} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.65}
$$

where we used the basis $(L_u, R_u, L_\nu, R_\nu, L_d, R_d, L_e, R_e)$. Taking in account the conditions (2.61) for μ_L and μ_R, we obtain:

$$
M^2_{LR_u} = \frac{v_R^2}{2}\lambda_{LR2}\begin{pmatrix} -1 & \frac{v_L}{v_R} \\ \frac{v_L}{v_R} & -\frac{v_L^2}{v_R^2} \end{pmatrix} \tag{2.66}
$$

$$
M^2_{LR_\nu} = v_R^2\begin{pmatrix} (\lambda_{L1}+\lambda_{L2})\frac{v_L^2}{v_R^2} & \frac{(\lambda_{LR1}+\lambda_{LR2})}{2}\frac{v_L}{v_R} \\ \frac{(\lambda_{LR1}+\lambda_{LR2})}{2}\frac{v_L}{v_R} & (\lambda_{R1}+\lambda_{R2}) \end{pmatrix} \tag{2.67}
$$

$$
M^2_{LR_d} = v_R^2\begin{pmatrix} -(\lambda_{L2}\frac{v_L^2}{v_R^2} + \frac{\lambda_{LR2}}{2}) & \lambda_{LR3}\frac{v_L}{v_R} \\ \lambda_{LR3}\frac{v_L}{v_R} & -(\frac{\lambda_{LR2}}{2}\frac{v_L^2}{v_R^2} + \lambda_{R2}) \end{pmatrix}. \tag{2.68}
$$

It is important to observe that no mass terms are present in the sector related to the charged leptons.

To find new conditions for the Higgs potential, we can ask that the matrix M^2_{LR} is positive defined so that, around the VEVs, the second derivatives have the concavity upwards. The matrix $M^2_{LR_u}$ don't give any conditions because $\det(M^2_{LR_u}) = 0$; instead from the Higgs sector related to the neutrinos, demanding $\det(M^2_{LR_\nu}) > 0$, we find a condition on the λ_i parameters given by:

$$(\lambda_{L1} + \lambda_{L2})(\lambda_{R1} + \lambda_{R2}) > \frac{1}{4}(\lambda_{LR1} + \lambda_{LR2})^2 \,. \tag{2.69}$$

To this we must add the conditions coming from the down-type quarks sector of the Higgs:

$$\det(M^2_{LR_d}) = \frac{v_R^4}{4}\lambda_{LR2}\left[\frac{1}{2}\lambda_{L2}\frac{v_L^4}{v_R^4} + \frac{1}{2}\lambda_{R2} + \frac{v_L^2}{v_R^2}\left(\frac{\lambda_{L2}\lambda_{R2}}{\lambda_{LR2}} + \frac{1}{4}\lambda_{LR2} - \frac{\lambda_{LR3}^2}{\lambda_{LR2}}\right)\right] > 0. \tag{2.70}$$

Considering that $v_L/v_R \ll 1$, we expand (2.70):

$$\det(M^2_{LR_d}) \approx \frac{v_R^4}{8}\lambda_{R2}\lambda_{LR2} > 0 \,, \tag{2.71}$$

that gives the condition: $\lambda_{R2}\lambda_{LR2} > 0$.

Another condition to take into account is that the mass term for a complex scalar field has the form: $-M^2\Phi\Phi^\dagger$, therefore if a mass eigenvalue of M^2_{LR} is negative the minus sign can not be reabsorbed as a phase factor in a field redefinition, as it happens for a fermionic mass term $m_{Dirac}\bar{\psi}_L\psi_R$ or $m_{Majorana}\psi^T\psi$, so we have to demand that also the single mass squared eigenvalues are positive. Thus, from the matrix $M^2_{LR_u}$ we calculate the eigenvalues:

$$M^2_{H_u} = (M^2_{LR_u})^{diag} = O^T_{H_u} M^2_{LR_u} O_{H_u} = -\frac{\lambda_{LR2}}{2}\begin{pmatrix} 0 & 0 \\ 0 & v_R^2 + v_L^2 \end{pmatrix} \,, \tag{2.72}$$

where the orthogonal matrix O_{H_u} is given by:

$$O_{H_u} = \begin{pmatrix} \cos\theta_u & -\sin\theta_u \\ \sin\theta_u & \cos\theta_u \end{pmatrix} = \frac{1}{\sqrt{v_L^2 + v_R^2}}\begin{pmatrix} v_L & -v_R \\ v_R & v_L \end{pmatrix} \Rightarrow \tan(2\theta_u) = -\frac{2v_L v_R}{v_R^2 - v_L^2} \,. \tag{2.73}$$

The squared mass eigenvalues for the Higgs related to neutrinos instead are:

$$M^2_{H_\nu} = (M^2_{LR_\nu})^{diag} = -C_\nu^2\Delta_\nu\begin{pmatrix} \frac{2(\lambda_{R1}+\lambda_{R2})\Gamma_\nu}{(\lambda_{LR1}+\lambda_{LR2})^2 v_L^2} + 1 & 0 \\ 0 & \frac{2(\lambda_{L1}+\lambda_{L2})\Gamma_\nu}{(\lambda_{LR1}+\lambda_{LR2})^2 v_R^2} - 1 \end{pmatrix} \,, \tag{2.74}$$

with

$$C_\nu = \sqrt{\frac{(\lambda_{LR1}+\lambda_{LR2})^2 v_L^2 v_R^2}{2\Delta_\nu^2 + 2\Delta_\nu\Big[(\lambda_{L1}+\lambda_{L2})v_L^2 - (\lambda_{R1}+\lambda_{R2})v_R^2\Big]}}\,; \tag{2.75}$$

$$\Gamma_\nu = (\lambda_{R1}+\lambda_{R2})v_R^2 - (\lambda_{L1}+\lambda_{L2})v_L^2 - \Delta_\nu\,; \tag{2.76}$$

$$\Delta_\nu = \sqrt{\Big[(\lambda_{R1}+\lambda_{R2})v_R^2 - (\lambda_{L1}+\lambda_{L2})v_L^2\Big]^2 + (\lambda_{LR1}+\lambda_{LR2})^2 v_L^2 v_R^2}\,. \tag{2.77}$$

Using the expansion formula: $\sqrt{ax^2+bx+c} \approx \sqrt{c} + \frac{b}{2\sqrt{c}}x$ for $x \to 0$, and assuming $v_L/v_R \ll 1$ it is possible to rewrite (2.74) as:

$$M^2_{H_\nu} \approx \begin{pmatrix} (\lambda_{R1}+\lambda_{R2})v_R^2 + \frac{(\lambda_{LR1}+\lambda_{LR2})^2}{4(\lambda_{R1}+\lambda_{R2})}v_L^2 & 0 \\ 0 & (\lambda_{L1}+\lambda_{L2})v_L^2 - \frac{(\lambda_{LR1}+\lambda_{LR2})^2}{4(\lambda_{R1}+\lambda_{R2})}v_L^2 \end{pmatrix}\,, \tag{2.78}$$

where the lightest eigenvalue (the second one) represents the physical Higgs of the SM. While the orthogonal matrix that relate the interaction basis to the mass one is:

$$\begin{aligned} O_{H_\nu} = C_\nu &\begin{pmatrix} 1 & \frac{-(\lambda_{R1}+\lambda_{R2})v_R^2+(\lambda_{L1}+\lambda_{L2})v_L^2+\Delta_\nu}{(\lambda_{LR1}+\lambda_{LR2})v_L v_R} \\ \frac{(\lambda_{R1}+\lambda_{R2})v_R^2-(\lambda_{L1}+\lambda_{L2})v_L^2-\Delta_\nu}{(\lambda_{LR1}+\lambda_{LR2})v_L v_R} & 1 \end{pmatrix} = \\ = &\begin{pmatrix} \cos\theta_\nu & -\sin\theta_\nu \\ \sin\theta_\nu & \cos\theta_\nu \end{pmatrix} \quad \Longrightarrow \tan(2\theta_\nu) = -\frac{(\lambda_{LR1}+\lambda_{LR2})v_L v_R}{(\lambda_{R1}+\lambda_{R2})v_R^2 - (\lambda_{L1}+\lambda_{L2})v_L^2}\,, \end{aligned} \tag{2.79}$$

where the mixing is given by:

$$\cos\theta_\nu \sin\theta_\nu = \frac{(\lambda_{LR1}+\lambda_{LR2})v_L v_R}{-2\Delta_\nu} \approx \frac{-\frac{(\lambda_{LR1}+\lambda_{LR2})}{2(\lambda_{R1}+\lambda_{R2})}\frac{v_L}{v_R}}{1+\Big[\frac{(\lambda_{L1}+\lambda_{L2})}{(\lambda_{R1}+\lambda_{R2})} - \frac{(\lambda_{LR1}+\lambda_{LR2})^2}{(\lambda_{R1}+\lambda_{R2})^2}\Big]\frac{v_L^2}{v_R^2}}\,. \tag{2.80}$$

The last Higgs squared mass eigenvalues matrix we have to evaluate is that Higgs part related to the down-type quarks, i.e. $(M^2_{LR_d})^{diag} = M^2_{H_d}$:

$$M^2_{H_d} = \frac{-4\lambda^2_{LR3}\Delta_d v_L^2 v_R^2}{2\Gamma_d}\begin{pmatrix} \frac{2\lambda_{R2}\Gamma_d v_R^2+\lambda_{LR2}\Gamma_d v_L^2}{4\lambda^2_{LR3}\Delta_d v_L^2 v_R^2}+1 & 0 \\ 0 & \frac{2\lambda_{L2}\Gamma_d v_L^2+\lambda_{LR2}\Gamma_d v_R^2}{4\lambda^2_{LR3}\Delta_d v_L^2 v_R^2}-1 \end{pmatrix}$$
$$\approx -\begin{pmatrix} \lambda_{R2}v_R^2+\left[\frac{\lambda_{LR2}}{2}+\frac{\lambda^2_{LR3}}{\lambda_{R2}-\frac{\lambda_{LR2}}{2}}\right]v_L^2 & 0 \\ 0 & \lambda_{L2}v_L^2+\left[\frac{\lambda_{LR2}}{2}-\frac{\lambda^2_{LR3}}{\lambda_{R2}-\frac{\lambda_{LR2}}{2}}\right]v_R^2 \end{pmatrix}, \tag{2.81}$$

with

$$\Gamma_d = \Delta_d^2 + \Delta_d\Big[v_L^2\Big(\frac{\lambda_{LR2}}{2}-\lambda_{L2}\Big)-v_R^2\Big(\frac{\lambda_{LR2}}{2}-\lambda_{R2}\Big)\Big]; \tag{2.82}$$

$$\Delta_d = \sqrt{4\lambda^2_{LR3}v_L^2v_R^2+\Big[v_R^2\Big(\frac{\lambda_{LR2}}{2}-\lambda_{R2}\Big)-v_L^2\Big(\frac{\lambda_{LR2}}{2}-\lambda_{L2}\Big)\Big]^2}. \tag{2.83}$$

To complete the study, the orthogonal matrix of this sector has the form:

$$O_{H_d} = C_d\begin{pmatrix} 1 & \frac{-(\frac{\lambda_{LR2}}{2}-\lambda_{R2})v_R^2+(\frac{\lambda_{LR2}}{2}-\lambda_{L2})v_L^2+\Delta_d}{2\lambda_{LR3}v_Lv_R} \\ \frac{(\frac{\lambda_{LR2}}{2}-\lambda_{R2})v_R^2-(\frac{\lambda_{LR2}}{2}-\lambda_{L2})v_L^2-\Delta_d}{2\lambda_{LR3}v_Lv_R} & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} \cos\theta_d & -\sin\theta_d \\ \sin\theta_d & \cos\theta_d \end{pmatrix} \Longrightarrow \tan(2\theta_d) = -\frac{2\lambda_{LR3}v_Lv_R}{(\lambda_{R2}-\frac{\lambda_{LR2}}{2})v_R^2-(\lambda_{L2}-\frac{\lambda_{LR2}}{2})v_L^2}, \tag{2.84}$$

with

$$C_d = \sqrt{\frac{4\lambda^2_{LR3}v_L^2v_R^2}{2\Gamma_d}}. \tag{2.85}$$

At this point we can complete the condition over the λ_i parameters of the Higgs potential demanding that all the Higgs mass eigenvalues are positive. The only eigenvalue that gives new conditions is the second in (2.81):

$$-\lambda_{L2}v_L^2-\Big[\frac{\lambda_{LR2}}{2}-\frac{\lambda^2_{LR3}}{\lambda_{R2}-\frac{\lambda_{LR2}}{2}}\Big]v_R^2 \geq 0, \tag{2.86}$$

which gives the systems:

$$\begin{cases} 2\lambda_{R2}-\lambda_{LR2}>0 \\ \lambda^2_{LR3}\geq\frac{\lambda_{LR2}}{4}(2\lambda_{R2}-\lambda_{LR2}) \end{cases} \quad \text{or} \quad \begin{cases} 2\lambda_{R2}-\lambda_{LR2}<0 \\ \lambda^2_{LR3}\leq\frac{\lambda_{LR2}}{4}(2\lambda_{R2}-\lambda_{LR2}) \end{cases}, \tag{2.87}$$

with the solutions:

$$2\lambda_{R2} - \lambda_{LR2} > 0 \quad \text{or} \quad 2\lambda_{R2} - \lambda_{LR2} \leq \frac{4\lambda_{LR3}^2}{\lambda_{LR2}} < 0\,. \tag{2.88}$$

Therefore, putting all together, the conditions that we were able to find to minimize the Higgs potential are:

$$\begin{cases} \lambda_{R2} < 0 \\ \lambda_{LR2} < 0 \\ \lambda_{R1} + \lambda_{R2} > 0 \\ \lambda_{LR1} + \lambda_{LR2} < 0 \\ \lambda_{L1} + \lambda_{L2} > \frac{(\lambda_{LR1}+\lambda_{LR2})^2}{4(\lambda_{R1}+\lambda_{R2})} > 0 \\ 2\lambda_{R2} - \lambda_{LR2} > 0 \quad \vee \quad 2\lambda_{R2} - \lambda_{LR2} \leq \frac{4\lambda_{LR3}^2}{\lambda_{LR2}} < 0 \end{cases} . \tag{2.89}$$

2.2.1 Interlude on the Goldstone bosons

Because the symmetry breaking pattern is: $SU(4) \otimes SU(2)_L \otimes SU(2)_R \to SU(3)_C \otimes U(1)_Q$, we need to give mass to $(15 + 3 + 3) - (8 + 1) = 12$ gauge bosons; so we have to look for twelve Goldstone bosons. The most obvious place to start looking at is the Higgs part related to the charged leptons for which it does not exist any mass term:

$$\begin{pmatrix} L_e & R_e \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} L_e \\ R_e \end{pmatrix} \Rightarrow \begin{cases} m_{L_e}^2 \Big(| \Re(L_e) |^2 + | \Im(L_e) |^2 \Big) = 0 \\ m_{R_e}^2 \Big(| \Re(R_e) |^2 + | \Im(R_e) |^2 \Big) = 0 \end{cases}, \tag{2.90}$$

so we have four Goldstone bosons that are absorbed to give mass to the $W_{\mu R}^{\pm}$ and $W_{\mu L}^{\pm}$ vector bosons by means of the Higgs mechanism. We have seen above that other massless scalar fields are in the Higgs-up sector:

$$\begin{aligned} \begin{pmatrix} L_u^a & R_u^a \end{pmatrix}^* \begin{pmatrix} -A^2 & AB \\ AB & -B^2 \end{pmatrix} \begin{pmatrix} L_u^a \\ R_u^a \end{pmatrix} &= \begin{pmatrix} H_{1u}^a & H_{2u}^a \end{pmatrix}^* \begin{pmatrix} 0 & 0 \\ 0 & C^2 \end{pmatrix} \begin{pmatrix} H_{1u}^a \\ H_{2u}^a \end{pmatrix} \\ \Rightarrow &\begin{cases} m_{H_{1u}}^2 \Big(| \Re(H_{1u}^a) |^2 + | \Im(H_{1u}^a) |^2 \Big) = 0 \\ m_{H_{2u}}^2 \Big(| \Re(H_{2u}^a) |^2 + | \Im(H_{2u}^a) |^2 \Big) = C^2 (H_{2u}^a)^* H_{2u}^a \end{cases}, \end{aligned} \tag{2.91}$$

and, taking into account the colour number, this gives three copies, so we have $2 \times 3 = 6$ Goldstone bosons that are eaten by the vector bosons $X_\mu^{a\pm}$ (with $a = 1, 2, 3$). The last two

Goldstone bosons which remain to find, to give mass to the vector bosons Z_μ and Z'_μ, can not be a combination of Higgs-down since the colour number would produce three copies of these complex fields; therefore the last two Goldstone bosons must be a combination of the complex fields L_ν and R_ν. This is similar to what happen in the Standard Model (see Appendix C).

2.3 Gauge bosons masses

After studying the mass spectra of the scalar fields, the next step is to evaluate the mass spectra for the gauge bosons of the Pati-Salam group. As usual the masses for the vector bosons are obtained by means of the covariant derivatives in the kinetic term of the Higgs fields. So for the left-handed multiplet we have:

$$\begin{aligned} D^\mu L^{i\alpha} D_\mu L_{i\alpha} &= \left(\delta_i^j \delta_\alpha^\beta \partial_\mu + i g_4 \delta_i^j G^\beta_{\mu\alpha} + i g_L \delta_\alpha^\beta W^j_{\mu L i}\right) L_{j\beta} \times \\ &\quad \left(\delta_k^i \delta_\gamma^\alpha \partial^\mu - i g_4 \delta_k^i G^{\mu\alpha}_\gamma - i g_L \delta_\gamma^\alpha W^{\mu i}_{Lk}\right) L^{k\gamma} \\ &= \partial_\mu L_{i\alpha} \partial^\mu L^{i\alpha} + 2 g_L g_4 G^\beta_{\mu\alpha} L_{i\beta} L^{j\alpha} W^{\mu i}_{Lj} + g_4^2 G^\beta_{\mu\alpha} L_{i\beta} L^{i\gamma} G^{\mu\alpha}_\gamma \\ &\quad + g_L^2 W^j_{\mu L i} L_{j\alpha} L^{k\alpha} W^{\mu i}_{Lk}\,, \end{aligned} \tag{2.92}$$

and then using $\langle L_{14}\rangle = v_L/\sqrt{2}$ we obtain the mass terms:

$$\begin{aligned} & g_L g_4 v_L^2 W^{\mu 1}_{L1} G^4_{\mu 4} + g_4^2 \frac{v_L^2}{2} G^4_{\mu\alpha} G^{\mu\alpha}_4 + g_L^2 \frac{v_L^2}{2} W^1_{\mu L i} W^{\mu i}_{L1} = \frac{g_L^2 v_L^2}{8}\left[W^0_{\mu L} W^{\mu 0}_L + 2 W^+_{\mu L} W^{\mu -}_L\right] \\ & + \frac{g_4^2 v_L^2}{4}\left[X^-_{1\mu} X^{\mu +}_1 + X^-_{2\mu} X^{\mu +}_2 + X^-_{3\mu} X^{\mu +}_3 + \frac{3}{4} B_\mu B^\mu\right] - \frac{3 g_L g_4 v_L^2}{4\sqrt{6}} B^\mu W^0_{\mu L}\,. \end{aligned} \tag{2.93}$$

The same result in (2.93) is obtained from the kinetic term $D^\mu R^{i\alpha} D_\mu R_{i\alpha}$ for the right-handed Higgs multiplet, once the replacements $g_L \to g_R$, $v_L \to v_R$ and $W^\mu_L \to W^\mu_R$ has been made. First of all from (2.93), and its right-handed version $(2.93)_{L\to R}$, we can easily obtain the values of the masses for the charged gauge bosons:

$$M^2_{W_L} = g_L^2 \frac{v_L^2}{4}\,, \quad M^2_{W_R} = g_R^2 \frac{v_R^2}{4} \quad \text{and} \quad M^2_X = g_4^2 \frac{v_R^2 + v_L^2}{4}\,. \tag{2.94}$$

Because the bosons $W_R^{\mu\pm}$ and $X^{\mu\pm}$ were not observed to date experimentally, their masses have to be much bigger than the electroweak scale, so we must have $v_R \gg v_L$.

For the neutral gauge bosons, instead, we can extrapolate from (2.93) + $(2.93)_{L\to R}$ the mass squared matrix:

$$M_0^2 = \frac{1}{8}\begin{pmatrix} g_L^2 v_L^2 & 0 & -\frac{3}{\sqrt{6}} g_4 g_L v_L^2 \\ 0 & g_R^2 v_R^2 & -\frac{3}{\sqrt{6}} g_4 g_R v_R^2 \\ -\frac{3}{\sqrt{6}} g_4 g_L v_L^2 & -\frac{3}{\sqrt{6}} g_4 g_R v_R^2 & \frac{3}{2} g_4^2 \left(v_R^2 + v_L^2\right) \end{pmatrix}, \tag{2.95}$$

where we considered the basis $(W^0_{\mu L}, W^0_{\mu R}, B_\mu)$. The matrix M_0^2 has a zero eigenvalue, related to the photon A_μ, and two non-zero eigenvalues:

$$\begin{aligned} \frac{m_\pm^2}{8} = \frac{1}{16}\left(\frac{3}{2} g_4^2 \left(v_R^2 + v_L^2\right) + g_R^2 v_R^2 + g_L^2 v_L^2\right) \pm \\ \pm \frac{1}{16}\sqrt{\left(\frac{3}{2} g_4^2 \left(v_R^2 - v_L^2\right) + g_R^2 v_R^2 - g_L^2 v_L^2\right)^2 + 9 g_4^4 v_R^2 v_L^2}\,, \end{aligned} \tag{2.96}$$

related to the bosons Z_μ and Z'_μ. Reminding that for the neutral vector bosons the mass terms are of the form: $\frac{M^2}{2} Z^2$, we obtain for Z_μ and Z'_μ the masses:

$$M_Z^2 = \frac{m_-^2}{4} \approx \left[g_L^2 + \frac{3 g_4^2 g_R^2}{3 g_4^2 + 2 g_R^2}\right] \frac{v_L^2}{4} \tag{2.97}$$

$$M_{Z'}^2 = \frac{m_+^2}{4} \approx \left[g_R^2 + \frac{3}{2} g_4^2\right] \frac{v_R^2}{4} + \left[\frac{9 g_4^4}{3 g_4^2 + 2 g_R^2}\right] \frac{v_L^2}{8} \tag{2.98}$$

at first order in $(v_L/v_R)^2$. From the matrix M_0^2 we get the mass eigenstate:

$$\begin{pmatrix} A_\mu \\ Z_\mu \\ Z'_\mu \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{3}{2}}\frac{g_4}{g_L} C_0 & \sqrt{\frac{3}{2}}\frac{g_4}{g_R} C_0 & C_0 \\ \frac{-2\sqrt{6} g_L g_4 v_L^2 C_-}{3 g_4 (v_R^2 + v_L^2) + 2 g_R^2 v_R^2 - 2 g_L^2 v_L^2 - 2\Delta} & \frac{-2\sqrt{6} g_L g_4 v_L^2 C_-}{3 g_4 (v_R^2 + v_L^2) - 2 g_R^2 v_R^2 + 2 g_L^2 v_L^2 - 2\Delta} & C_- \\ \frac{-2\sqrt{6} g_L g_4 v_L^2 C_+}{3 g_4 (v_R^2 + v_L^2) + 2 g_R^2 v_R^2 - 2 g_L^2 v_L^2 + 2\Delta} & \frac{-2\sqrt{6} g_L g_4 v_L^2 C_+}{3 g_4 (v_R^2 + v_L^2) - 2 g_R^2 v_R^2 + 2 g_L^2 v_L^2 + 2\Delta} & C_+ \end{pmatrix} \begin{pmatrix} W^0_{\mu L} \\ W^0_{\mu R} \\ B_\mu \end{pmatrix} \tag{2.99}$$

where we defined the normalization constants:

$$C_0 = \sqrt{\frac{2 g_R^2 g_L^2}{3 g_R^2 g_4^2 + 2 g_R^2 g_L^2 + 3 g_L^2 g_4^2}}\,; \tag{2.100}$$

$$C_\pm = \sqrt{\frac{p_\pm^2 q_\pm^2}{24 g_4^2 (g_L^2 v_L^4 q_\pm^2 + g_R^2 v_R^4 p_\pm^2) + p_\pm^2 q_\pm^2}}\,; \tag{2.101}$$

with

$$\begin{cases} p_\pm = 3g_4^2(v_R^2+v_L^2)+2g_R^2v_R^2-2g_L^2v_L^2 \pm 2\Delta \\ q_\pm = 3g_4^2(v_R^2+v_L^2)-2g_R^2v_R^2+2g_L^2v_L^2 \pm 2\Delta \\ \Delta = 9g_4^4v_R^2v_L^2 + \left[\frac{3}{2}g_4^2(v_R^2-v_L^2)+g_R^2v_R^2-g_L^2v_L^2\right]^2 . \end{cases} \tag{2.102}$$

A general 3×3 trigonometric orthogonal matrix has a form of the type:

$$\begin{pmatrix} \cos\alpha\cos\gamma-\sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta & -\cos\alpha\sin\gamma-\sin\alpha\sin\beta\cos\gamma \\ \cos\alpha\sin\beta\sin\gamma+\sin\alpha\cos\gamma & \cos\alpha\cos\beta & \cos\alpha\sin\beta\cos\gamma-\sin\alpha\sin\gamma \\ \cos\beta\sin\gamma & -\sin\beta & \cos\beta\cos\gamma \end{pmatrix}, \tag{2.103}$$

with α, β and γ the mixing angles. Assuming $g_L = g_R = g_2$ and $v_L \ll v_R$, the orthogonal mixing matrix in (2.99) can be rewritten as:

$$\begin{pmatrix} A_\mu \\ Z_\mu \\ Z'_\mu \end{pmatrix} \approx \begin{pmatrix} \sin\theta'_W & \sin\theta'_W & \sqrt{\cos 2\theta'_W} \\ \cos\theta'_W & -\tan\theta'_W\sin\theta'_W & -\tan\theta'_W\sqrt{\cos 2\theta'_W} \\ 0 & -\frac{\sqrt{\cos 2\theta'_W}}{\cos\theta'_W} & \tan\theta'_W \end{pmatrix} \begin{pmatrix} W^0_{\mu L} \\ W^0_{\mu R} \\ B_\mu \end{pmatrix}, \tag{2.104}$$

where we use the definition:

$$\sin\theta'_W = \sqrt{\frac{3}{2}}\frac{g_4}{g_2}C_0 = \sqrt{\frac{3g_4^2}{6g_4^2+2g_2^2}} . \tag{2.105}$$

Therefore we can write the orthogonal matrix in (2.99) as a function of the angle θ'_W only, like in SM where the weak mixing:

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \sin\theta_W & \cos\theta_W \\ \cos\theta_W & -\sin\theta_W \end{pmatrix} \begin{pmatrix} W^0_{\mu L} \\ B_\mu \end{pmatrix} \tag{2.106}$$

is a function of the Weinberg angle θ_W only, with $\sin\theta_W = \sqrt{\frac{g_Y^2}{g_L^2+g_Y^2}}$. If at this point we compare the photon-fermions interaction coming from the covariant derivative $i\bar{\Psi}\gamma_\mu D^\mu\Psi$ with the charge definition given in (2.14):

$$\frac{1}{2}\left[g_L\sigma_{3L}\sin\theta_W + g_R\sigma_{3R}\sin\theta_W + g_4\lambda_{15}\sqrt{1-2\sin^2\theta_W}\right]A_\mu = eQA_\mu\,, \tag{2.107}$$

we obtain the following relations:

$$\begin{cases} e = g_L \sin\theta_W \\ e = g_R \sin\theta_W \\ e\sqrt{\frac{2}{3}} = g_4\sqrt{1-2\sin^2\theta_W} \end{cases}, \tag{2.108}$$

that, besides being consistent with the assumption $g_L = g_R = g_2$, it give us the new definitions of the Weinberg angle and the electric charge:

$$\sin\theta'_W = \frac{e}{g_2} = \sqrt{\frac{1}{2} - \frac{e^2}{3g_4^2}} \quad \Longrightarrow \quad e = \sqrt{\frac{3g_2^2 g_4^2}{6g_4^2 + 2g_2^2}}\,. \tag{2.109}$$

2.4 RGE for the gauge coupling constants

In this section we study the running of the gauge coupling costants evaluating the 1-loop solution of the Callan-Symanzik equation:

$$\alpha_i^{-1}(M) = \alpha_i^{-1}(m) - \frac{a_i}{4\pi}\log\frac{M}{m}\,, \tag{2.110}$$

where $\alpha_i = \frac{g_i^2}{4\pi}$, with g_i the coupling constant and a_i some coefficients coming from group calculations. Under a non-abelian group $SU(N_i)$, in particular, these coefficients are given by:

$$a_i = \frac{4}{3}n_g - \frac{11}{3}N_i + \frac{1}{3}\eta\frac{T(N_i)}{N_i}\prod_k N_k\,, \tag{2.111}$$

where n_g is the number of fermion generations, η is a real number equal to 1 for complex scalar field or 1/2 for real scalar field and $T(N_i)$ is the Dynkin index. So considering the SM, for the group $SU(3)_C$ we have:

$$a_{3C} = \frac{4}{3}\cdot 3 - \frac{11}{3}\cdot 3 = -7\,; \tag{2.112}$$

instead for $SU(2)_L$, given that the SM Higgs multiplet is $\varphi \equiv (1,2)$ under the group $SU(3)_C \otimes SU(2)_L$, we find:

$$a_{2L} = \frac{4}{3}\cdot 3 - \frac{11}{3}\cdot 2 + \frac{1}{3}\cdot 1\cdot\frac{1/2}{2}\cdot(1\cdot 2) = -\frac{19}{6}\,; \tag{2.113}$$

while for the abelian group $U(1)_Y$, using the hypercharge redefinition (D.6) in Appendix D, we have:

$$\begin{aligned} a_{1Y} &= \frac{2}{3} \sum_{fermions} \frac{Y_f'^2}{4} + \frac{1}{3} \sum_{scalars} \frac{Y_s'^2}{4} = \\ &= \frac{2}{3} \cdot \frac{3}{5} n_g \left[N_c \left(2 \cdot \frac{1}{36} + \frac{4}{9} + \frac{1}{9} \right) + \left(2 \cdot \frac{1}{4} + 1 \right) \right] + \frac{1}{3} \cdot \frac{3}{5} \cdot 2 \cdot \frac{1}{4} = \frac{41}{10} \,. \end{aligned} \tag{2.114}$$

In Fig. 2.1 we show the evolution of the gauge coupling constants of the Standard Model.

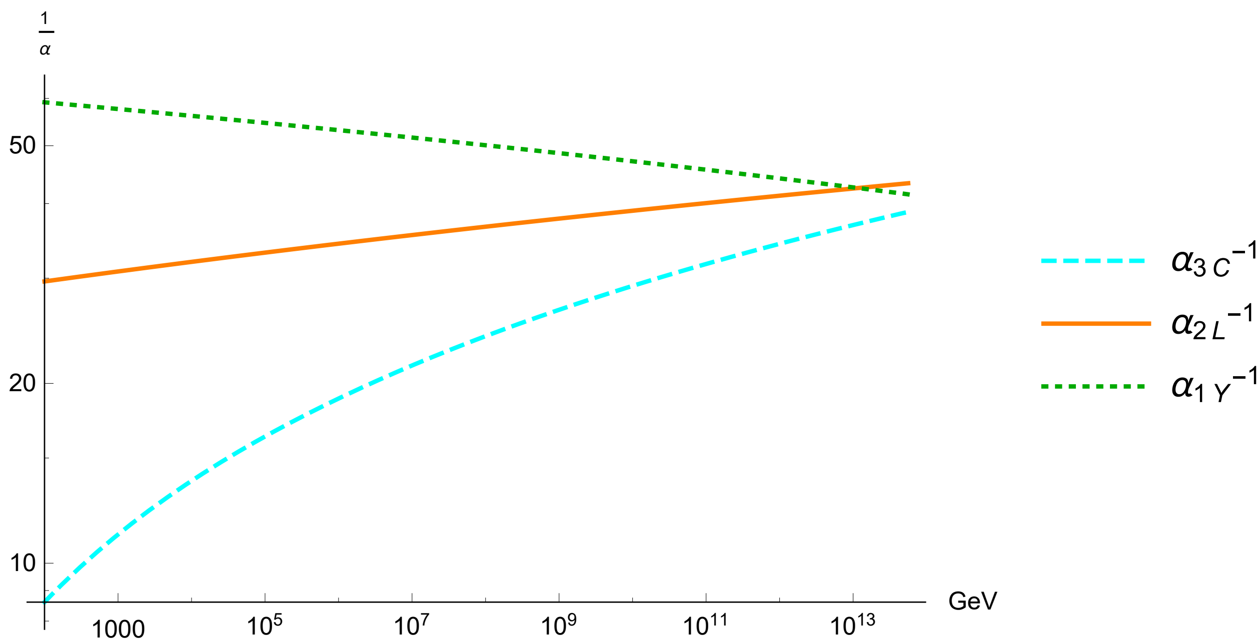


Figure 2.1: *Running of the coupling constants $1/\alpha_i$ of the SM as a function of the energy scale, where the green dotted, orange solid and cyan dashed lines represent the evolutions of the coupling costants for the groups $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$, respectively.*

We can perform the same analysis for the model 1. In this case we have two Higgs multiplets that, under the group $SU(4) \otimes SU(2)_L \otimes SU(2)_R$, are in the representations $L \equiv (4, 2, 1)$ and $R \equiv (4, 1, 2)$, therefore the coefficients of the Callan-Symanzik equation are:

$$\begin{cases} a_{2L} = \frac{4}{3} \cdot 3 - \frac{11}{3} \cdot 2 + \frac{1}{3} \cdot 1 \cdot \frac{1/2}{2} \cdot (4 \cdot 2 \cdot 1) = -\frac{8}{3} \\ a_{2R} = \frac{4}{3} \cdot 3 - \frac{11}{3} \cdot 2 + \frac{1}{3} \cdot 1 \cdot \frac{1/2}{2} \cdot (4 \cdot 1 \cdot 2) = -\frac{8}{3} \\ a_{4C} = \frac{4}{3} \cdot 3 - \frac{11}{3} \cdot 4 + 2 \cdot \left[\frac{1}{3} \cdot 1 \cdot \frac{1/2}{4} \cdot (4 \cdot 2 \cdot 1) \right] = -10 \end{cases} . \tag{2.115}$$

At this point, in order to evaluate the partial-unification scale M_U, to which there is the transition between the Standard Model to the Pati-Salam model, we ask that at this energy scale the gauge coupling constants respect the matching conditions (E.15) in Appendix E, coming from the hypercharge definition, and that the assumption $g_L = g_R$ is verified:

$$\begin{cases} \alpha_{1Y}^{-1}(M_U) = \frac{3}{5}\alpha_{2R}^{-1}(M_U) + \frac{2}{5}\alpha_{4C}^{-1}(M_U) \\ \alpha_{2R}(M_U) = \alpha_{2L}(M_U) \end{cases} . \tag{2.116}$$

From the running of the coupling constants, we obtain the following system of equations:

$$\begin{cases} \alpha_{3C}^{-1}(M_U) = \alpha_{3C}^{-1}(M_Z) + \frac{7}{2\pi}\ln\frac{M_U}{M_Z} = \alpha_{4C}^{-1}(M_U) \\ \alpha_{2L}(M_U) = \alpha_{2L}^{-1}(M_Z) + \frac{19}{12\pi}\ln\frac{M_U}{M_Z} = \alpha_{2R}^{-1}(M_U) \\ \alpha_{1Y}^{-1}(M_U) = \alpha_{1Y}^{-1}(M_Z) - \frac{41}{20\pi}\ln\frac{M_U}{M_Z} = \frac{3}{5}\alpha_{2R}^{-1}(M_U) + \frac{2}{5}\alpha_{4C}^{-1}(M_U) \end{cases} , \tag{2.117}$$

where we use the experimental values for the SM coupling costants at the electro-weak scale $M_Z \approx 91.1876$ GeV:

$$\begin{cases} \alpha_{1Y}(M_Z) \approx 0.0169 \\ \alpha_{2L}(M_Z) \approx 0.0338 \\ \alpha_{3C}(M_Z) \approx 0.1176 \end{cases} . \tag{2.118}$$

From (2.117) it is simple to estimate $M_U \approx 5.6103 \cdot 10^{13}$ GeV that represents, besides the value of the Pati-Salam scale, the order of magnitude of the VEV $\langle R \rangle = v_R$ too. In Fig. 2.2 it is exhibited the evolution of the gauge coupling constants through the partial-unification scale M_U and the fact that the evolution of g_L is the same as that of g_R, since the model respects the left-right symmetry for energies above M_U.

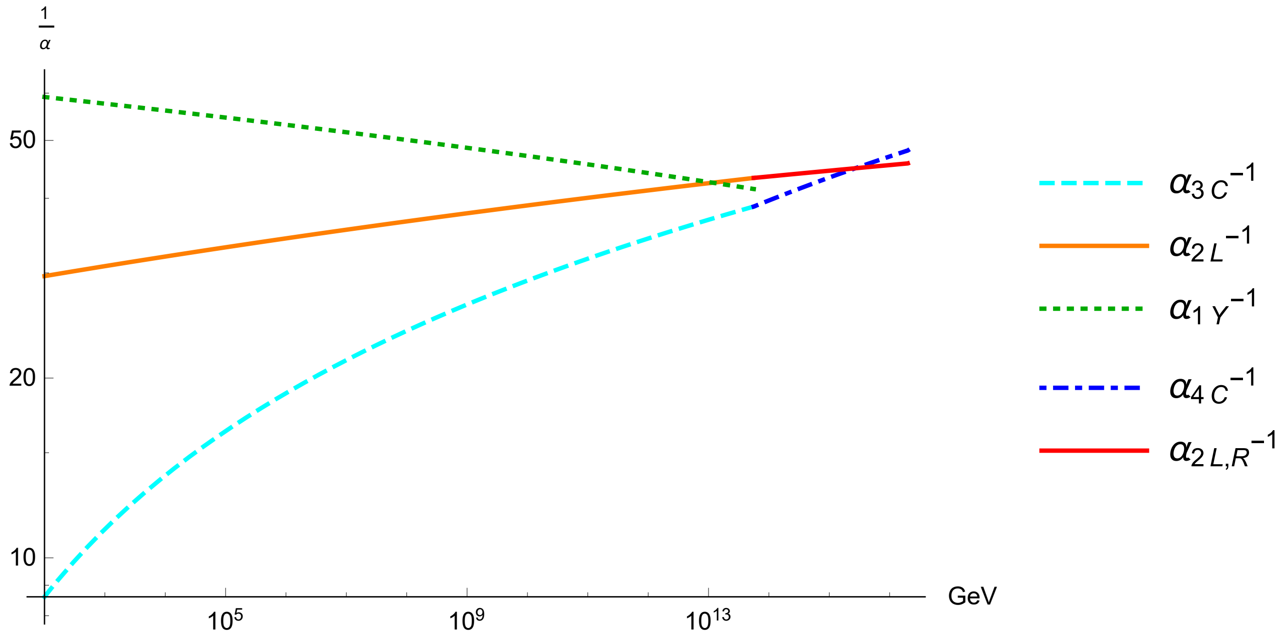
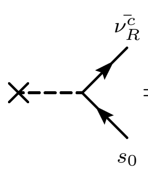


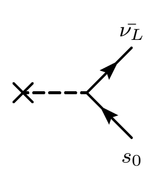
Figure 2.2: *Running of the coupling constants $1/\alpha_i$ of the SM and Pati-Salam model, as a function of the energy scale, through the partial-unification, where the green dotted, orange solid and cyan dashed lines represent the evolutions of the coupling costants for the groups $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$ respectively, while the red solid and blue dotdashed lines represent the evolutions of the coupling costants for the groups $SU(2)_{L/R}$ and $SU(4)$, respectively.*

2.5 Neutrinos masses

As we mentioned at the beginnig, in this model the only fermions that get a tree level mass are the neutrinos. So let us analyze the terms in the Lagrangian (2.11) that produces these contributions. When the Higgs fields get the VEV, from the Yukawa interaction we obtain the tree level mass terms of the form:



$$= -ik_R \frac{v_R}{\sqrt{2}} \quad \text{and}$$

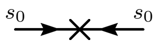


$$= -ik_L \frac{v_L}{\sqrt{2}}\,, \tag{2.119}$$

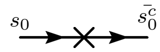
while from the explicit mass term for the Majorana fermion $S = \begin{pmatrix} s_0^c \\ s_0 \end{pmatrix}$:

$$\begin{aligned} m_0 \bar{S} S = m_0 \left[(s_0)^\dagger s_0^c + (s_0^c)^\dagger s_0 \right] \equiv m_0 \left[\bar{s}_0 s_0^c + \bar{s^c}_0 s_0 \right] = \\ = m_0 \left[\bar{s}_0 (i\sigma_2) \bar{s}_0^T + s_0^T (i\sigma_2)^\dagger s_0 \right] \,, \end{aligned} \tag{2.120}$$

we extrapolate the Majorana mass insertion vertex for the singlet s_0, which is given by the form:



$$ = -im_0(i\sigma_2)^\dagger \quad \text{or} $$



$$ = -im_0\,, \tag{2.121}$$

where we consider m_0 a real parameter. If for the neutrino fields we use the basis definition:

$$\Psi_\nu = \begin{pmatrix} \nu_L \\ \nu_R^c \\ s_0^c \end{pmatrix} \quad \text{and} \quad \Psi_\nu^c = \begin{pmatrix} \nu_L^c \\ \nu_R \\ s_0 \end{pmatrix}, \tag{2.122}$$

we can write the total mass lagrangian for the neutrinos as:

$$\mathcal{L}_{mass}^{\nu} = -\bar{\Psi}_\nu M_\nu \Psi_\nu^c - \bar{\Psi_\nu^c} M_\nu^\dagger \Psi_\nu\,, \tag{2.123}$$

where the neutrino mass matrix M_ν is defined as:

$$M_\nu = \begin{pmatrix} 0 & 0 & \frac{k_L v_L}{2\sqrt{2}} \\ 0 & 0 & \frac{k_R v_R}{2\sqrt{2}} \\ \frac{k_L v_L}{2\sqrt{2}} & \frac{k_R v_R}{2\sqrt{2}} & m_0 \end{pmatrix} = M_\nu^\dagger\,. \tag{2.124}$$

We note that the factor 1/2 in M_ν definition comes from the fact that we consider real VEVs and Yukawa couplings, therefore we have for example:

$$\frac{k_L v_L}{2\sqrt{2}} \bar{\nu}_L s_0 = \frac{k_L v_L}{2\sqrt{2}} \bar{s_0^c} \nu_L^c\,. \tag{2.125}$$

The eigenvalues λ of the neutrino mass matrix M_ν in (2.124) are given by:

$$(M_\nu)^{diag} = \begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_+ & 0 \\ 0 & 0 & \lambda_- \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{m_0}{2} + \frac{\Delta}{2\sqrt{2}} & 0 \\ 0 & 0 & \frac{m_0}{2} - \frac{\Delta}{2\sqrt{2}} \end{pmatrix}, \tag{2.126}$$

with $\Delta = \sqrt{2m_0^2 + k_R^2 v_R^2 + k_L^2 v_L^2}$. The physical mass of the neutrinos are the absolute values of $(M_\nu)^{diag}$, where some possible overall minus sign can be reabsorbed by means of field redefinitions of the neutrino mass eigenstates ν_i. In this regard, we recall that it is possible to pass from the interaction basis to the mass eigenstate basis thanks to the orthogonal matrix:

$$O_\nu = \begin{pmatrix} \frac{k_R v_R}{\sqrt{\Delta^2-2m_0^2}} & \frac{k_L v_L}{\sqrt{(\sqrt{2}m_0+\Delta)^2+\Delta^2-2m_0^2}} & \frac{k_L v_L}{\sqrt{(\sqrt{2}m_0-\Delta)^2+\Delta^2-2m_0^2}} \\ -\frac{k_L v_L}{\sqrt{\Delta^2-2m_0^2}} & \frac{k_R v_R}{\sqrt{(\sqrt{2}m_0+\Delta)^2+\Delta^2-2m_0^2}} & \frac{k_R v_R}{\sqrt{(\sqrt{2}m_0-\Delta)^2+\Delta^2-2m_0^2}} \\ 0 & \frac{\sqrt{2}m_0+\Delta}{\sqrt{(\sqrt{2}m_0+\Delta)^2+\Delta^2-2m_0^2}} & \frac{\sqrt{2}m_0-\Delta}{\sqrt{(\sqrt{2}m_0-\Delta)^2+\Delta^2-2m_0^2}} \end{pmatrix}, \tag{2.127}$$

where the columns of O_ν are the normalized eigenvectors of M_ν. In this way it is possible to write for the neutrino mass term the relation:

$$\bar{\Psi}_\nu M_\nu \Psi_\nu^c = \bar{\Psi}_\nu O_\nu \left[O_\nu^T M_\nu O_\nu\right] O_\nu^T \Psi_\nu^c = \begin{pmatrix} \bar{\nu_1} & \bar{\nu_2} & \bar{\nu_3} \end{pmatrix} (M_\nu)^{diag} \begin{pmatrix} \nu_1^c \\ \nu_2^c \\ \nu_3^c \end{pmatrix}; \tag{2.128}$$

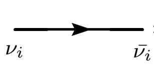
in particular, the sterile neutrino mass insertion can be written in the mass eigenbasis:

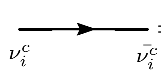
$$\begin{aligned} \bar{s^c}_0 m_0 s_0 &= \bar{\nu_i^c}(O_\nu^T)_{i3} \left[(O_\nu)_{3k}(M_\nu)^{diag}_{kl}(O_\nu^T)_{l3}\right](O_\nu)_{3j}\nu_j = \\ &= \bar{\nu_i^c}(O_\nu^T)_{i3} \left[(O_\nu)_{3k} m_k \delta_{kl}(O_\nu^T)_{l3}\right](O_\nu)_{3j}\nu_j = \\ &= \bar{\nu_i^c}(O_\nu^T)_{i3} \left[(O_\nu)^2_{3k} m_k\right](O_\nu)_{3j}\nu_j\,, \end{aligned} \tag{2.129}$$

or, to simplify the notation, we can rewrite the lagrangian related to the neutrinos as:

$$\begin{aligned} \mathcal{L}^\nu &= i(\bar{\Psi}_\nu)_a \gamma_\mu \partial^\mu (\Psi_\nu)_a + i(\bar{\Psi_\nu^c})_a \gamma_\mu \partial^\mu (\Psi_\nu^c)_a - \left[(\bar{\Psi}_\nu)_a (M_\nu)_{ab} (\Psi_\nu^c)_b + h.c.\right] = \\ &= i\bar{\nu}_i \gamma_\mu \partial^\mu \nu_i + i\bar{\nu_i^c}\gamma_\mu \partial^\mu \nu_i^c - \left[m_i \bar{\nu}_i \nu_i^c + h.c.\right], \end{aligned} \tag{2.130}$$

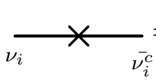
so that we can use the propagators relative to the massive Majorana fermion ν_i:

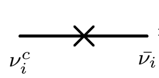


$$= \frac{-ip_\mu\gamma_\mu}{p^2-m_i^2+i\epsilon}$$ and


$$= \frac{ip_\mu\gamma_\mu}{p^2-m_i^2+i\epsilon}, \tag{2.131}$$

where the sign difference is due to the different chirality between the fields ν_i and ν_i^c. While concerning the propagators with chirality flip, i.e. the mass terms, they are given by:



$$= \frac{im_i}{p^2-m_i^2+i\epsilon}$$ and


$$= \frac{im_i}{p^2-m_i^2+i\epsilon}. \tag{2.132}$$

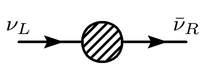
See Appendix F and Appendix G for more details about the study of the fermion mass terms and the fermionic propagators.

Considering that we have three fermionic families ($f = 3$), then there are three matrices M_ν with the same structure but different values for k_L, k_R and m_0. Therefore we will have a zero tree-level mass for each of the three SM neutrinos, that should get the correct small radiative corrections at one-loop; while we expect the others six mass eigenstates to be of order v_R, so that the mixing with the SM ones will be very small. In the limit $m_0 \gg v_R$, the eigenvalues would be: $\lambda_0 = 0$, $\lambda_+ \approx m_0$ and $\lambda_- \approx 0$; therefore there would be three massless neutrinos. While if $m_0 \sim v_R$ or $m_0 \ll v_R$ we get: $\lambda_0 = 0$, $\lambda_+ \sim v_R$ and $\lambda_- \sim -v_R$, where the minus sign can be reabsorbed into a phase factor in a field redefinitions. However, in the rest of the thesis, we assume $m_0 \sim v_R$ because this is the natural scale of the sterile neutrino mass in the theory; in this way we have also a second parameter, together with v_R, in the seesaw mechanism to mantain the SM neutrinos light.

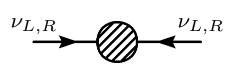
I conclude this section anticipating the form that the neutrino mass matrix M_ν will have at one-loop, once the corrections are taken in account:

$$M_\nu^{1-loop} = \begin{pmatrix} \left(m_{\nu_L}^M\right)^* & \left(\frac{m_\nu^D}{2}\right)^* & \frac{k_L v_L}{2\sqrt{2}} \\ \left(\frac{m_\nu^D}{2}\right)^\dagger & m_{\nu_R}^M & \frac{k_R v_R}{2\sqrt{2}} \\ \frac{k_L v_L}{2\sqrt{2}} & \frac{k_R v_R}{2\sqrt{2}} & m_0 \end{pmatrix} . \tag{2.133}$$

The neutrino masses radiatively generated in (2.133) are defined by the one-loop diagram:



$= -im_\nu^{Dirac}$ and



$= -im_{\nu_{L,R}}^{Majorana}$, (2.134)

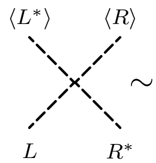
which, as we shall see, will contain loops with scalar and gauge bosons exchanges.

2.6 Loops

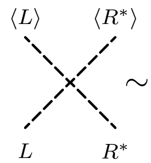
Beside what we have just seen for the neutrinos, in this model no tree level mass term for quarks and charged leptons exists. The neutrino mass corrections in (2.134) and all the other fermion mass contributions will be therefore radiatively generated at one-loop. In particular, we have scalar and gauge loops responsible for the generation of Dirac masses for quarks and leptons, and also masses of Majorana type for the neutrinos ν_L and ν_R. Since the loops we are going to analyze will generate terms that are not present in the bare Lagrangian (2.11), that is terms which have not to be renormalized at one-loop, we expect these contributions to be finite.

2.6.1 Scalar loop

The first contributions we take into account are those given by loops containing an Higgs boson exchange. In order to generate a mass term for the fermions of the form $-m\bar{\psi}_R\psi_L$ we need, from the scalar potential (2.28), the four-point Higgs interaction vertices:

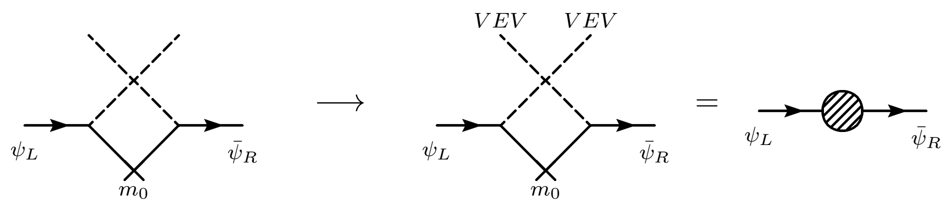


$\sim (\lambda_{LR1} + \lambda_{LR2})$ and



$\sim \lambda_{LR3}\,;$ (2.135)

in fact, if we want to attach fermionic fields to the free Higgs legs in the vertices above we can do it by means of the Yukawa interactions $-k_R \Psi_R^{i\alpha} R_{i\alpha} s_0^c - k_L \bar{s}_0 L^{i\alpha} \Psi_{Li\alpha}$ and the sterile neutrino mass term. In this way, once the remaining Higgs fields get VEV:



, (2.136)

we get the one-loop generated fermion mass term. The only non-zero contribution for (2.136) that we can obtain from the VEVs $v_R = \langle R_{14}\rangle$ and $v_L = \langle L_{14}\rangle$ in the four-point scalar interaction proportional to the parameter λ_{LR1} in (2.28) is given by:

$$\Psi_R^{k\mu} R_{k\mu} \left[R^{i\alpha} R_{i\alpha} L^{j\beta} L_{j\beta}\right] L^{l\nu} \Psi_{Ll\nu} \;\rightarrow\; \Psi_R^{14} \langle R_{14} R^{14}\rangle \langle R_{14}\rangle \langle L^{14}\rangle \langle L_{14} L^{14}\rangle \Psi_{L14}\,, \qquad (2.137)$$

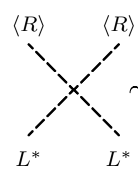
where $\langle L^{i\alpha} L_{i\alpha}\rangle$ and $\langle R^{i\alpha} R_{i\alpha}\rangle$ represent the scalar propagators; this means that the λ_{LR1} interaction contributes only to the neutrino masses. From the interaction proportional to λ_{LR2}, instead, we have:

$$\Psi_R^{k\mu} R_{k\mu} \left[R^{i\alpha} L^{j\beta} R_{i\beta} L_{j\alpha}\right] L^{l\nu} \Psi_{Ll\nu} \;\rightarrow\; \Psi_R^{1\alpha} \langle R_{1\alpha} R^{1\alpha}\rangle \langle L^{14}\rangle \langle R_{14}\rangle \langle L_{1\alpha} L^{1\alpha}\rangle \Psi_{L1\alpha}\,, \qquad (2.138)$$

therefore this gives a contribution to the neutrino and up-type quark masses. On the other hand from the interaction proportional to λ_{LR3} we obtain a contribution only for the down-type quark masses:

$$\begin{aligned}
&\Psi_R^{k\mu} R_{k\mu} \left[R^{m\mu} R^{n\nu} \epsilon_{mn} \epsilon^{pq} L_{p\mu} L_{q\nu}\right] L^{l\nu} \Psi_{Ll\nu} = \\
&= \frac{1}{4} \Psi_R^{k\mu} R_{k\mu} \left[R^{i\alpha}(\delta_i^m \delta_\alpha^\mu R^{n\nu} + R^{m\mu} \delta_i^n \delta_\alpha^\nu) \epsilon_{mn} \epsilon^{pq} (\delta_p^j \delta_\mu^\beta L_{q\nu} + L_{p\mu} \delta_q^j \delta_\nu^\beta) L_{j\beta}\right] L^{l\nu} \Psi_{Ll\nu} = \\
&= \frac{1}{2} \Psi_R^{k\mu} R_{k\mu} R^{i\alpha} \epsilon_{ik} \epsilon^{jl} \left[\delta_\alpha^\beta R^{k\eta} L_{l\eta} - R^{k\beta} L_{l\alpha}\right] L_{j\beta} L^{l\nu} \Psi_{Ll\nu} \longrightarrow \\
&\longrightarrow \quad \frac{\langle R^{14}\rangle \langle L_{14}\rangle}{2} \left[\Psi_R^{2\alpha} \langle R_{2\alpha} R^{2\alpha}\rangle \langle L_{2\alpha} L^{2\alpha}\rangle \Psi_{L2\alpha} - \Psi_R^{24} \langle R_{24} R^{24}\rangle \langle L_{24} L^{24}\rangle \Psi_{L24}\right] = \\
&\qquad = \sum_{\alpha=1}^{3} \Psi_R^{2\alpha} \langle R_{2\alpha} R^{2\alpha}\rangle \frac{\langle R^{14}\rangle \langle L_{14}\rangle}{2} \langle L_{2\alpha} L^{2\alpha}\rangle \Psi_{L2\alpha}\,.
\end{aligned} \tag{2.139}$$

In particular we can note that we do not have any scalar loop contribution of this type for the masses of the charged leptons e, μ and τ. The results of this analysis are consistent with what we calculated in Sec. 2.2, where all the interactions that we have just written can be also found in the non-diagonal elements of the Higgs squared mass matrices (2.66), (2.67) and (2.68). To be complete we consider also the possible contribution to a Majorana mass term that should be given by a vertex of the type:



$$\sim \lambda_{LR3}\,, \tag{2.140}$$

but this is zero because of the product between an anti-symmetric tensor with a symmetric one:

$$\lambda_{LR3} \Psi_L^{k\mu} L_{k\mu} \Big[L^{i\alpha} R_{p\nu} R_{n\mu} \epsilon^{np} \epsilon_{mk} \left(\delta_i^k \delta_\alpha^\nu \delta_j^m \delta_\beta^\mu + \delta_i^m \delta_\alpha^\mu \delta_j^k \delta_\beta^\nu\right) L^{j\beta}\Big] L_{l\nu} \Psi_L^{l\nu} = 0\,. \tag{2.141}$$

Putting together the Higgs self interactions with the sterile neutrino mass insertion we can generate the diagram depicted in Fig. 2.3, that represents the radiative generation of the Dirac mass term: $-m_H^{(\psi)} \bar{\psi}_R \psi_L$, with $\psi = u, d, \nu$.

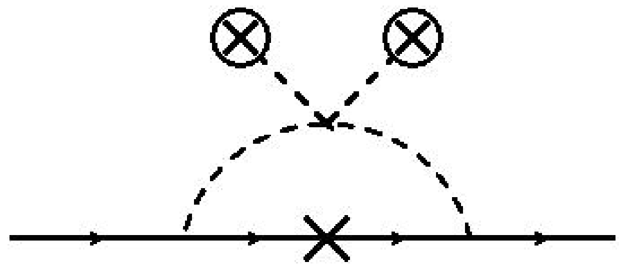
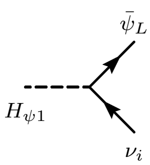


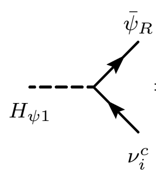
Figure 2.3: Scalar loop for the radiative generation of the fermion mass term $-m_H^{(\psi)}\bar{\psi}_R\psi_L$, with Higgs VEV (represented by a circled cross) and sterile neutrino mass insertions (represented by a cross).

We observe that the diagram in Fig. 2.3 contains: two fermionic and two scalar propagators, two VEVs and a mass insertions, from a dimensional point of view the count is $O(m^3/p^6)$, so this diagram has to come out finite. Let us see in details its evaluation. For simplicity we pass from the interaction basis to the mass basis for the Higgs and the sterile neutrino fields, therefore we rewrite the Yukawa interaction terms as:

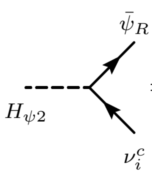
$$\begin{aligned}\mathcal{L}_{Yukawa} &= -k_L\bar{\psi}_L L_\psi s_0 - k_R\bar{\psi}_R R_\psi s_0^c + h.c. = \\ &= -k_L\bar{\psi}_L\left[H_{\psi 1}\cos\theta_\psi - H_{\psi 2}\sin\theta_\psi\right]\left[(O_\nu)_{3i}\nu_i\right] - \\ &\quad - k_R\bar{\psi}_R\left[H_{\psi 2}\cos\theta_\psi + H_{\psi 1}\sin\theta_\psi\right]\left[(O_\nu)_{3i}\nu_i^c\right] + h.c. \quad ,\end{aligned} \tag{2.142}$$

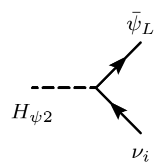
from which we extrapolate the following Feynman rules:



$= -ik_L\cos\theta_\psi(O_\nu)_{3i}$,


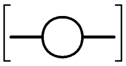
$= -ik_R\sin\theta_\psi(O_\nu)_{3i}$, (2.143)



$= -ik_R\cos\theta_\psi(O_\nu)_{3i}$,


$= +ik_L\sin\theta_\psi(O_\nu)_{3i}$. (2.144)

The mass $m_H^{(\psi)}$ generated by the diagram in Fig. 2.4 can thus be written as:

$m_H^{(\psi)} = i\Big[$  $\Big]_H^{(\psi)} = k_Lk_R\sin\theta_\psi\cos\theta_\psi\sum_{i=1}^{3}(O_\nu)_{3i}^2 I_H^{(\psi)\,i}(p)$, (2.145)

where $I_H^{(\psi)\,i}(p)$ is the loop integral. Because we treat the radiatively generated mass $m_H^{(\psi)}$ as a coupling between states with different chirality (see Appendix G), its value has to be identified, from the normalization conditions, with the value of the loop diagram in the limit of zero external momentum p, thus:

$$I_H^{(\psi)\,i}(p=0) = -i \int \frac{d^4q}{(2\pi)^4} \frac{im_i}{q^2 - m_i^2 + i\epsilon} \left[\frac{i}{q^2 - M^2_{H_{\psi1}} + i\epsilon} - \frac{i}{q^2 - M^2_{H_{\psi2}} + i\epsilon} \right] , \tag{2.146}$$

where as propagators we used for the neutrinos the first in (2.132), while the scalar propagator has the form:

$$\underset{H_{\psi k}\qquad\qquad H^*_{\psi k}}{\text{- - - - - - - - -}} = \frac{i}{q^2 - M^2_{H_{\psi k}} + i\epsilon} \quad ; \tag{2.147}$$

if the propagating Higgs field $H_{\psi k}$ in the loops is a Goldstone boson (so $M_{H_{\psi k}} = 0$) then its propagator is of the form:

$$\underset{H_{k}\qquad\qquad H^*_{k}}{\text{- - - - - - - - -}} = \frac{i}{q^2 - \xi M^2_{G_k} + i\epsilon} \tag{2.148}$$

where ξ is the gauge parameter and M_{G_k} is the mass of the gauge boson $G_{\mu k}$ that acquires mass by means of the Higgs mechanism "eating" the Goldstone boson H_k.

Because the loop integral is in $d^4q = \pi^2 q^2 d(q^2)$ and the integration extremes are $\pm\infty$, we can neglect any odd functions in the integrated momentum q. Then, considering that the neutrinos mass eigenstates are $m_1 = 0$ and $m_{2,3} \sim v_R$, we have that the contribution coming from the mass eigenstates ν_1 is zero because its propagator $\langle \nu_1 \bar{\nu}_1 \rangle = iq_\mu \gamma^\mu / q^2$ is an odd function, or null if of the type as in (2.132). Thus we can reduce the sum over the indices of the neutrino mass eigenstates in the mass formula:

$$m_H^{(\psi)} = k_L k_R \sin\theta_\psi \cos\theta_\psi \sum_{i=2}^{3} (O_\nu)^2_{3i} I_H^{(\psi)\,i}(0) \,. \tag{2.149}$$

To simplify the calculation we go to the Euclidean space, so we redefine the time-component of the four-momentum q_μ as $q_0 = iq_4$, obtaining the identity $q^2 = -q_E^2$ and $d^4q = id^4q_E$. The loop integral written in the Euclidean space then take the form:

$$\begin{aligned} I_H^{(\psi)\,i}(0) &= \int \frac{d^4q_E}{(2\pi)^4} \frac{m_i}{q_E^2 + m_i^2} \left[\frac{1}{q_E^2 + M^2_{H_{\psi2}}} - \frac{1}{q_E^2 - M^2_{H_{\psi1}}} \right] = \\ &= I_{H\,2}^{(\psi)\,i} - I_{H\,1}^{(\psi)\,i} \,. \end{aligned} \tag{2.150}$$

Using the dimensional regularization in dimension $D = 4 - \epsilon$, the integral can be evaluated by means of the formulas:

$$\int \frac{d^D p}{(2\pi)^D} \frac{(p^2)^\beta}{(p^2+m^2)^\alpha} = \frac{\Gamma(\beta+\frac{D}{2})\Gamma(\alpha-\beta-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}\Gamma(\alpha)\Gamma(\frac{D}{2})}(m^2)^{\frac{D}{2}-\alpha+\beta} \tag{2.151}$$

and

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[Ax+B(1-x)]^2}\,, \tag{2.152}$$

where $\Gamma(x)$ is the Eulero gamma function, which has the properties:

$$\begin{aligned} &\Gamma(n+1) = n\Gamma(n) = n! \qquad \text{for} \quad n \in \mathbb{N}\,; \\ &\Gamma(z) = \frac{1}{z} - \gamma_E + O(z) \qquad \text{for} \quad z \to 0\,; \end{aligned} \tag{2.153}$$

with $\gamma_E \approx 0.5772$ the Eulero-Mascheroni constant. We obtain the result:

$$\begin{aligned} I^{(\psi)\,i}_{H\,k} &= \mu^\epsilon \int \frac{d^D q}{(2\pi)^4} \frac{m_i}{(q^2+m_i^2)} \frac{1}{(q^2+M^2_{H_{\psi k}})} = \\ &= \mu^\epsilon \int_0^1 dx \int \frac{d^D q}{(2\pi)^4} \frac{m_i}{\left[q^2+m_i^2 x + M^2_{H_{\psi k}}(1-x)\right]^2} = \\ &= m_i \mu^\epsilon \int_0^1 dx \frac{\Gamma(\frac{D}{2})\Gamma(2-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}\Gamma(2)\Gamma(\frac{D}{2})} \left[m_i^2 x + M^2_{H_{\psi k}}(1-x)\right]^{\frac{D}{2}-2} = \\ &= m_i \int_0^1 dx \frac{\Gamma(\frac{\epsilon}{2})}{(4\pi)^{2-\frac{\epsilon}{2}}} \left[\frac{m_i^2 x + M^2_{H_{\psi k}}(1-x)}{\mu^2}\right]^{-\frac{\epsilon}{2}} \approx \\ &\approx \frac{m_i}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma_E + O(\epsilon)\right) \int_0^1 dx \left(1 - \frac{\epsilon}{2}\ln\left[\frac{m_i^2 x + M^2_{H_{\psi k}}(1-x)}{\mu^2}\right]\right) = \\ &= \frac{2m_i}{(4\pi)^2\epsilon} + O(1)\,, \end{aligned} \tag{2.154}$$

where the energy scale μ was introduced in order to mantain each term of the Lagrangian at dimension D in unit of mass. In fact, as we can see analyzing the Yukawa interaction term, in dimension $D = 4-\epsilon$ the Yukawa couplings, k_L and k_R, get a non-zero dimensionality:

$$D = [\bar{\psi}] + [\psi] + [H] + [k] = 2\left(\frac{D-1}{2}\right) + \frac{D-2}{2} + [k] \quad \Longrightarrow \quad [k] = \frac{4-D}{2} = \frac{\epsilon}{2}\,, \tag{2.155}$$

where the dimensionality of the fermionic and Higgs fields are obtained from their kinetic terms:

$$D = [\bar{\psi}] + [\partial_\mu] + [\psi] = 2[\psi] + 1 \qquad \text{and} \qquad D = [H] + 2[\partial_\mu] + [H^*] = 2[H] + 2\,. \quad (2.156)$$

It is possible to mantain k_L and k_R dimensionless, as in the case $D = 4$, redefining the couplings in the Yukawa interaction as:

$$\mathcal{L}^{(D=4)}_{Yukawa} = -k\bar{\psi}\psi H \quad \rightarrow \quad \mathcal{L}^{(D=4-\epsilon)}_{Yukawa} = -k\mu^{\frac{\epsilon}{2}}\bar{\psi}\psi H\,. \tag{2.157}$$

In the limit in which we remove the regularization parameter, that is $\epsilon \rightarrow 0$, the divergent part of the integral (2.150) is zero:

$$\left[I^{(\psi)\,i}_{H}(0)\right]_{div} = \left[I^{(\psi)\,i}_{H\,2} - I^{(\psi)\,i}_{H\,1}\right]_{div} = \frac{2m_i}{(4\pi)^2\epsilon} - \frac{2m_i}{(4\pi)^2\epsilon} = 0\,, \tag{2.158}$$

this means that the diagram in Fig. 2.4 results to be finite and the non-zero contribution of its loop integral is given by:

$$\begin{aligned} I^{(\psi)\,i}_{H}(0) &= \frac{m_i}{(4\pi)^2}\left(\frac{2}{\epsilon}\right)\int_0^1 dx\frac{\epsilon}{2}\ln\left[\frac{m_i^2 x + M^2_{H_{\psi 1}}(1-x)}{m_i^2 x + M^2_{H_{\psi 2}}(1-x)}\right] = \\ &= \frac{m_i}{(4\pi)^2}\left[\frac{M^2_{H_{\psi 1}}}{m_i^2 - M^2_{H_{\psi 1}}}\ln\left(\frac{m_i^2}{M^2_{H_{\psi 1}}}\right) - \frac{M^2_{H_{\psi 2}}}{m_i^2 - M^2_{H_{\psi 2}}}\ln\left(\frac{m_i^2}{M^2_{H_{\psi 2}}}\right)\right]\,, \end{aligned} \tag{2.159}$$

where we used the identity:

$$\begin{aligned} \int_0^1 dx \ln\left[(m^2 - M^2)x + M^2\right] &= \frac{1}{m^2 - M^2}\int_{M^2}^{m^2} dy\,\ln(y) = \\ &= \left[\frac{m^2\ln(m^2) - m^2 - M^2\ln(M^2) + M^2}{m^2 - M^2}\right]. \end{aligned} \tag{2.160}$$

2.6.2 Gauge loop

The second contribution to the radiative generation of the fermion masses that we are going to study, comes from loop diagrams containing a gauge boson exchange. We start this analysis considering the fact that the neutrinos are the only fermions that can propagate in a loop connecting two different chiralities of a fermion for which we want to generate a Dirac mass term; this is possible thanks to the sterile neutrino mass insertion which allows the spin flip. Consequently we need of a vector bosons which can propagate between a Left and a Right fermionic state and that is connected to the neutrino fields. These requests restrict the possibilities to loops with the exchange of the bosons Z_μ, Z'_μ, for a neutrino Dirac mass generation, and the leptoquarks $X^\pm_\mu$, for the up-type quarks. Moreover we can note that the sterile neutrino mass insertion (2.121) together with the neutral gauge bosons Z_μ and Z'_μ exchanges make it also possible to generate radiatively masses of Majorana type, but only for the neutrino fields ν_L and ν_R. Therefore the masses that will receive a contribution from the gauge loop are only those related to the up-type quarks and the neutrinos. All these types of radiative mass generation are depicted in Fig. 2.4.

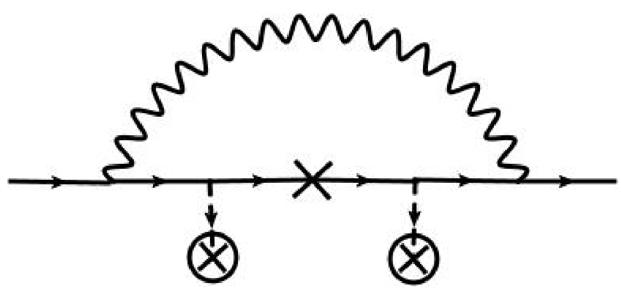


Figure 2.4: *Gauge loop for the radiative generation of the fermion mass term $-m_H^{(\psi)}\bar{\psi}_R\psi_L$, with Higgs VEV (represented by a circled cross) and sterile neutrino mass insertions (represented by a cross).*

This particular diagram is again finite because we have: four fermionic and one vector boson propagators, one sterile neutrino mass and two Higgs VEVs insertions, so the diagram is $O(m^3/p^6)$ as the previous one in Fig. 2.3. As we will see the diagram in Fig. 2.4 is able to generates fermion masses of Dirac and Majorana type. Let us concentrate our attention, for the moment, to the contribution given by this diagram to the mass of the fermionic field $\chi = u, \nu$ of the Dirac type only, which will be given by the formula:

$$m_G^{(\chi)D} = i\Big[\text{—}\bigcirc\text{—}\Big]_G^{(\chi)D} = \sum_{i=1}^{3} g_B \left(\frac{k_L v_L}{2\sqrt{2}}\right)(O_\nu)_{3i}(O_\nu^T)_{i3}\left(\frac{k_R v_R}{2\sqrt{2}}\right) g_A\, I_G^{(\chi)\,i}(p)\,, \qquad (2.161)$$

where g_A and g_B are the coupling constants of the gauge boson $G_\mu = X^\pm_\mu, Z'_\mu, Z_\mu$ with the fermion fields χ_R and χ_L. Using the definitions in Sec. 2.5, the mass expression in (2.161) can be rearranged rewriting the orthogonal matrix elements $(O_\nu)_{3i}$, used to convert the sterile neutrino field s_0 in the basis of mass eigenstates ν_i, and the VEV insertions, correspondig to the non-diagonal elements of the neutrino mass matrix (2.124), in the following form:

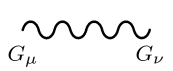
$$
\begin{aligned}
m_G^{(\chi)D} &= \sum_{i=1}^{3} g_A g_B (M_\nu)_{23} (O_\nu)^2_{3i} (M_\nu)_{31}\, I_G^{(\chi)\,i}(p) = \\
&= \sum_{i=1}^{3} g_A g_B \left[O_\nu M_\nu^{diag} O_\nu^T\right]_{2k} (O_\nu)^2_{ki} \left[O_\nu M_\nu^{diag} O_\nu^T\right]_{k1} I_G^{(\chi)\,i}(p) = \\
&= \sum_{i=1}^{3} g_A g_B (O_\nu)_{2j} (M_\nu^{diag})_{jj} (O_\nu^T O_\nu)_{ij} (O_\nu^T O_\nu)_{il} (M_\nu^{diag})_{ll} (O_\nu^T)_{l1}\, I_G^{(\chi)\,i}(p) = \\
&= \sum_{i=2}^{3} g_A g_B (O_\nu)_{2i} m_i^2 (O_\nu)_{i1}\, I_G^{(\chi)\,i}(p)\,,
\end{aligned}
\tag{2.162}
$$

with m_i the neutrino mass eigenvalues. Instead, for the loop integral, that we take again with a zero external momentum, we have:

$$
I_G^{(\chi)\,i}(p=0) = i \int \frac{d^4q}{(2\pi)^4} \gamma^\mu \frac{-iq_\alpha\gamma^\alpha}{q^2+i\epsilon} \frac{im_i}{q^2-m_i^2+i\epsilon} \frac{iq_\rho\gamma^\rho}{q^2+i\epsilon} \gamma^\nu \langle G_\mu^+ G_\nu^- \rangle\,, \tag{2.163}
$$

where the propagator for the generic massive vector boson G_μ is, as usual, of the form:

$$
\langle G_\mu G_\nu \rangle =
$$



$$
= \frac{-i}{q^2 - M_G^2} \left[g_{\mu\nu} + (\xi - 1) \frac{q_\mu q_\nu}{q^2 - \xi M_G^2} \right]\,, \tag{2.164}
$$

with ξ the gauge parameter and M_G the mass of the boson G_μ; instead for the two most external massless fermion propagators $\langle \nu_R \bar{\nu}_R \rangle$ and $\langle \nu_L \bar{\nu}_L \rangle$ we used respectively the first and the second expression in (2.131), while for the middle one we used the first in (2.132). As seen for the scalar loop, we can neglect the odd functions of integrated momentum and pass for simplicity from Minkowskian to Euclidean space:

$$
\begin{aligned}
I_G^{(\chi)\,i} &= \int \frac{d^4q}{(2\pi)^4} \frac{-m_i \gamma_\mu q^2 \gamma_\nu}{(q^2+i\epsilon)^2 (q^2-m_i^2+i\epsilon)} \frac{-i}{q^2-M_G^2} \left[g_{\mu\nu} + (\xi-1) \frac{q_\mu q_\nu}{q^2-\xi M_G^2} \right] = \\
&= \int \frac{d^4q}{(2\pi)^4} \frac{im_i}{(q^2+i\epsilon)^2 (q^2-m_i^2+i\epsilon)(q^2-M_G^2)} \left[\gamma_\mu q^2 \gamma^\mu + \frac{(\xi-1)q^4}{q^2-\xi M_G^2} \right] = \\
&= \int \frac{d^4q_E}{(2\pi)^4} \frac{-m_i}{(-q_E^2)(-q_E^2-m_i^2)(-q_E^2-M_G^2)} \left[\gamma_\mu\gamma_\mu + \frac{(\xi-1)q_E^2}{q_E^2+\xi M_G^2} \right] = \\
&= \int \frac{d^4q_E}{(2\pi)^4} \frac{m_i q_E^2 \gamma_\mu \gamma_\nu}{q_E^4 (q_E^2+m_i^2)} \langle G^\mu G^\nu \rangle_{Eucl}\,,
\end{aligned}
\tag{2.165}
$$

where the Euclidean propagator for a gauge boson G_μ is given by:

$$\langle G_\mu G_\nu \rangle_{Eucl} = \frac{1}{q^2+M_G^2}\left[\delta_{\mu\nu} + (\xi-1)\frac{q_\mu q_\nu}{q^2+\xi M_G^2}\right] . \tag{2.166}$$

In order to simplify the calculations let us use again the dimensional regularization, so that from the identities:

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^{1-x} dy \frac{1}{\left[Ax+By+C(1-x-y)\right]^3} \tag{2.167}$$

and $\gamma_\mu\gamma_\mu = D = 4-\varepsilon$, for the loop integral (2.165) we obtain the form:

$$\begin{aligned} I_G^{(\chi)\,i} &= \mu^\epsilon \int \frac{d^Dq}{(2\pi)^D}\frac{m_i\gamma_\mu\gamma_\nu}{q^2(q^2+m_i^2)}\frac{1}{q^2+M_G^2}\left[\delta_{\mu\nu}+(\xi-1)\frac{q_\mu q_\nu}{q^2+\xi M_G^2}\right] = \\ &= \mu^\epsilon \int \frac{d^Dq}{(2\pi)^D}\frac{m_i}{q^2+m_i^2}\left[\frac{D}{q^2(q^2+M_G^2)} + \frac{(\xi-1)}{(q^2+M_G^2)(q^2+\xi M_G^2)}\right] = \\ &= \frac{m_i\Gamma(3-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}}\mu^\epsilon \int_0^1 dx\int_0^{1-x} dy \left\{ D\left[Q(x,y)\right]^{\frac{D}{2}-3} + \left[Q(x,y)+\xi M_G^2 x\right]^{\frac{D}{2}-3}\right\} \end{aligned} \tag{2.168}$$

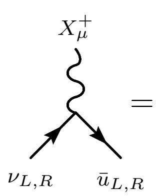
with $Q(x,y) = (m_i^2 - M_G^2)y + M_G^2(1-x)$. At the end we obtain the result:

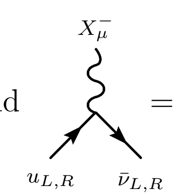
$$I_G^{(\chi)\,i} = \frac{1}{(4\pi)^2}\frac{m_i}{m_i^2-M_G^2}\left[\frac{(3+\xi)m_i^2 - 4\xi M_G^2}{m_i^2-\xi M_G^2}\ln\left(\frac{m_i^2}{M_G^2}\right) - \frac{m_i^2-M_G^2}{m_i^2-\xi M_G^2}\xi\ln(\xi)\right] . \tag{2.169}$$

The last element is to define the couplings g_A and g_B. Looking at the fermion kinetic terms, containing the covariant derivative:

$$i\Psi_{\mathbf{K}}^{i\alpha}(D_\mu)_{i\alpha}^{j\beta}\gamma^\mu\Psi_{\mathbf{K}j\beta} = i\Psi_{\mathbf{K}}^{i\alpha}\left[\delta_i^j\delta_\alpha^\beta\partial_\mu + ig_4\delta_i^j G_{\mu\alpha}^\beta + g_2\delta_\alpha^\beta W_{\mathbf{K}\mu i}^j\right]\gamma^\mu\Psi_{\mathbf{K}j\beta} \tag{2.170}$$

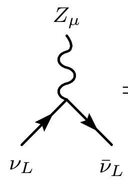
(with $\mathbf{K} = L, R$), we can determine the gauge interaction vertices needed in the diagram in Fig. 2.4. To be precise, the contribution to the up-type quark Dirac masses is generated by means of the vertices containing the leptoquarks:



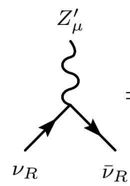
$= -\frac{i}{\sqrt{2}}g_2\gamma_\mu$ and


$= -\frac{i}{\sqrt{2}}g_2\gamma_\mu$, (2.171)

therefore in this case we have $g_A = g_B = g_2/\sqrt{2}$. While for the neutrinos we can have three different contributions:

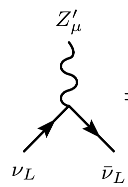


$$= -\frac{i}{2}\left(g_4\sqrt{\frac{3}{2}\cos 2\theta_W}\tan\theta_W + g_2\cos\theta_W\right)\gamma_\mu\,, \tag{2.172}$$



$$= \frac{i}{2}\left(g_4\sqrt{\frac{3}{2}}\tan\theta_W + \frac{\sqrt{\cos 2\theta_W}}{\cos\theta_W}\right)\gamma_\mu \tag{2.173}$$

and



$$= \frac{i}{2}\left(g_4\sqrt{\frac{3}{2}}\tan\theta_W\right)\gamma_\mu\,. \tag{2.174}$$

In fact, the propagation of the Z'_μ contributes to the neutrino mass generation of Dirac, Majorana-Left and Majorana-Right types, while the propagation of the Z_μ boson is related only to the Majorana-Left mass generation. So at the end, generalizing the loop calculation done for the Dirac case also to the Majorana masses, we can write the general formula for the mass contributions radiatively generated by the gauge-loop for the fields $\chi = u, \nu$ as:

$$m_G^{(\chi)} = g_A g_B \sum_{i=2}^{3}(O_\nu)_{xi}(O_\nu)_{yi} m_i^2\, I_G^{(\chi)\,i}\,, \tag{2.175}$$

where $x = 1$ and $y = 2$ (or $x = 2$ and $y = 1$) stands for the Dirac mass, $x = 1$ and $y = 1$ stands for the Majorana-Left mass and $x = 2$ and $y = 2$ for the Majorana-Right mass.

2.6.3 Summary of the one-loop formulas for radiative fermion masses

Let us recap all the formulae found above and explicit the contributions to the radiative mass generation for each fermion. To semplify the notation we write the masses only for one fermion family, those related to the other two families can be obtained with different Yukawa couplings and sterile neutrino mass.

Down-type quark sector

As we saw, for the down-type quark there are no gauge vertices with neutrinos (which are the only fermions that can propagate between states left and right thanks to a non-zero tree-level mass term) so a radiative mass generation by means of a gauge loop is not allowed. The down-type quark mass is therefore generated only by means of scalar loops:

$$\begin{aligned} m_d = m_H^{(d)} = \frac{-\lambda_{LR3} k_L v_L k_R v_R}{(4\pi)^2 \Delta_d} \sum_{i=2}^{3} (O_\nu)^2_{3i} m_i \Bigg[\frac{M^2_{H_d1}}{m_i^2 - M^2_{H_d1}} \ln\left(\frac{m_i^2}{M^2_{H_d1}}\right) - \\ - \frac{M^2_{H_d2}}{m_i^2 - M^2_{H_d2}} \ln\left(\frac{m_i^2}{M^2_{H_d2}}\right)\Bigg], \end{aligned} \tag{2.176}$$

where Δ_d is definited in (2.83).

Up-type quark sector

Concerning the hierarchy of mass between the up-type and the down-type quarks we note that for the up-type quark we have scalar loop contributions as for the down-type quark and, even if we have a different set of Higgs bosons, the masses of the Higgs related to the up and down sector are all of the same order $O(v_R)$, so the radiative contributions are quite similar. Although the shift of the masses in the first family (up/down) is small enough to be corrected at higher order in the loops, for the other two families (charm/strange and top/bottom) we look for a sensitive mass difference at one-loop. The model helps us because for the up-type quark sector we have also a contribution from gauge loops with leptoquarks exchanges:

$$\begin{aligned} m_u &= m_G^{(u)} + m_H^{(u)} = \\ &= \frac{g_4^2}{2(4\pi)^2} \sum_{i=2}^{3} \frac{(O_\nu)_{1i}(O_\nu)_{2i} m_i^3}{m_i^2 - \xi M_X^2} \left[\frac{(3+\xi) m_i^2 - 4\xi M_X^2}{m_i^2 - M_X^2} \ln\left(\frac{m_i^2}{M_X^2}\right) - \xi \ln(\xi)\right] + \\ &\quad + \frac{k_L v_L k_R v_R}{(4\pi)^2 (v_R^2 + v_L^2)} \sum_{i=2}^{3} (O_\nu)^2_{3i} m_i \Bigg[\frac{\xi M_X^2}{m_i^2 - \xi M_X^2} \ln\left(\frac{m_i^2}{\xi M_X^2}\right) - \\ &\quad - \frac{M^2_{H_u2}}{m_i^2 - M^2_{H_u2}} \ln\left(\frac{m_i^2}{M^2_{H_u2}}\right)\Bigg]. \end{aligned} \tag{2.177}$$

This difference in the loop contribution between up and down sector should explain their mass difference.

Charged lepton sector

Considering one-loop corrections, as evidenced in Sec. 2.6.1 for the charged lepton massess we can not have any loop corrections of the type in Fig. 2.3. Furthermore, although the charged leptons have gauge interactions together with neutrinos, unlike the down-type quarks, these vertices are related to the gauge bosons W_L^μ and W_R^μ that do not mix with each other; therefore we can not connect the left part of any charged lepton with the right ones in a gauge loop of the type in Fig. 2.4. So at one-loop the charged lepton masses stay at zero ($m_e = m_\mu = m_\tau = 0$) in this model.

Neutrino sector

The last step left is to give the explicit expressions for the neutrino mass matrix elements of (2.133). We start with the Dirac mass generated by the contributions of a gauge and a scalar loops:

$$\begin{aligned} m_\nu^D = & \frac{g_4 \mathbf{A}\sqrt{3}\tan\theta_W}{2\sqrt{2}(4\pi)^2} \sum_{i=2}^{3} \frac{(O_\nu)_{1i}(O_\nu)_{2i} m_i^3}{m_i^2 - \xi M_{Z'}^2} \left[\frac{(3+\xi)m_i^2 - 4\xi M_{Z'}^2}{m_i^2 - M_{Z'}^2} \ln\left(\frac{m_i^2}{M_{Z'}^2}\right) - \right. \\ & \left. - \xi \ln(\xi) \right] + \frac{k_L k_R}{(4\pi)^2} \cos\theta_\nu \sin\theta_\nu \sum_{i=2}^{3} (O_\nu)_{3i}^2 m_i \left[\frac{M_{H_\nu 1}^2}{m_i^2 - M_{H_\nu 1}^2} \ln\left(\frac{m_i^2}{M_{H_\nu 1}^2}\right) - \right. \\ & \left. - \frac{M_{H_\nu 2}^2}{m_i^2 - M_{H_\nu 2}^2} \ln\left(\frac{m_i^2}{M_{H_\nu 2}^2}\right) \right] . \end{aligned} \tag{2.178}$$

Then we have the Majorana mass for the left neutrino, obtained by the gauge loops with the propagation of the Z'_μ and Z bosons:

$$\begin{aligned} m_{\nu L}^M = & \frac{\mathbf{A}^2}{(4\pi)^2} \sum_{i=2}^{3} \frac{(O_\nu)_{1i}^2 m_i^3}{m_i^2 - \xi M_Z^2} \left[\frac{(3+\xi)m_i^2 - 4\xi M_Z^2}{m_i^2 - M_Z^2} \ln\left(\frac{m_i^2}{M_Z^2}\right) - \xi \ln(\xi) \right] + \\ & + \left(\frac{g_4\sqrt{3}\tan\theta_W}{2\sqrt{2}(4\pi)^2} \right)^2 \sum_{i=2}^{3} \frac{(O_\nu)_{1i}^2 m_i^3}{m_i^2 - \xi M_{Z'}^2} \left[\frac{(3+\xi)m_i^2 - 4\xi M_{Z'}^2}{m_i^2 - M_{Z'}^2} \ln\left(\frac{m_i^2}{M_{Z'}^2}\right) - \right. \\ & \left. - \xi \ln(\xi) \right] , \end{aligned} \tag{2.179}$$

with

$$\mathbf{A} = g_4 \sqrt{\frac{3}{2}} \frac{\tan\theta_W}{2} + \frac{g_2}{2} \frac{\sqrt{\cos 2\theta_W}}{\cos\theta_W} \,. \tag{2.180}$$

While the gauge loops with the exchange of the boson Z'_μ and Z give us the Majorana mass for the right neutrino:

$$m^M_{\nu R} = \frac{\mathbf{A}^2}{(4\pi)^2} \sum_{i=2}^{3} \frac{(O_\nu)^2_{2i} m_i^3}{m_i^2 - \xi M^2_{Z'}} \left[\frac{(3+\xi) m_i^2 - 4\xi M^2_{Z'}}{m_i^2 - M^2_{Z'}} \ln\left(\frac{m_i^2}{M^2_{Z'}} \right) - \xi \ln(\xi) \right] . \tag{2.181}$$

2.7 Numerical results

Although we know that the model, exposed till now, does not admit any mass for the charged leptons, it is time to test the results this model can produce. Obviously we are interested to see if the formulas above are able to generate as faithfully as possible the other fermion masses of the Standard Model. In particular we try to reproduce the fermion mass values at the Pati-Salam scale $M_U \sim 10^{14}$ GeV. The experimental values that we use as reference are listed in [18], but because these values are fermion masses runned from the electroweak scale up to the scales $\mu = 10^{12}$ GeV and $\mu = 10^{16}$ GeV, we will consider the mean values of the masses between these two scales:

$$\begin{aligned}
m_u &= 0.53^{+0.16}_{-0.13}\,\text{MeV} \\
m_d &= 1.24^{+0.39}_{-0.37}\,\text{MeV} \\
m_s &= 24.00^{+5.32}_{-4.61}\,\text{MeV} \\
m_c &= 258^{+27.34}_{-26.63}\,\text{MeV} \\
m_b &= 1.105 \pm 0.032\,\text{GeV} \\
m_t &= 80.35^{+28.28}_{-26.52}\,\text{GeV} \\
m_e &= 0.480254 \pm 0.048025\,\text{MeV} \\
m_\mu &= 101.385 \pm 10.138\,\text{MeV} \\
m_\tau &= 1.723 \pm 0.17\,\text{GeV}
\end{aligned} \tag{2.182}$$

for the sake of completenees, we also list the masses of the charged leptons that will be used in Sec. 2.9. For the standard deviation related to the lepton sector we took the 10% of the mass value, while for those related to the quark sector we used the propagation error formula:

$$\sigma(\mu \approx 10^{14}\text{GeV}) = \frac{1}{2}\sqrt{\sigma^2(\mu = 10^{12}\text{GeV}) + \sigma^2(\mu = 10^{16}\text{GeV})}\,. \tag{2.183}$$

For the neutrino sector, instead, we consider the squared mass ratio in normal hierarchy defined as:

$$r = \frac{\Delta m^2_{21}}{\Delta m^2_{32}} = \frac{m_2^2 - m_1^2}{m_3^2 - m_2^2} \quad , \tag{2.184}$$

where these neutrino masses ($m_1 < m_2 < m_3$) are the lightest among the nine mass eigenstates generated by the model, three for each of the three fermion family. In inverted hierarchy, instead, the ratio is defined as:

$$r_{inv} = \frac{\Delta m^2_{21}}{\Delta m^2_{13}} = \frac{m_2^2 - m_1^2}{m_1^2 - m_3^2} \quad . \tag{2.185}$$

As reference we use the value at $\mu = 10^{16}$ GeV [22]:

$$r \simeq 0.031 \pm 0.001\,; \tag{2.186}$$

since its experimental value, in normal hierarchy, varies little during the running, as we can evaluate taking in account the values of r at different energy scales [18]:

$$r \simeq \begin{cases} 0.0320 & \text{at } \mu = M_Z \\ 0.0314 & \text{at } \mu = 1 \text{ TeV} \\ 0.0309 & \text{at } \mu = 10^9 \text{ GeV} \\ 0.0330 & \text{at } \mu = 10^{12} \text{ GeV} \end{cases} \quad . \tag{2.187}$$

In order to find, if it exists, a parameter set which is able to generate a correct mass spectrum for the fermions, as a first approach we built a code with the Mathematica software to scan the parameter space, just to test the model which we already know to be not complete yet (no mass for the charged leptons). The parameters taken in account are the λ_i of the scalar potential, the Yukawa couplings λ_i, k_{La} and k_{Ra}, k_L and k_R, and a parameter m_a related to the sterile neutrino mass through the identity $m_{0a} = m_a v_R$, with $a = 1, 2, 3$ the fermion family index. We have to specify that the parameter scan code is a random scan code, in the sense that, at each cycle, the parameters λ_i, k_{La}, k_{Ra} and m_a take randomly a real value (because we do not consider CP violation) in the interval

$[-3.5, 3.5]$; for the parameter m_a, instead, we consider the same interval but the values are extracted to a Gaussian distribution with mean $\mu = 1$ and $\sigma = 0.5$. The interval $[-3.5, 3.5]$ is chosen imposing, for the Yukawa and Higgs couplings, the following constraints for the reliability of perturbative calculations (as in Ref. [2]):

$$\frac{k^2}{4\pi} < 1 \qquad \text{and} \qquad \frac{\lambda}{2} v_R^2 \sim M_H^2 < 2 v_R^2 \,. \tag{2.188}$$

The only parameters retained at the end of the simulations are those that respect both conditions below:

1. the parameters λ_i have to respect the system (2.89);

2. the resulting masses have to be in the ranges:

$$\begin{cases} |m_{up}| \leq 1 \text{ MeV} \\ |m_{down}| \leq 1.6 \text{ MeV} \\ |m_{strange}| \leq 30 \text{ MeV} \\ 125 \text{ MeV} \leq |m_{charm}| \leq 350 \text{ MeV} \\ 750 \text{ MeV} \leq |m_{bottom}| \leq 1.5 \text{ GeV} \end{cases} . \tag{2.189}$$

Using a random scan code, the probability to find the exact parameters which generate the experimental values in (2.182) is very small, so we take in account the mass intervals (2.189) containing the experimental masses in (2.182) within the $2\sigma \div 5\sigma$ range, a part for the bottom quark for which we left a larger interval since it belongs to the same family of the top quark, for which we have not been able, at this stage, to generate a realistic mass. In fact, we let note that in (2.189) we did not consider the top quark because, also after many trials, the simulations containing also the condition over the top mass did not able to produce any result. This is, however, in agreement with what is stated in Ref. [2].

Before displaying some examples of the obtained results, let us quote some numbers that we used as constants in the code. The first are the VEVs related the left-handed Higgs multiplet $L_{i\alpha}$ that we let it to coincide with the one in the Standard Model, and that related to the right-handed Higgs multiplet $R_{i\alpha}$, that we assume of the order of the Pati-Salam scale M_U found in Sec. 2.4:

$$v_L = 246 \, \text{GeV} \qquad \text{and} \qquad v_R = 10^{14} \, \text{GeV} \,. \tag{2.190}$$

From the formulas in Sec. 2.4 we can also extrapolate the values of the gauge couplings for the groups $SU(2)_{L,R}$ and $SU(4)$ at the scale M_U, which are respectively:

$$g_2 = 0.5388 \qquad \text{and} \qquad g_4 = 0.5694\,. \tag{2.191}$$

With the quantities v_L, v_R, g_4 and g_2 we can evaluate the gauge bosons masses, defined in Sec. 2.3, finding for the new gauge bosons:

$$\begin{gathered} M_{Z'}(M_U) = 4.4 \cdot 10^{13}\,\text{GeV}\,, \quad M_{W_R}(M_U) = 2.6 \cdot 10^{13}\,\text{GeV} \\ \text{and} \quad M_X = 2.8 \cdot 10^{13}\,\text{GeV}\,, \end{gathered} \tag{2.192}$$

while for those present also in the SM we find:

$$M_Z(M_U) = 84.5\,\text{GeV} \qquad \text{and} \qquad M_{W_L}(M_U) = 66.2\,\text{GeV}\,. \tag{2.193}$$

Furthermore we have for the "pseudo-Weinberg" angle defined in (2.105):

$$\sin\theta'_W = 0.62\,. \tag{2.194}$$

All the calculation are made in Feynamn gauge ($\xi = 1$).

The scan code we used is built up in the following way: it performs a total of 10000 cycles over the scalar parameters in order to find the parameters λ_i which verify the conditions (2.89); when the program finds a good set of λ_is it starts to span the parameter space for k_L, k_R and m_a, at most for 1000 times for each fermion family before to exit from the cycle, in order to find the set of Yukawa couplings and the sterile neutrino masses which are able to generate some quark masses inside (2.189). With the values of the VEVs, the gauge couplings and the gauge boson masses discussed above we obtained a total of 20 results depicted in Fig. 2.5 and Fig. 2.6, where we do not see any evident correlation among the quark masses. Among these we take only two cases as examples, choosing the results with the lowest χ^2 value:

$$\chi^2 = \sum_q \frac{\left(m_q^{(res)} - m_q^{(exp)}\right)^2}{m_q^{(exp)}}\,, \tag{2.195}$$

with $m_q^{(exp)}$ the experimental quark masses in (2.182) and $m_q^{(res)}$ the related values obtained from the simulations. For the first example we consider only the first family, therefore in (2.195) we have $q = u, d$; while in the second example we consider also the second family, so $q = u, d, s, c$. In the following we list the masses, obtained from the simulations, for the first two quark families with the related χ^2 values (in bold the lowest).

$m_u^{(res)}$ (MeV)	$m_d^{(res)}$ (MeV)	$m_s^{(res)}$ (MeV)	$m_c^{(res)}$ (MeV)	χ^2 $(q = u, d)$	χ^2 $(q = u, d, s, c)$
0.18	0.04	28.94	132.32	1.39	63.62
0.1	0.01	24.36	228.19	1.57	**5.02**
0.21	0.04	22.52	140.79	1.36	54.70
0.32	1.13	28.92	141.14	**0.09**	54.03
0.12	1.32	28.18	133.87	0.32	60.77
0.77	1.08	17.33	176.04	0.13	28.02
0.46	0.06	17.93	126.30	1.12	69.88
0.79	1.45	29.48	128.19	0.17	66.73
0.90	0.12	29.47	206.83	1.27	12.66
0.25	0.04	25.91	217.55	1.31	7.80
0.54	0.05	16.82	189.92	1.14	21.26
0.40	0.10	28.39	141.51	1.08	54.48
0.79	0.18	24.59	158.22	1.03	39.64
0.75	0.10	25.88	218.70	1.14	7.28
0.36	0.08	27.86	136.84	1.14	58.66
0.87	0.14	21.56	128.58	1.19	66.36
0.01	0.002	24.42	140.95	1.75	54.86
0.19	0.01	9.20	155.06	1.43	51.63
0.81	0.12	23.56	154.40	1.15	42.76
0.30	0.06	27.94	128.33	1.22	67.04

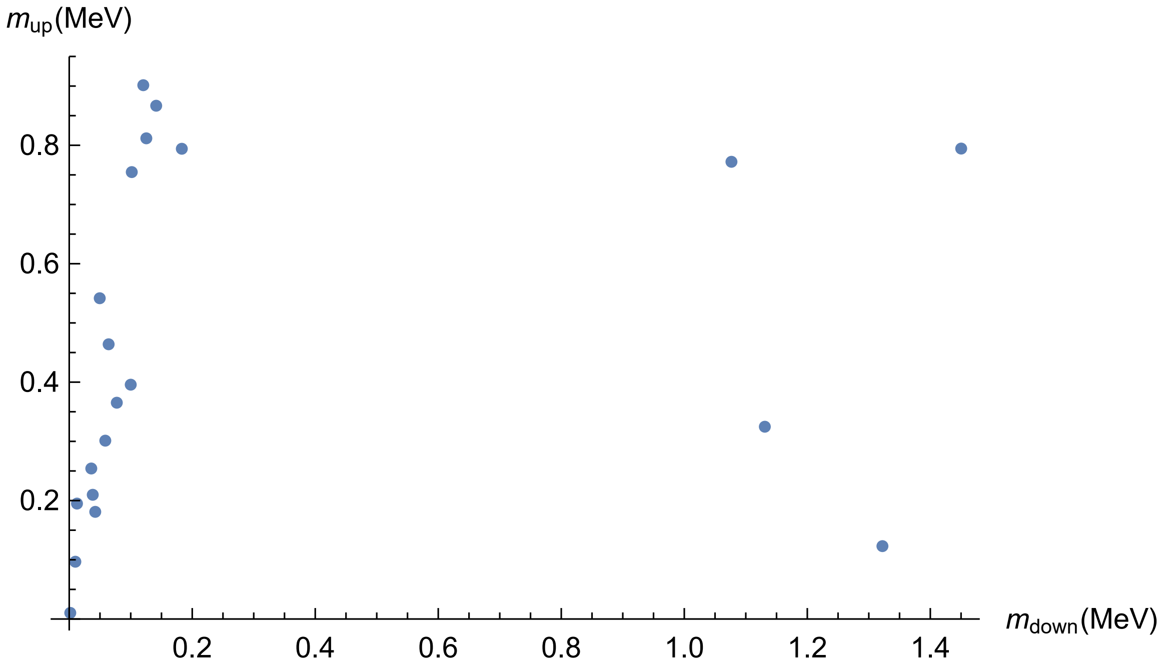


Figure 2.5: *Correlation among the up (vertical axis) and down (horizontal axis) quark masses.*

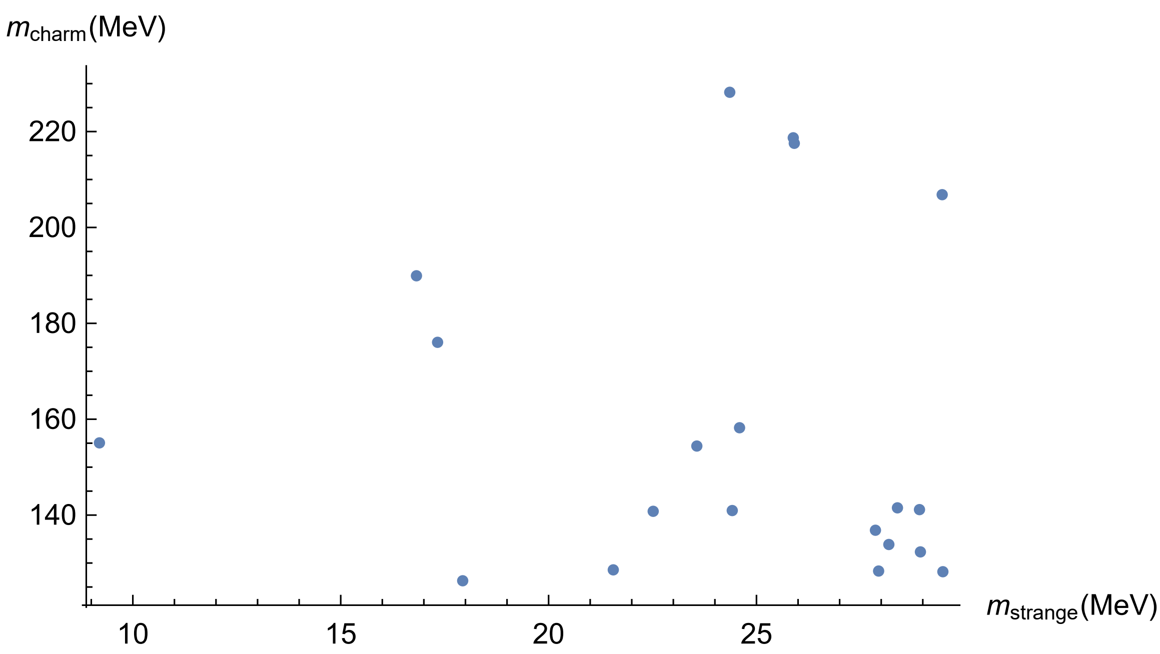


Figure 2.6: *Correlation among the charm (vertical axis) and strange (horizontal axis) quark masses.*

Example 1

In this first example, for the parameters of the scalar potential, we have the values:

$$\begin{cases} \lambda_{LR1} = 2.79 \\ \lambda_{LR2} = -2.89 \\ \lambda_{LR3} = -0.33 \\ \lambda_{L1} = 1.32 \\ \lambda_{L2} = 0.72 \\ \lambda_{R1} = 1.99 \\ \lambda_{R2} = -0.35 \end{cases} , \tag{2.196}$$

For the charged Higgs related to the down-type quark sector we find:

$$M_{H_{d1}} = 0.42 \cdot 10^{14}\,\text{GeV} \qquad \text{and} \qquad M_{H_{d2}} = 0.85 \cdot 10^{14}\,\text{GeV}\,, \tag{2.197}$$

with the mixing:

$$\cos\theta_d \sin\theta_d = 0.74 \cdot 10^{-12}\,, \tag{2.198}$$

as expected from the theory of the order $O(v_L/v_R)$ (see Sec. 2.2). For the Higgs related the up-type quark sector, instead, we have:

$$M_{H_{u2}} = 1.2 \cdot 10^{14}\,\text{GeV}\,, \tag{2.199}$$

while the Goldstone boson H_{u1} has been replaced in the loops by the leptoquark mass. The mixing between H_{u1} and H_{u2} is:

$$\cos\theta_u \sin\theta_u = \frac{v_L v_R}{\sqrt{v_L^2 + v_R^2}} = 2.46 \cdot 10^{-12}\,. \tag{2.200}$$

To complete the spectrum we have the Higgs linked to the neutrinos that get the masses:

$$M_{H_{\nu1}} = 1.28 \cdot 10^{14}\,\text{GeV} \qquad \text{and} \qquad M_{H_{\nu2}} = 350.92\,\text{GeV}\,, \tag{2.201}$$

with the mixing:

$$\cos\theta_\nu \sin\theta_\nu = 0.76 \cdot 10^{-13}\,. \tag{2.202}$$

We have to note that, while $M_{H_{\nu 1}}$ is of the order of the Pati-Salam scale, $M_{H_{\nu 2}}$ is proportional to v_L as expected, even if the value is bigger than the experimental mass of the Standard Model Higgs. To conclude this example, it remains to see which masses can be generated for quarks and neutrinos given the above spectra of bosons. With the following extracted values for the Yukawa couplings:

$$k_L = 0.09 \qquad \text{and} \qquad k_R = 0.23\,, \tag{2.203}$$

and for the sterile neutrino mass:

$$m_0 = 4.11 \cdot 10^{12}\,\text{GeV}\,, \tag{2.204}$$

the model generates the masses for the first fermion family:

$$m_{up} = 0.32\,\text{MeV} \qquad \text{and} \qquad m_{down} = 1.13\,\text{MeV}\,. \tag{2.205}$$

Instead for the second family from our numerical procedure we get the following values of the Yukawa parameters:

$$k_L = -0.12 \qquad \text{and} \qquad k_R = 1.24\,, \tag{2.206}$$

and for the second sterile neutrino mass:

$$m_0 = 0.68 \cdot 10^{14}\,\text{GeV}\,; \tag{2.207}$$

this choice produces the masses:

$$m_{charm} = 141.14\,\text{MeV} \quad \text{and} \quad m_{strange} = 28.92\,\text{MeV}\,. \tag{2.208}$$

We can see that for the first two families the values of the masses generated by the model are within the 2σ range compared to the experimental values (2.182). So it seems that the model works fairly well, even though at this stage we know that the charged lepton masses cannot be generated at all. However, for the third family the model has a big problem. In fact, with the choice for the Yukawa couplings:

$$k_L = 2.62 \qquad \text{and} \qquad k_R = 1.79\,, \tag{2.209}$$

and for the last sterile neutrino mass:

$$m_0 = 0.91 \cdot 10^{14}\,\text{GeV}\,, \tag{2.210}$$

we find the masses:

$$m_{top} = 5127.19\,\text{MeV} \qquad \text{and} \qquad m_{bottom} = 983.60\,\text{MeV}\,. \tag{2.211}$$

We can note that, although the result for the bottom mass is not as good as the results for the first and second families (it is within the 4σ range from the experimental values), the top mass is completely unrealistic. We will see later that there is the possibility to alleviate this problem, together with the mass generation for the charged leptons, extending the model with new Higgs bosons. The remaining observables that we have to consider is the result related to the neutrinos. For each family ($\alpha = e, \mu, \tau$) we have three mass eigenstates for the neutrino mass matrix (2.133): $m_{\alpha 1}$ and $m_{\alpha 2}$ are of the order of v_R, while $m_{\alpha 3}$ is much lighter. Taking the lightest mass for each family, and defining in this case:

$$m_1 = m_{\mu 3}\,, \quad m_2 = m_{e3} \quad \text{and} \quad m_3 = m_{\tau 3}\,, \tag{2.212}$$

we obtain the ratio:

$$r = 0.8 \cdot 10^{-4}\,. \tag{2.213}$$

We can note that the result for r is not good compared to the experimental value (2.186). However we want to emphasize that such a small number is the consequence of the Mathematica machine precision which is not always able to handle with the huge difference between the heavy and light neutrino eigenstates ($\geq 10^{14}$), sometimes treating as zero very small numbers.

Example 2

In this second example the new set of scalar parameters is:

$$\begin{cases} \lambda_{LR1} = -3.13 \\ \lambda_{LR2} = -2.40 \\ \lambda_{LR3} = -0.18 \\ \lambda_{L1} = 3.34 \\ \lambda_{L2} = 2.32 \\ \lambda_{R1} = 2.50 \\ \lambda_{R2} = -0.80 \end{cases} . \tag{2.214}$$

These values produce the following down-type Higgs masses:

$$M_{H_{d1}} = 0.63 \cdot 10^{14}\,\text{GeV} \qquad \text{and} \qquad M_{H_{d2}} = 0.77 \cdot 10^{14}\,\text{GeV} \tag{2.215}$$

with the mixing:

$$\cos\theta_d \sin\theta_d = 1.11 \cdot 10^{-12}\,; \tag{2.216}$$

while the massive up-type Higgs receives the mass:

$$M_{H_{u2}} = 1.09 \cdot 10^{14}\,\text{GeV}\,. \tag{2.217}$$

To complete the scalar spectrum there are the Higgs related to the neutrinos, that take the masses:

$$M_{H_{\nu1}} = 1.30 \cdot 10^{14}\,\text{GeV} \qquad \text{and} \qquad M_{H_{\nu2}} = 267.16\,\text{GeV}\,, \tag{2.218}$$

where we see again that the mass for the Higgs corresponding to the SM one is a bit too large compared to the experimental values; their mixing is given by the value:

$$\cos\theta_\nu \sin\theta_\nu = 3.99 \cdot 10^{-12}\,. \tag{2.219}$$

Regarding the fermion masses we have that with the Yukawa couplings:

$$k_L = 0.01\,, \quad k_R = 0.01\,, \tag{2.220}$$

and a sterile neutrino mass:

$$m_0 = 1.49 \cdot 10^{14}\,\mathrm{GeV}\,, \tag{2.221}$$

as inputs, we generate for the first family:

$$m_{up} = 0.1\,\mathrm{MeV} \quad \text{and} \quad m_{down} = 0.01\,\mathrm{MeV}\,; \tag{2.222}$$

they are small with respect to the experimental values, however inside 3.5σ. Much better is the situation for the second family, for which taking:

$$k_L = 0.12\,, \quad k_R = -1.87\,, \tag{2.223}$$

and

$$m_0 = 1.17 \cdot 10^{14}\,\mathrm{GeV}\,, \tag{2.224}$$

we obtain the results:

$$m_{charm} = 228.19\,\mathrm{MeV} \quad \text{and} \quad m_{strange} = 24.36\,\mathrm{MeV}\,, \tag{2.225}$$

within the 1.5σ range from the experimental values. For the third family, instead, we can find again the same problem discussed above; in fact using the Yukawa couplings:

$$k_L = 3.18\,, \quad k_R = 3.37\,, \tag{2.226}$$

and the sterile neutrino mass:

$$m_0 = 1.69 \cdot 10^{14}\,\mathrm{GeV}\,, \tag{2.227}$$

we get the quark masses:

$$m_{top} = 11069.76\,\mathrm{MeV} \quad \text{and} \quad m_{bottom} = 1132.90\,\mathrm{MeV}\,, \tag{2.228}$$

where the bottom mass is within the 1σ range from the experimental value, while the top mass is again completely unrealistic, although a little better than before. We complete this second example with the results for neutrinos for which we encountered the same problems as the previous example. Assuming:

$$m_1 = m_{e3}\,, \quad m_2 = m_{\mu 3} \quad \text{and} \quad m_3 = m_{\tau 3}\,, \tag{2.229}$$

we find the result:

$$r = 1.43 \cdot 10^{-5}\,, \tag{2.230}$$

again a too small value.

2.8 Mixing

Before discussing the modifications of the minimal model needed to solve the problems related to the charged leptons mass generations and the too low top quark mass, in this section we want to introduce the inter-family mixing. In order to do that we have to promote the Yukawa couplings to 3×3 matrices in the Lagrangian (2.11), so the Yukawa interaction terms become of the form:

$$\mathcal{L}_{Yukawa} = -k_L^{ab}\Big[\bar{\Psi}_{La}^{i\alpha} L_{i\alpha} s_{0b} + h.c.\Big] - k_R^{ab}\Big[\bar{\Psi}_{Ra}^{i\alpha} R_{i\alpha} s_{0b}^c + h.c.\Big] \tag{2.231}$$

where a and b are the fermion family indices. We again disregard for the moment any source of CP violation, therefore the Yukawa matrices will be taken real, as well as for the VEVs. For the sterile neutrinos, instead, we continue to demand that there is no explicit lepton flavour violation, therefore the sterile neutrino masses will be represented by a diagonal matrix:

$$\bar{s}_{0a} M_0^{ab} s_{0b}^c = \begin{pmatrix} \bar{s}_{0e} & \bar{s}_{0\mu} & \bar{s}_{0\tau} \end{pmatrix} \begin{pmatrix} m_{0e} & 0 & 0 \\ 0 & m_{0\mu} & 0 \\ 0 & 0 & m_{0\tau} \end{pmatrix} \begin{pmatrix} s_{0e}^c \\ s_{0\mu}^c \\ s_{0\tau}^c \end{pmatrix}. \tag{2.232}$$

Once we introduced the inter-family mixing, the tree-level neutrino mass matrix M_ν, as well as the orthogonal matrix O_ν, will be 9×9 matrices; in the rest of this thesis we use the following neutrino basis:

$$\Psi_\nu = \begin{pmatrix} \nu_{Le} & \nu_{L\mu} & \nu_{L\tau} & \nu_{Re}^c & \nu_{R\mu}^c & \nu_{R\tau}^c & s_{0e}^c & s_{0\mu}^c & s_{0\tau}^c \end{pmatrix}^T. \tag{2.233}$$

Consequently the elements of the one-loop neutrino Dirac mass matrix (2.133) get the forms:

$$m_\nu^D = \begin{pmatrix} (m_\nu^D)_{ee} & (m_\nu^D)_{e\mu} & (m_\nu^D)_{e\tau} \\ (m_\nu^D)_{\mu e} & (m_\nu^D)_{\mu\mu} & (m_\nu^D)_{\mu\tau} \\ (m_\nu^D)_{\tau e} & (m_\nu^D)_{\tau\mu} & (m_\nu^D)_{\tau\tau} \end{pmatrix}, \tag{2.234}$$

while the Majorana Left and Right masses are given by:

$$m_{\nu L,R}^M = \begin{pmatrix} (m_{\nu L,R}^M)_{ee} & \frac{(m_{\nu L,R}^M)_{e\mu}}{2} & (m_{\nu L,R}^M)_{e\tau} \\ \frac{(m_{\nu L,R}^M)_{e\mu}}{2} & (m_{\nu L,R}^M)_{\mu\mu} & \frac{(m_{\nu L,R}^M)_{\mu\tau}}{2} \\ \frac{(m_{\nu L,R}^M)_{e\tau}}{2} & \frac{(m_{\nu L,R}^M)_{\mu\tau}}{2} & (m_{\nu L,R}^M)_{\tau\tau} \end{pmatrix}. \tag{2.235}$$

For the fermion mass term $m\bar{\psi}_R\psi_L$, the parameter m is promoted to a 3×3 matrix:

$$\bar{\psi}_R^a m_\psi^{ab}\psi_L^b = \bar{\psi}_R^a[m_H^{(\psi)ab} + m_G^{(\psi)ab}]\psi_L^b\,, \tag{2.236}$$

where the radiatively generated masses $m_H^{(\psi)ab}$ and $m_G^{(\psi)ab}$ are obtained generalizing the formulas in Sec. 2.6.1 and 2.6.2 to the case of inter-family mixing. In particular from the scalar loop we obtain:

$$m_H^{(\psi)ab} = \sum_{m,n=1}^{3} k_R^{am}k_L^{bn}\sin\theta_\psi\cos\theta_\psi\sum_{i=1}^{9}(O_\nu)_{m+6,i}(O_\nu)_{n+6,i}I_H^{(\psi)\,i}\,, \tag{2.237}$$

where $I_H^{(\psi)\,i}$ is given by (2.159); while from the gauge loop we have:

$$m_G^{(\psi)ab} = g_A g_B\sum_{i=1}^{9}(O_\nu)_{xi}(O_\nu)_{yi}m_i^2\,I_G^{(\psi)\,i}\,, \tag{2.238}$$

where $I_G^{(\psi)\,i}$ is given by (2.169). Again, for the gauge loop we can distinguish three different cases: with $x=a$ and $y=b+3$ we generate a Dirac mass; with $x=a$ and $y=b$ we get a Majorana Left mass; while for a Majorana Right mass we take $x=a+3$ and $y=b+3$. We note that the particular form of the indices of O_ν is due to the order of the neutrinos in the basis (2.233). We remember that in (2.237) and (2.238) in the cases with $m_i=0$ the contributions are zero because, as we saw before, the loop-integral calculation gives us a null result.

2.8.1 CKM and PMNS matrices

We now remind how the CKM matrix is defined, remembering that, since we consider real Yukawa couplings, the entries of the mixing matrices are real numbers. Let us start from the quark fields q_{ui} of the up-type sector, with $i = 1, 2, 3$ the family index. The mass term (2.236) can be written in the mass eigenstates basis performing the following rotations:

$$\begin{aligned}\begin{pmatrix}\bar{q}_{u1} & \bar{q}_{u2} & \bar{q}_{u3}\end{pmatrix}^{\alpha}_{R} M_u \begin{pmatrix} q_{u1} \\ q_{u2} \\ q_{u3}\end{pmatrix}^{\alpha}_{L} &= \left[\begin{pmatrix}\bar{q}_{u1} & \bar{q}_{u2} & \bar{q}_{u3}\end{pmatrix}^{\alpha}_{R} C\right] C^{\dagger} M_u A \left[A^{\dagger}\begin{pmatrix} q_{u1} \\ q_{u2} \\ q_{u3}\end{pmatrix}^{\alpha}_{L}\right] = \\ &= \begin{pmatrix}\bar{u} & \bar{c} & \bar{t}\end{pmatrix}^{\alpha}_{R}\left[C^{\dagger} M_u A\right]\begin{pmatrix} u \\ c \\ t\end{pmatrix}^{\alpha}_{L},\end{aligned} \tag{2.239}$$

where α is the colour index of $SU(3)$, while A and C are unitary matrices ($UU^{\dagger} = U^{\dagger}U = \mathbb{I}$) defined in such a way that:

$$A^{\dagger} M_u^{\dagger} M_u A = |M_u^{diag}|^2 = C^{\dagger} M_u M_u^{\dagger} C\,, \tag{2.240}$$

with the diagonal mass matrix:

$$M_u^{diag} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t\end{pmatrix}, \tag{2.241}$$

containing the mass eigenvalues for the up, charm and top quarks. The same rotation, from the interaction basis to the mass eigenbasis, can be applied also to the quark fields q_{di} for the down-type sector:

$$\begin{aligned}\begin{pmatrix}\bar{q}_{d1} & \bar{q}_{d2} & \bar{q}_{d3}\end{pmatrix}^{\alpha}_{R} M_d \begin{pmatrix} q_{d1} \\ q_{d2} \\ q_{d3}\end{pmatrix}^{\alpha}_{L} &= \left[\begin{pmatrix}\bar{q}_{d1} & \bar{q}_{d2} & \bar{q}_{d3}\end{pmatrix}^{\alpha}_{R} D\right] D^{\dagger} M_d B \left[B^{\dagger}\begin{pmatrix} q_{d1} \\ q_{d2} \\ q_{d3}\end{pmatrix}^{\alpha}_{L}\right] = \\ &= \begin{pmatrix}\bar{d} & \bar{s} & \bar{b}\end{pmatrix}^{\alpha}_{R}\left[D^{\dagger} M_d B\right]\begin{pmatrix} d \\ s \\ b\end{pmatrix}^{\alpha}_{L},\end{aligned} \tag{2.242}$$

where in this case the unitary matrices B and D are defined demanding:

$$B^\dagger M_d^\dagger M_d B = |M_d^{diag}|^2 = D^\dagger M_d M_d^\dagger D\,, \tag{2.243}$$

with the diagonal mass matrix given by:

$$M_d^{diag} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}\,, \tag{2.244}$$

containig the mass eigenvalues for the down, strange and bottom quarks. So at the end we define the CKM matrix as:

$$V_{CKM} = A^\dagger B = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{2.245}$$

where the absolute value $|V_{ij}|^2$ gives the transition probability from a quark of type i to a quark of type j. For the experimental values we consider the running in [18, 22] of the following CKM elements from the electro-weak scale $\mu = M_Z$:

$$\begin{aligned} |V_{us}| &= 0.2257 \pm 0.0021\,; \\ |V_{cb}| &= 0.0416 \pm 0.006\,; \\ |V_{ub}| &= 0.00431 \pm 0.0003\,; \end{aligned} \tag{2.246}$$

to the GUT scale $\mu = 2 \cdot 10^{16}$ GeV:

$$\begin{aligned} |V_{us}| &= 0.2254 \pm 0.0006\,; \\ |V_{cb}| &= 0.04194 \pm 0.0006\,; \\ |V_{ub}| &= 0.00369 \pm 0.00013\,; \end{aligned} \tag{2.247}$$

which we use as a reference in our numerical simulations.
In a similar way one can define the PMNS matrix for the leptonic sector. As for the CKM matrix, we need two unitary matrix U_l and U_ν that transform the interaction basis in the

mass eigenstates basis, respectively for the charged leptons and the neutrinos:

$$\begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_L = U_l \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}_L \qquad \text{and} \qquad \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}_L = U_\nu \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_L . \tag{2.248}$$

As for the quarks case in (2.240) and (2.243), from a general 3×3 mass matrix M_l for the charged leptons we can obtain the mass eigenvalues using the unitary matrix U_l in the following way:

$$U_l^\dagger M_l^\dagger M_l U_l = \begin{pmatrix} |m_e|^2 & 0 & 0 \\ 0 & |m_\mu|^2 & 0 \\ 0 & 0 & |m_\tau|^2 \end{pmatrix} . \tag{2.249}$$

We point out that, although in the Standard Model the rotation matrix U_l coincides with the identiy because there is no lepton-flavour violation, in a general model it is not so trivial, in fact the general PMNS matrix is defined as:

$$U_{PMNS} = U_l^\dagger U_\nu = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} . \tag{2.250}$$

However, because at this stage our model does not generate mass for the charged leptons, we do not have any problem to define the matrix U_l which coincides with the identity matrix, and so the PMNS matrix will be given only by the matrix U_ν that we must define carefully. Concerning the neutrino sector, we have to take in account that in our model the SM neutrinos (ν_{Le}, $\nu_{L\mu}$ and $\nu_{L\tau}$) are collected in the basis Ψ_ν and the transformation from this to the mass eigenstates basis is generated by:

$$\Psi_\nu = O_\nu \begin{pmatrix} \nu_9 & \nu_8 & \nu_7 & \nu_6 & \nu_5 & \nu_4 & \nu_3 & \nu_2 & \nu_1 \end{pmatrix}^T , \tag{2.251}$$

where the mass eigenstates ν_i (with $i = 1, \ldots 9$) are taken in increasing order of absolute value of mass: $|m_{\nu 9}| > |m_{\nu 8}| > \cdots > |m_{\nu 1}|$. To define the PMNS matrix we have to consider only the elements of O_ν that link the three SM neutrino fields ν_{La}, with $a = e, \mu, \tau$, to the three lightest mass eigenstates ν_{iL}, with $i = 1, 2, 3$. Therefore in the normal hierarchy the PMNS matrix will be given by:

$$\big[U_{PMNS}\big]_{i,j} = \big[U_\nu\big]_{i,j} = \big[O_\nu\big]_{i,10-j} \quad \text{with} \quad i,j = 1,2,3\,; \tag{2.252}$$

while if we consider the inverted hierarchy the PMNS matrix gets the form:

$$U_{PMNS} = \begin{pmatrix} (O_\nu)_{18} & (O_\nu)_{17} & (O_\nu)_{19} \\ (O_\nu)_{28} & (O_\nu)_{27} & (O_\nu)_{29} \\ (O_\nu)_{38} & (O_\nu)_{37} & (O_\nu)_{39} \end{pmatrix} . \tag{2.253}$$

The indices of O_ν selected in (2.252) and (2.253) are due to the fact that the light mass eigenstates, which appear in the basis (ν_1, ν_2, ν_3) for the normal hierarchy (or in the basis (ν_2, ν_3, ν_1) for the inverted hierarchy), occupy respectively the position 9, 8 and 7 (or 8, 7 and 9) in the mass eigenstates vector $O_\nu^T \Psi_\nu$.

A typical parametrization for the PMNS matrix is of the form:

$$U_{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{2.254}$$

whose mixing angles can be written as:

$$\theta_{12} = \arctan\left(\frac{|U_{e2}|}{|U_{e1}|}\right) , \quad \theta_{23} = \arctan\left(\frac{|U_{\mu 3}|}{|U_{\tau 3}|}\right) \quad \text{and} \quad \theta_{13} = \arcsin\Big(|U_{e3}|\Big) . \tag{2.255}$$

To match the results given by our numerical simulations with the experimental values, we will use as again the values in [22] at the scale $\mu = 2 \cdot 10^{16}$ GeV:

$$\begin{aligned} \sin^2\theta_{12} &= 0.308 \pm 0.017\,; \\ \sin^2\theta_{23} &= 0.3875 \pm 0.0225\,; \\ \sin^2\theta_{13} &= 0.0241 \pm 0.0025\,. \end{aligned} \tag{2.256}$$

2.8.2 Numerical results

Let us test the present model to generate, in addition to the fermion masses, also the CKM and the PMNS matrices, just as a first try before to improve the model. To do that we extend the parameters scan to include the inter-family mixing. Since the model becomes more complex with the mixing, the tests take much more time, therefore we have only one example to show. With the set of scalar parameters λ_i given by:

$$\begin{cases} \lambda_{LR1} = 0.64 \\ \lambda_{LR2} = -1.32 \\ \lambda_{LR3} = -0.92 \\ \lambda_{L1} = 1.83 \\ \lambda_{L2} = 3.19 \\ \lambda_{R1} = 0.83 \\ \lambda_{R3} = -0.23 \end{cases} , \tag{2.257}$$

we obtain for the down-type Higgs the masses:

$$M_{H_{d1}} = 0.34 \cdot 10^{14}\,\text{GeV} \qquad \text{and} \qquad M_{H_{d2}} = 0.57 \cdot 10^{14}\,\text{GeV}\,, \tag{2.258}$$

with the mixing:

$$\cos\theta_d \sin\theta_d = 0.53 \cdot 10^{-11}\,; \tag{2.259}$$

for the up-type Higgs, instead, we have:

$$M_{H_{u2}} = 0.81 \cdot 10^{14}\,\text{GeV}\,; \tag{2.260}$$

and for the neutrino-type Higgs:

$$M_{H_{\nu1}} = 0.78 \cdot 10^{14}\,\text{GeV} \qquad \text{and} \qquad M_{H_{\nu2}} = 540.549\,\text{GeV} \tag{2.261}$$

with the mixing:

$$\cos\theta_\nu \sin\theta_\nu = 1.37 \cdot 10^{-12}\,. \tag{2.262}$$

For the sterile neutrino masses and the 3×3 Yukawa matrices (Left and Right), instead, the program found the values:

$$M_0 = \begin{pmatrix} 1.26 & 0 & 0 \\ 0 & 0.66 & 0 \\ 0 & 0 & 1.41 \end{pmatrix} \cdot 10^{14}\,\text{GeV}\,; \tag{2.263}$$

$$k_L = \begin{pmatrix} -0.90 & -0.93 & -0.92 \\ -0.21 & -0.55 & -0.32 \\ -0.19 & 0.72 & 0.09 \end{pmatrix} ; \tag{2.264}$$

$$k_R = \begin{pmatrix} 0.14 & -0.72 & 0.16 \\ 0.81 & -0.72 & -0.65 \\ -0.38 & -0.56 & 0.69 \end{pmatrix} . \tag{2.265}$$

With this set of parameters we get the following quark masses:

$$\begin{aligned} m_{up} &= 0.46\,\text{MeV}\,; \\ m_{down} &= 0.24\,\text{MeV}\,; \\ m_{charm} &= 166.47\,\text{MeV}\,; \\ m_{strange} &= 13.1\,\text{MeV}\,; \\ m_{top} &= 2959.73\,\text{MeV}\,; \\ m_{bottom} &= 948.96\,\text{MeV}\,. \end{aligned} \tag{2.266}$$

All these values are within the range of 5σ from the experimental values (2.182), except for the top mass that is again completely unrealistic; for this reason we can not expect to have a good result for the CKM matrix and in fact we found:

$$V_{CKM} = \begin{pmatrix} -0.723 & \mathbf{0.676} & -\mathbf{0.143} \\ 0.501 & 0.656 & \mathbf{0.564} \\ 0.475 & 0.337 & -0.813 \end{pmatrix} , \tag{2.267}$$

where we enlighted in bold the CKM elements corresponding to the experimental values (2.247). As we can see, not only the bold elements (especially $|V_{ub}|$) but also the diagonal elements, that we would expect to be of the order $O(1)$, are far from the experimental values. The last results we show are related to the neutrino sector, in which we have again the same precision problems in the Mathematica calculations. For the mass squared difference ratio we obtain again a too small result:

$$r = 2.34 \cdot 10^{-5}\,; \tag{2.268}$$

while for the PMNS matrix, which is obtained only from a submatrix of the orthogonal matrix O_ν (since the present model does not generate masses for the charged leptons), we get the following result in the normal hierarchy, then using the basis (ν_1, ν_2, ν_3):

$$U_{PMNS} = \begin{pmatrix} 0.323 & -0.048 & 0.945 \\ -0.943 & -0.106 & 0.316 \\ 0.085 & -0.993 & -0.080 \end{pmatrix} . \quad (2.269)$$

From these PMNS elements we obtain the following values for the mixing angles:

$$\sin^2\theta_{13} = 0.894\,, \quad \sin^2\theta_{12} = 0.022 \quad \text{and} \quad \sin^2\theta_{23} = 0.940\,, \quad (2.270)$$

which are values far (much more than 15σ) from the experimental values in (2.256), especially for the mixing angle θ_{13} which is bigger by almost two orders of magnitude than expected. While if we take in account the inverted hierarchy, so we use the basis (ν_2, ν_3, ν_1), we obtain:

$$U_{PMNS} = \begin{pmatrix} -0.0481 & 0.945 & 0.323 \\ -0.106 & 0.316 & -0.943 \\ -0.993 & -0.080 & 0.08 \end{pmatrix} , \quad (2.271)$$

which give us the following mixing angles:

$$\sin^2\theta_{13} = 0.104\,, \quad \sin^2\theta_{12} = 0.997 \quad \text{and} \quad \sin^2\theta_{23} = 0.992\,, \quad (2.272)$$

which, although we got a better result for the mixing angle θ_{13}, are values within the range of more than 25σ from the experimental values. Therefore we can argue that, at the present status, the model is not able to reproduce realistic mixings for quarks and neutrinos, as well as the masses for the top and the light neutrinos.

2.8.3 Other one-loop corrections

Before improving the model to solve its theoretical problems (charged leptons massless and unrealistic masses for the top quark and the light neutrinos), in this section we briefly show some other diagrams that contribute to the masses corrections at one-loop. The first one is shown in Fig. 2.7 and it represents the one-loop correction to the sterile neutrino mass term $\bar{s}_0 m_0 s_0^c$. The evaluation of this diagram is similar to the gauge loop seen above, and it gives the result:

$$\delta m_0 = \frac{k_L k_R \sin\theta_\nu \cos\theta_\nu}{(4\pi)^2} \sum_{i=2}^{3} (O_\nu)_{1i}(O_\nu)_{2i} \left[\frac{m_i^3}{m_i^2 - M_{H_{\nu 2}}^2} \ln\left(\frac{m_i^2}{M_{H_{\nu 2}}^2}\right) - \right. \\ \left. - \frac{m_i^3}{m_i^2 - M_{H_{\nu 1}}^2} \ln\left(\frac{m_i^2}{M_{H_{\nu 1}}^2}\right) \right] \sim \frac{v_L^2}{v_R} \ln\left(\frac{v_R^2}{v_L^2}\right). \tag{2.273}$$

We neglect it because this correction is small compared to the tree level value of the sterile neutrino mass of the order of the Pati-Salam scale ($\sim 10^{14}$ GeV).

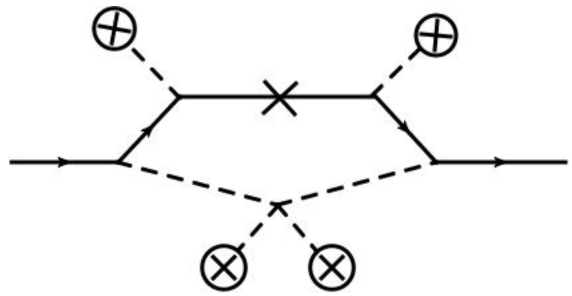


Figure 2.7: *Diagram for the one-loop correction of the sterile neutrino mass m_0.*

The second type of diagram is related to the correction to the elements of the neutrino mass matrix of the type $k_{L,R}v_{L,R}$, as we can see in Fig. 2.8. However, because of the particular structure of the Higgs insertions in this loop (the same, represented in (2.141), needed for a general Majorana mass term $m_M \Psi_{L,R}\Psi_{L,R}$ generation) we know that this contribution is zero.

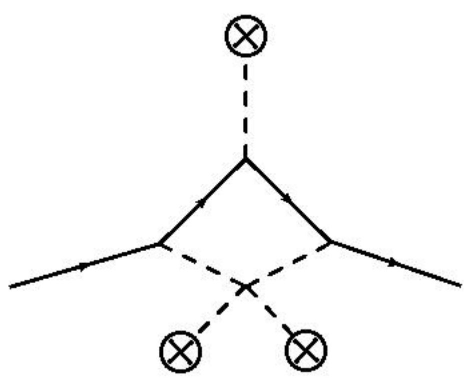


Figure 2.8: *Diagram for the one-loop correction of the neutrino mass terms: $-\frac{k_L v_L}{\sqrt{2}}\bar{\nu}_L s_0$ and $-\frac{k_R v_R}{\sqrt{2}}\bar{\nu}_R s_0^c$.*

Other one-loop diagrams can be built inserting in the loops, of all the diagrams seen till now, more and more VEVs and sterile neutrino masses, but these contributions will be more and more suppressed and so we neglect them in the rest of the thesis.

2.9 Extended model 1

As we have repeatedly emphasized, at the present status the model 1 has two big problems: the first one is the impossibility to generate a mass for the charged leptons, because with the particular scalar structure of the model we cannot connect the left charged lepton with its right part in a scalar loop of type in Fig. 2.3, and the same happens in a gauge loop of type in Fig. 2.4 where the bosons W_L and W_R do not mix; the second problem is the too light mass obtained for the top quark. In order to solve both these issues we extend the model introducing two more Higgs multiplets:

$$\Lambda = (4,2,1) \qquad \text{and} \qquad T = (4,1,2)\,, \tag{2.274}$$

analogs to the multiplets L and R, but with the difference that these new scalars do not get any VEV:

$$\langle\Lambda\rangle = 0 \qquad \text{and} \qquad \langle T\rangle = 0\,. \tag{2.275}$$

2.9.1 New RGE

In this section we spend few words to note that, with the introduction of new scalar fields, the running of the coupling constants, related to the groups $SU(2)_{L,R}$ and $SU(4)$, will change compared to that shown in Sec. 2.4. In particular in the Callan-Symanzik equation (2.110) we have to replace the coefficients (2.111) with:

$$a_{2L} = a_{2R} = -2 \qquad \text{and} \qquad a_{4C} = -\frac{28}{3}\,, \tag{2.276}$$

obtaining the new running depicted in Fig. 2.9, where we compare the evolutions of the gauge coupling constants of the groups $SU(4)$ and $SU(2)_{L,R}$ in the model 1 (where we have only the two Higgs multiplets L and R) with their evolutions in the extended model 1 (where we have two more Higgs multiplets Λ and T). We can see that, although the introduction of two more Higgs multiplets changes the inclinations of the lines representing the evolutions of $g_4(\mu)$ and $g_2(\mu)$, the energy scale at which the intersection occurs between the two gauge coupling constants is mantained at $\mu = 2.7 \cdot 10^{15}$ GeV.

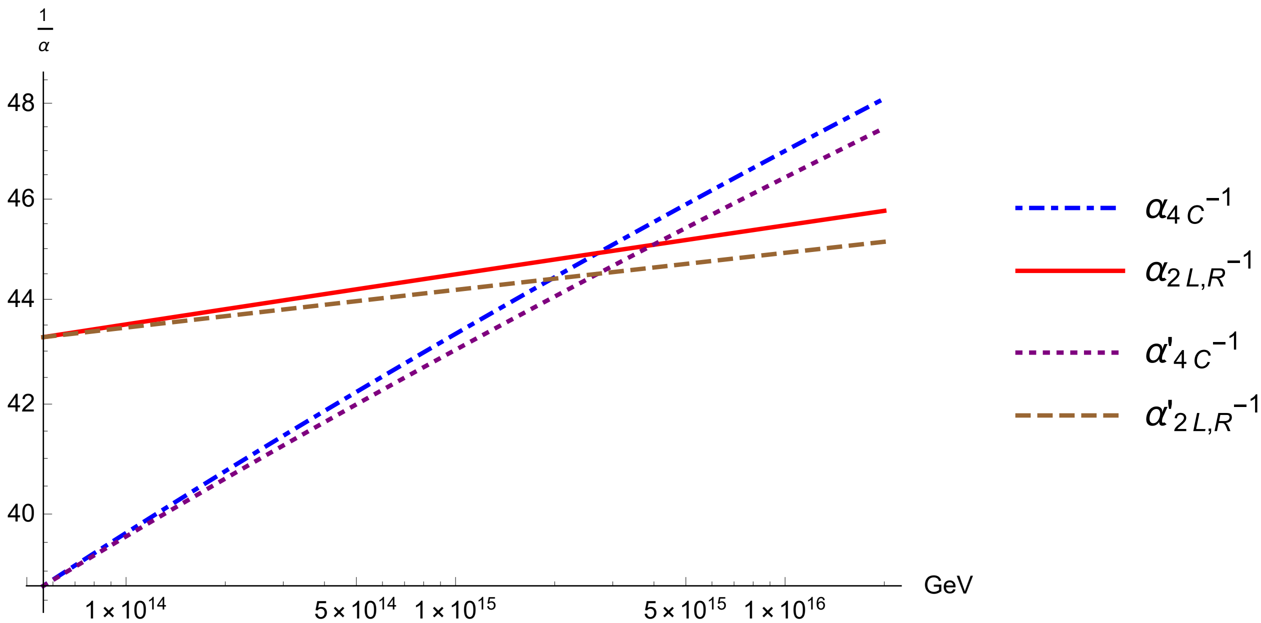


Figure 2.9: *Comparison between the running of the coupling constants $1/\alpha_i$ of the Pati-Salam Model for the groups $SU(2)_{L,R}$ and $SU(4)$. The red solid and blue dot-dashed lines refer, respectively, to the case with only the Higgs multiplets L and R; while the brown dashed and violet dotted lines refer, respectively, to the case with also the multiplets Λ and T.*

2.9.2 New Higgs potential and mass spectrum

As we can imagine, with two other multiplets, the scalar potential becomes more complex than the case with only two multiplets. For simplicity we do not write the complete scalar potential $V(L, R, \Lambda, T)$, but only the terms which contribute to the Higgs masses and then to the scalar loops of the type in Fig. 2.3, in which we need four-point scalar interactions with two VEV insertions in order to connect a left-handed fermion with the right-handed one and obtain a fermion mass term. So having (2.275), the terms of the scalar potential which we consider are only those with less than three multiplets Λ and/or T, since they are those can contribute to the Higgs masses:

$$M_H^2 \propto \left[\frac{\partial^2 V(L, R, \Lambda, T)}{\partial H_{i\alpha} \partial H^{j\beta}}\right]_{VEV} \qquad \text{with} \qquad H = L, R, \Lambda, T\,. \tag{2.277}$$

Again, for the Higgs fields, we use the following notations: greek letters for the $SU(4)$ indices and latin letters for the $SU(2)_{L,R}$ indices, paying attention not to mix the Right and Left indices. Before writing the potential let us make some considerations about the independent terms we would have. We limit ourselves to the case of $(4, 2, 1)$, all the reasonings can be repeated also for $(4,1,2)$. Possible contractions between three fields $L_{i\alpha}$ and one field $\Lambda_{i\alpha}$ are given below:

$$L^{i\alpha}L_{i\alpha}L_{j\beta}\Lambda^{j\beta}\,; \tag{2.278a}$$
$$L^{i\alpha}L_{i\beta}L_{j\alpha}\Lambda^{j\beta}\,; \tag{2.278b}$$
$$L^{i\alpha}L_{j\alpha}L^{j}_{\beta}\Lambda^{\beta}_{i}\,; \tag{2.278c}$$

notice that the last term can be rewritten as (2.278c)=(2.278a)−(2.278b) [2]. Now we pass to consider the terms with two fields $L_{i\alpha}$ and two fields $\Lambda_{i\alpha}$, whose singlets are:

$$L^{i\alpha}L_{i\alpha}\Lambda^{j\beta}\Lambda_{j\beta}\,; \tag{2.279a}$$
$$L^{i\alpha}L_{j\alpha}\Lambda^{j\beta}\Lambda_{i\beta}\,; \tag{2.279b}$$
$$L^{i\alpha}L_{j\alpha}\Lambda^{j}_{\beta}\Lambda^{\beta}_{i}\,; \tag{2.279c}$$
$$L^{i\alpha}\Lambda_{i\alpha}L^{j\beta}\Lambda_{j\beta}\,; \tag{2.279d}$$
$$L^{i\alpha}\Lambda_{j\alpha}L^{j\beta}\Lambda_{i\beta}\,; \tag{2.279e}$$
$$L^{i\alpha}\Lambda_{j\alpha}L^{\beta}_{i}\Lambda^{j}_{\beta}\,; \tag{2.279f}$$
$$L^{i\alpha}\Lambda_{i\alpha}L_{j\beta}\Lambda^{j\beta}\,; \tag{2.279g}$$
$$L^{i\alpha}\Lambda_{j\alpha}L_{i\beta}\Lambda^{j\beta}\,; \tag{2.279h}$$
$$L^{i\alpha}\Lambda_{j\alpha}L^{j}_{\beta}\Lambda^{\beta}_{i}\,. \tag{2.279i}$$

Again some of the terms in (2.279) are not independent, as they can be written as a linear combination of others; in particular we have that (2.279c)=(2.279a)−(2.279b), (2.279f)=(2.279d)−(2.279e) and (2.279i)=(2.279g)−(2.279h).

What we have shown can be easily repeated for the Right multiplets $(4,1,2)$, and similarly to the case with mixing between the Left and the Right multiplets, paying attention to the fact that the $SU(2)_L$ indices of $L_{i\alpha}$ and $\Lambda_{i\alpha}$ can not be contracted with the $SU(2)_R$ indices of $R_{i\alpha}$ and $T_{i\alpha}$. At this point, putting all together, the part of the scalar potential $\tilde{\mathbf{V}}$ with less than three fields $\Lambda_{i\alpha}$ and $T_{i\alpha}$ takes the form written in (H.1) in Appendix H.

The minimum of the potential can be evaluated, as usual, solving the equations for the

[2] This is possible considering the identity $\epsilon^{ij}\epsilon_{kl}=\delta^i_k\delta^j_l-\delta^i_l\delta^j_k$, as seen in Sec. 2.1 for the terms related to the parameter λ_{L3} and λ_{R3}.

first derivatives in the Higgs fields:

$$\left[\frac{\partial V(L,R,\Lambda,T)}{\partial H_{i\alpha}}\right]_{VEV} = 0 \qquad \text{with} \qquad H = L, R, \Lambda, T\,, \tag{2.280}$$

but since any term with more than one field Λ or T gives a zero contribution in (2.280), we can use instead of $V(L,R,\Lambda,T)$ the potential $\tilde{\mathbf{V}}$. From (2.280) we obtain the conditions:

$$\begin{cases} 2\mu_L^2 = (\lambda_{L1}+\lambda_{L2})v_L^2 + (\lambda_{LR1}+\lambda_{LR2})\frac{v_R^2}{2} \\ \mu_Z^2 = (\lambda_{XZ1}+\lambda_{XZ2})v_L^2 + (\lambda_{HK1}+\lambda_{HK2})v_R^2 \\ 2\mu_R^2 = (\lambda_{R1}+\lambda_{R2})v_L^2 + (\lambda_{LR1}+\lambda_{LR2})\frac{v_R^2}{2} \\ \mu_C^2 = (\lambda_{HI1}+\lambda_{HI2})v_L^2 + (\lambda_{AC1}+\lambda_{AC2})v_R^2 \end{cases} , \tag{2.281}$$

that give the same form for the minimum found in Sec. 2.1.1:

$$\tilde{\mathbf{V}}_{min} = -(\lambda_{L1}+\lambda_{L2})\frac{v_L^4}{4} - (\lambda_{R1}+\lambda_{R2})\frac{v_R^4}{4} - (\lambda_{LR1}+\lambda_{LR2})\frac{v_L^2 v_R^2}{4} \quad . \tag{2.282}$$

Even if the presence of the two new Higgs multiplets does not change the form of the minimum, certainly it modifies the mass spectrum of the Higgs fields. In fact, although every mass matrix for a particular type of Higgs fields continues to be decoupled from each others as before:

$$M^2_{LR\Lambda T} = \begin{pmatrix} M^2_{LR\Lambda T_u} & 0 & 0 & 0 \\ 0 & M^2_{LR\Lambda T_d} & 0 & 0 \\ 0 & 0 & M^2_{LR\Lambda T_\nu} & 0 \\ 0 & 0 & 0 & M^2_{LR\Lambda T_e} \end{pmatrix} , \tag{2.283}$$

we find that, using the basis $(L_u, R_u, \Lambda_u, T_u)$, the single squared mass matrix for the up-type Higgs fields is now of the form:

$$M^2_{LR\Lambda T_u} = \frac{v_R^2}{2} \begin{pmatrix} -\lambda_{LR2} & \lambda_{LR2}\frac{v_L}{v_R} & -\lambda_{HK2} & \lambda_{HI2}\frac{v_L}{v_R} \\ \lambda_{LR2}\frac{v_L}{v_R} & -\lambda_{LR2}\frac{v_L^2}{v_R^2} & -\lambda_{HK2}\frac{v_L}{v_R} & -\lambda_{HI2}\frac{v_L^2}{v_R^2} \\ -\lambda_{HK2} & -\lambda_{HK2}\frac{v_L}{v_R} & p_u & \lambda_{HJ2}\frac{v_L}{v_R} \\ \lambda_{HI2}\frac{v_L}{v_R} & -\lambda_{HI2}\frac{v_L^2}{v_R^2} & \lambda_{HJ2}\frac{v_L}{v_R} & q_u \end{pmatrix} , \tag{2.284}$$

with

$$\begin{cases} p_u = -4\frac{\mu_Y^2}{v_R^2} + (\lambda_{XY1} + \lambda_{XY2})\frac{v_L^2}{v_R^2} + \lambda_{KK1} \\ q_u = -4\frac{\mu_B^2}{v_R^2} + (\lambda_{AB1} + \lambda_{AB2}) + \lambda_{II1}\frac{v_L^2}{v_R^2} \end{cases} . \tag{2.285}$$

In the top left 2×2 submatrix we can recognize the previous squared mass matrix $M^2_{LR_u}$ evaluated in (2.66) for the simple model with only two Higgs multiplets. The same happens for the down-type Higgs where, using the basis $(L_d, R_d, \Lambda_d, T_d)$, we obtain the squared mass matrix:

$$M^2_{LR\Lambda T_d} = \frac{v_R^2}{2}\times$$
$$\begin{pmatrix} -2\lambda_{L2}\frac{v_L^2}{v_R^2} - \lambda_{LR2} & 2\lambda_{LR3}\frac{v_L}{v_R} & -\lambda_{XZ2}\frac{v_L^2}{v_R^2} - \lambda_{HK2} & \lambda_{HI3}\frac{v_L}{v_R} \\ 2\lambda_{LR3}\frac{v_L}{v_R} & -2\lambda_{R2} - \lambda_{LR2}\frac{v_L^2}{v_R^2} & \lambda_{HK3}\frac{v_L}{v_R} & -\lambda_{AC2} - \lambda_{HI2}\frac{v_L^2}{v_R^2} \\ -\lambda_{XZ2}\frac{v_L^2}{v_R^2} - \lambda_{HK2} & \lambda_{HK3}\frac{v_L}{v_R} & p_d & \lambda_{HJ3}\frac{v_L}{v_R} \\ \lambda_{HI3}\frac{v_L}{v_R} & -\lambda_{AC2} - \lambda_{HI2}\frac{v_L^2}{v_R^2} & \lambda_{HJ3}\frac{v_L}{v_R} & q_d \end{pmatrix}, \tag{2.286}$$

with

$$\begin{cases} p_d = -4\frac{\mu_Y^2}{v_R^2} + \lambda_{XY1}\frac{v_L^2}{v_R^2} + \lambda_{KK1} \\ q_d = -4\frac{\mu_B^2}{v_R^2} + \lambda_{AB1} + \lambda_{II1}\frac{v_L^2}{v_R^2} \end{cases} . \tag{2.287}$$

Eventually the Higgs fields related to the neutrino sector, with the basis $(L_\nu, R_\nu, \Lambda_\nu, T_\nu)$, has the following squared mass matrix:

$$M^2_{LR\Lambda T_\nu} = \frac{v_R^2}{2}\times$$
$$\begin{pmatrix} 2(\lambda_{L1} + \lambda_{L2})\frac{v_L^2}{v_R^2} & (\lambda_{LR1} + \lambda_{LR2})\frac{v_L}{v_R} & (\lambda_{XZ1} + \lambda_{XZ2})\frac{v_L^2}{v_R^2} & (\lambda_{HI1} + \lambda_{HI2})\frac{v_L}{v_R} \\ (\lambda_{LR1} + \lambda_{LR2})\frac{v_L}{v_R} & 2(\lambda_{R1} + \lambda_{R2}) & (\lambda_{HK1} + \lambda_{HK2})\frac{v_L}{v_R} & \lambda_{AC1} + \lambda_{AC2} \\ (\lambda_{XZ1} + \lambda_{XZ2})\frac{v_L^2}{v_R^2} & (\lambda_{HK1} + \lambda_{HK2})\frac{v_L}{v_R} & p_\nu & (\lambda_{IK1} + \lambda_{HJ2})\frac{v_L}{v_R} \\ (\lambda_{HI1} + \lambda_{HI2})\frac{v_L}{v_R} & \lambda_{AC1} + \lambda_{AC2} & (\lambda_{IK1} + \lambda_{HJ2})\frac{v_L}{v_R} & q_\nu \end{pmatrix}, \tag{2.288}$$

with

$$\begin{cases} p_\nu = -4\frac{\mu_Y^2}{v_R^2} + (\lambda_{XY1} + \lambda_{XY2} + \lambda_{ZZ} + 2\lambda_{ZZ3})\frac{v_L^2}{v_R^2} + (\lambda_{KK1} + \lambda_{KK2}) \\ q_\nu = -4\frac{\mu_B^2}{v_R^2} + (\lambda_{AB1} + \lambda_{AB2} + \lambda_{CC} + 2\lambda_{CC3}) + (\lambda_{II1} + \lambda_{II2})\frac{v_L^2}{v_R^2} \end{cases} . \tag{2.289}$$

As we will see later, the new Higgs multiplets, in addition to produce the modifications of the squared mass matrices, they will also cause an addition of new diagrams involving scalar loops, which will correct the radiatively generated masses of quarks and leptons. The biggest change is in the sector related to the charged leptons. In fact, using the basis $(L_e, R_e, \Lambda_e, T_e)$, we have that the squared mass matrix is non-zero and it has the form:

$$M^2_{LR\Lambda T_e} = \frac{v_R^2}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_e & (\lambda_{HJ3} - \lambda_{HI3})\frac{v_L}{v_R} \\ 0 & 0 & (\lambda_{HJ3} - \lambda_{HI3})\frac{v_L}{v_R} & q_e \end{pmatrix}, \tag{2.290}$$

with

$$\begin{cases} p_e = -4\frac{\mu_Y^2}{v_R^2} + (\lambda_{XY1} + 2\lambda_{ZZ3})\frac{v_L^2}{v_R^2} + (\lambda_{KK1} + \lambda_{KK2}) \\ q_e = -4\frac{\mu_B^2}{v_R^2} + (\lambda_{AB1} + 2\lambda_{CC3}) + (\lambda_{II1} + \lambda_{II2})\frac{v_L^2}{v_R^2} \end{cases}. \tag{2.291}$$

This means that, while in the simple model the mass eigenstates of the e-type Higgs squared mass matrix were pure Goldstone bosons which gave mass to the vector bosons W_L^μ and W_R^μ, in this extended model we obtain in general two massive eigenstates for the e-type Higgs, which mix the Left and the Right multiplet (Λ and T respectively) making possible to build up diagrams for the radiative mass generations for the charged leptons, totally absent till now.

2.9.3 New scalar loops

As anticipated above, we have a new set of diagrams for the radiative fermion mass generations. In fact, while the diagrams with loops involving gauge bosons remain the same seen for the simple model in Fig. 2.4, we have now to consider three more diagrams containing scalar loops, shown in Fig. 2.10. These new diagrams are given by all possible combinations between an Higgs multiplet $(4, 2, 1)$, L or Λ, and a multiplet $(4, 1, 2)$, R or T.

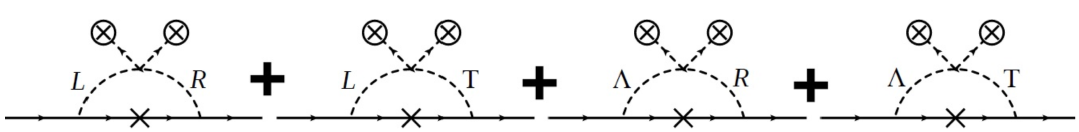


Figure 2.10: *Sum of all the possible scalar loops contributing to the radiative fermion mass generations.*

The scalar-loop contributions to the fermion masses, in analogy with that seen in

Sec. 2.6.1, will be given by the general formula:

$$m_H^{(\psi)} = \sum_{i=1}^{9} \frac{m_i^2 (O_\nu)_{3i}^2}{(4\pi)^2} \sum_{j=1}^{4} \Big[k_R (O_{H_\psi})_{2j} + k_T (O_{H_\psi})_{4j}\Big] \Big[k_L (O_{H_\psi})_{1j} + k_\Lambda (O_{H_\psi})_{3j}\Big] \times \\ \times \left[\frac{M_{H_{\psi j}}^2}{M_{H_{\psi j}}^2 - m_i^2} \ln \left(\frac{\tilde{m}_i^2}{M_{H_{\psi j}}^2} \right) \right], \tag{2.292}$$

where for the neutrino mass eigenstates in the logarithms we use the definition:

$$\tilde{m}_i = \begin{cases} m_i & \text{if} \quad m_i \neq 0 \\ const. \neq 0 & \text{if} \quad m_i = 0 \end{cases}, \tag{2.293}$$

while $M_{H_{\psi j}}^2$ represent the eigenvalues of the squared mass matrix $M_{LR\Lambda T_\psi}^2$, with $\psi = u, d, \nu$, and O_{H_ψ} are the 4×4 orthogonal matrices built up with their normalized eigenvectors, for which the following identity is valid:

$$\sum_{k=1}^{4} (O_{H_\psi})_{ik} (O_{H_\psi})_{jk} = \delta_{ij} \, . \tag{2.294}$$

We note, instead, that for the charged leptons ($\psi = e$) the formula (2.292) has to be restricted to the 2×2 non-zero submatrix of $M_{LR\Lambda T_e}^2$ related to the mixing of only the Higgs Λ_e and T_e:

$$m_H^{(e)} = \sum_{i=1}^{9} \frac{m_i^2 (O_\nu)_{3i}^2}{(4\pi)^2} \sum_{j=3}^{4} k_T (O_{H_e})_{4j} k_\Lambda (O_{H_e})_{3j} \left[\frac{M_{H_{ej}}^2}{M_{H_{ej}}^2 - m_i^2} \ln \left(\frac{\tilde{m}_i^2}{M_{H_{ej}}^2} \right) \right], \tag{2.295}$$

with

$$O_{H_e} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_{e34} & -\sin\theta_{e34} \\ 0 & 0 & \sin\theta_{e34} & \cos\theta_{e34} \end{pmatrix}. \tag{2.296}$$

2.9.4 Numerical results without mixing

Due to the larger parameter space, we need to rely on a different software to test numerically the extended model. So, instead of Mathematica, we implement a new scan code using MultiNest [24, 25, 26]. It is a Bayesian inference tool which explores the parameter space trying to maximize their likelihood given the experimental values (2.182) and (2.186). For the sake of simplicity, we perform two approximations in order to reduce the number of free parameters in the scan. First of all we neglect the mixing between the pairs (L, R) and (Λ, T). In this case we have two contributions coming from the scalar loops, in the same form as (2.145), one for the mass eigenstates $H_{\psi 1}$ and $H_{\psi 2}$, linear combinations of the fields L_ψ and R_ψ:

$$m_{H12}^{(\psi)} = k_L k_R \sin\theta_{\psi 12} \cos\theta_{\psi 12} \sum_{i=1}^{9} (O_\nu)^2_{3i} \frac{m_i}{(4\pi)^2} \left[\frac{M^2_{H_{\psi 1}}}{m_i^2 - M^2_{H_{\psi 1}}} \ln\left(\frac{\tilde{m}_i^2}{M^2_{H_{\psi 1}}}\right) - \frac{M^2_{H_{\psi 2}}}{m_i^2 - M^2_{H_{\psi 2}}} \ln\left(\frac{\tilde{m}_i^2}{M^2_{H_{\psi 2}}}\right)\right], \tag{2.297}$$

and the other for the mass eigenstates $H_{\psi 3}$ and $H_{\psi 4}$, linear combinations of the new fields Λ_ψ and T_ψ only:

$$m_{H34}^{(\psi)} = k_\Lambda k_T \sin\theta_{\psi 34} \cos\theta_{\psi 34} \sum_{i=1}^{9} (O_\nu)^2_{3i} \frac{m_i}{(4\pi)^2} \left[\frac{M^2_{H_{\psi 3}}}{m_i^2 - M^2_{H_{\psi 3}}} \ln\left(\frac{\tilde{m}_i^2}{M^2_{H_{\psi 3}}}\right) - \frac{M^2_{H_{\psi 4}}}{m_i^2 - M^2_{H_{\psi 4}}} \ln\left(\frac{\tilde{m}_i^2}{M^2_{H_{\psi 4}}}\right)\right]. \tag{2.298}$$

The total scalar loop contribution, for each family, is then given by: $m_H^{(\psi)} = m_{H12}^{(\psi)} + m_{H34}^{(\psi)}$, with $\psi = u, d, e, \nu$. In this way we have to take in account only two mixing angles. More to that, assuming to have in $\tilde{\mathbf{V}}$ enough parameters to generate any particular Higgs mass spectrum (a fine-tuning hypotesis [27, 28]), we consider as parameters to scan, instead of the λ_is, directly the Higgs squared mass and their mixings. Here below we show two examples of results, in the case without inter-family mixing, obtained with MultiNest. We let span the values for the Yukawa couplings (k_L^a, k_R^a, k_Λ^a and k_T^a with $a = 1, 2, 3$ the family index), and a parameter m_a, related to the sterile neutrino masses, by the identity: $m_{0a} = m_a v_R$, in the interval $[-10, 10]$, while we use the interval $(0, 10]$ for the parameters $\xi_{\psi i}$ and $\eta_{\psi xy}$ related to the Higgs squared masses and mixings, respectively, by the identities:

$$M^2_{H_{\psi i}} = \xi_{\psi i} v^2_{R,L} \qquad \text{and} \qquad \cos\theta_{\psi xy} = \eta_{\psi xy} \frac{v_L}{v_R}\,, \tag{2.299}$$

with $i = 1, 2, 3, 4$ and $xy = 12, 34$. Remembering, from the non-extended model, that: $M^2_{H_{u1}}$ is substitute in the loops with M^2_X; the mixing $\cos\theta_{u12}\sin\theta_{u12} = \frac{v_L v_R}{\sqrt{v_L^2 v_R^2}}$ is a constant; while $M^2_{H_{e1}} = M^2_{H_{e1}} = 0$, then $\cos\theta_{e12}\sin\theta_{e12} = 0$. Furthermore we note that it is possible to consider a positive interval where to search for the values of the parameters, not only those related to the Higgs squared masses, but also those related to the Higgs mixings.[3] The two examples we present have a Log-likelihood ~ -35, but they differ in the interval we use for the parameters $\xi_{\nu 2}$, in order to have $M_{H_{\nu 2}}$ of the order of the SM Higgs mass. So, in the first example we use the interval $(0, 10]$ common to all the other scalar parameters, while in the second example we use the smaller interval $[0.2, 0.3]$.

Example 1

In the first example, we get the following Higgs masses:

$$M_{H_{d1}} = 2.13 \cdot 10^{14}\,\text{GeV} \qquad \text{and} \qquad M_{H_{d2}} = 2.17 \cdot 10^{14}\,\text{GeV} \tag{2.300}$$

for the down-type Higgs, linear combinations of L_d and R_d with the mixing:

$$\cos\theta_{d12}\sin\theta_{d12} = 0.96 \cdot 10^{-12}\,; \tag{2.301}$$

while for the others two down-type Higgs, linear combinations of Λ_d and T_d, we have:

$$M_{H_{d3}} = 2.70 \cdot 10^{14}\,\text{GeV} \qquad \text{and} \qquad M_{H_{d4}} = 2.66 \cdot 10^{14}\,\text{GeV}\,, \tag{2.302}$$

and the mixing:

$$\cos\theta_{d34}\sin\theta_{d34} = 2.93 \cdot 10^{-12}\,. \tag{2.303}$$

For the up-type Higgs, combinations of L_u and R_u, we have:

$$M_{H_{u2}} = 1.67 \cdot 10^{14}\,\text{GeV} \tag{2.304}$$

where we remember that, being the same situation seen in the simple model, the other eigenstate H_{u1} is a Goldstone boson (so we use $M_{H_{u1}} = M_X$ in the scalar propagator) and

[3] This because, if we need a mixing $\cos\theta_{\psi xy}\sin\theta_{\psi xy} < 0$ the program has only to interchange the parameters related to $M^2_{H_{\psi 1,3}}$ with those related to $M^2_{H_{\psi 2,4}}$, which are all of order $O(v_R^2)$ except for the case $M^2_{H_{\nu 2}} = \xi_{\nu 2} v_L^2$. However we know from (2.80) that for the corresponding mixing we have: $\cos\theta_{\nu 12}\sin\theta_{\nu 12} \sim O(\frac{v_L}{v_R}) > 0$.

the mixing is given by the constant value (2.200). While for the other two up-type Higgses we get the values:

$$M_{H_{u3}} = 1.48 \cdot 10^{14}\,\text{GeV} \qquad \text{and} \qquad M_{H_{u4}} = 1.72 \cdot 10^{14}\,\text{GeV} \tag{2.305}$$

with the mixing:

$$\cos\theta_{u34}\sin\theta_{u34} = 1.47 \cdot 10^{-12}\,. \tag{2.306}$$

We go on with the ν-type Higgs, combination of L_ν and R_ν, for which the program select the values for the mixing:

$$\cos\theta_{\nu12}\sin\theta_{\nu12} = 1.17 \cdot 10^{-11} \tag{2.307}$$

and the masses:

$$M_{H_{\nu1}} = 2.31 \cdot 10^{14}\,\text{GeV} \qquad \text{and} \qquad M_{H_{\nu2}} = 611.88\,\text{GeV}\,, \tag{2.308}$$

where again, like in the previous results for the simple model, we can see that $M_{H_{\nu2}}$ is too big to represent the Standard Model Higgs. For the mass eigenstates coming from the combination of Λ_ν and T_ν, which have to be of the order of the Pati-Salam scale we get:

$$M_{H_{\nu3}} = 2.12 \cdot 10^{14}\,\text{GeV} \qquad \text{and} \qquad M_{H_{\nu4}} = 2.01 \cdot 10^{14}\,\text{GeV}\,, \tag{2.309}$$

with the mixing given by:

$$\cos\theta_{\nu34}\sin\theta_{\nu34} = 2.09 \cdot 10^{-11}\,. \tag{2.310}$$

To complete the Higgs mass spectrum there remain the massive Higgs related to the charged leptons sector:

$$M_{H_{e3}} = 2.04 \cdot 10^{14}\,\text{GeV} \qquad \text{and} \qquad M_{H_{e4}} = 2.31 \cdot 10^{14}\,\text{GeV} \tag{2.311}$$

that are combintation of Λ_e and T_e with the mixing:

$$\cos\theta_{e34}\sin\theta_{e34} = 1.93 \cdot 10^{-12}\,. \tag{2.312}$$

The remaining parameters are the four Yukawa couplings for the three families, that we collect in the following diagonal matrices:

$$k_L = \begin{pmatrix} -0.07 & 0 & 0 \\ 0 & 0.95 & 0 \\ 0 & 0 & 6.97 \end{pmatrix} ; \tag{2.313}$$

$$k_R = \begin{pmatrix} -0.05 & 0 & 0 \\ 0 & -0.16 & 0 \\ 0 & 0 & -5.41 \end{pmatrix} ; \tag{2.314}$$

$$k_\Lambda = \begin{pmatrix} 0.41 & 0 & 0 \\ 0 & -1.03 & 0 \\ 0 & 0 & 2.56 \end{pmatrix} ; \tag{2.315}$$

$$k_T = \begin{pmatrix} -0.14 & 0 & 0 \\ 0 & -0.30 & 0 \\ 0 & 0 & 2.01 \end{pmatrix} ; \tag{2.316}$$

and the three sterile neutrino masses, one for each fermion family:

$$M_0 = \begin{pmatrix} -2.22 \cdot 10^{-2} & 0 & 0 \\ 0 & 2.08 & 0 \\ 0 & 0 & 0.20 \end{pmatrix} \cdot 10^{14}\,\text{GeV}\,. \tag{2.317}$$

With this particular selection of parameters, as results for the quark masses, we obtain the values:

$$\begin{aligned} &m_{up} = 0.60\,\text{MeV}\,, \quad m_{down} = 0.09\,\text{MeV}\,, \\ &m_{charm} = 253.75\,\text{MeV}\,, \quad m_{strange} = 23.97\,\text{MeV}\,, \\ &m_{top} = 77.46\,\text{GeV}\,, \quad m_{bottom} = 1.13\,\text{GeV}\,, \end{aligned} \tag{2.318}$$

so that all masses are within the 1σ experimental ranges apart from m_{down}, which is reproduced within 3σ. A good result is also obtained for the charged lepton masses:

$$\begin{aligned} m_e &= 0.48\,\mathrm{MeV}\,, \\ m_\mu &= 103.63\,\mathrm{MeV}\,, \\ m_\tau &= 1663.42\,\mathrm{MeV}\,, \end{aligned} \tag{2.319}$$

where all the three masses are within the 1σ range from the experimental values. On the other hand, in the neutrino sector we still have a very small r ratio:

$$r = 3.94 \cdot 10^{-22}\,. \tag{2.320}$$

Example 2

The second example we show is defined with the following parameters selection:

$$M_{H_{d1}} = 2.12 \cdot 10^{14}\,\mathrm{GeV} \qquad \text{and} \qquad M_{H_{d2}} = 2.04 \cdot 10^{14}\,\mathrm{GeV}\,; \tag{2.321}$$

$$M_{H_{d3}} = 2.14 \cdot 10^{14}\,\mathrm{GeV} \qquad \text{and} \qquad M_{H_{d4}} = 2.09 \cdot 10^{14}\,\mathrm{GeV}\,; \tag{2.322}$$

for the down-type Higgs, with the mixings respectively:

$$\cos\theta_{d12}\sin\theta_{d12} = 1.10 \cdot 10^{-12} \qquad \text{and} \qquad \cos\theta_{d34}\sin\theta_{d34} = 1.96 \cdot 10^{-12}\,. \tag{2.323}$$

While for the up-type Higgs we have:

$$M_{H_{u2}} = 2.18 \cdot 10^{14}\,\mathrm{GeV}\,; \tag{2.324}$$

$$M_{H_{u3}} = 2.31 \cdot 10^{14}\,\mathrm{GeV} \qquad \text{and} \qquad M_{H_{u4}} = 2.68 \cdot 10^{14}\,\mathrm{GeV}\,, \tag{2.325}$$

with the mixing:

$$\cos\theta_{u34}\sin\theta_{u34} = 1.42 \cdot 10^{-12}\,. \tag{2.326}$$

To the Higgs related to the charged leptons, instead, the program assigns the masses:

$$M_{H_{e3}} = 2.41 \cdot 10^{14}\,\mathrm{GeV} \qquad \text{and} \qquad M_{H_{e4}} = 2.21 \cdot 10^{14}\,\mathrm{GeV}\,, \tag{2.327}$$

and the related mixing:

$$\cos\theta_{e34}\sin\theta_{e34} = 1.86\cdot 10^{-12}\,. \tag{2.328}$$

Next we get the mass values for the ν-type Higgs:

$$M_{H_{\nu 1}} = 2.08\cdot 10^{14}\,\mathrm{GeV} \qquad \text{and} \qquad M_{H_{\nu 2}} = 122.50\,\mathrm{GeV}\,; \tag{2.329}$$

$$M_{H_{\nu 3}} = 2.40\cdot 10^{14}\,\mathrm{GeV} \qquad \text{and} \qquad M_{H_{\nu 4}} = 2.36\cdot 10^{14}\,\mathrm{GeV}\,; \tag{2.330}$$

with the mixings:

$$\cos\theta_{\nu 12}\sin\theta_{\nu 12} = 1.20\cdot 10^{-11} \qquad \text{and} \qquad \cos\theta_{\nu 34}\sin\theta_{\nu 34} = 9.05\cdot 10^{-12}\,. \tag{2.331}$$

We can recognize in $M_{H_{\nu 2}}$ a more realistic SM Higgs mass, because in this second run we used for $M_{H_{\nu 2}}$ a smaller interval, in which to span the parameter scan code, than in the previous example, where was selected a $M_{H_{\nu 2}}$ too big.
Eventually we have the parameter selection for the Yukawa couplings:

$$k_L = \begin{pmatrix} -0.22 & 0 & 0 \\ 0 & 0.47 & 0 \\ 0 & 0 & 6.88 \end{pmatrix}\;; \tag{2.332}$$

$$k_R = \begin{pmatrix} 0.08 & 0 & 0 \\ 0 & 0.48 & 0 \\ 0 & 0 & 4.20 \end{pmatrix}\;; \tag{2.333}$$

$$k_\Lambda = \begin{pmatrix} 0.21 & 0 & 0 \\ 0 & -0.86 & 0 \\ 0 & 0 & -1.90 \end{pmatrix}\;; \tag{2.334}$$

$$k_T = \begin{pmatrix} -0.38 & 0 & 0 \\ 0 & -1.68 & 0 \\ 0 & 0 & 4.37 \end{pmatrix}\;; \tag{2.335}$$

and the sterile neutrino masses:

$$M_0 = \begin{pmatrix} -2.17 \cdot 10^{-3} & 0 & 0 \\ 0 & 0.35 & 0 \\ 0 & 0 & 1.10 \end{pmatrix} \cdot 10^{14}\,\text{GeV}\,. \tag{2.336}$$

With this particular set of parameters we get the results:

$$\begin{aligned} m_{up} &= 0.56\,\text{MeV}\,, \quad m_{down} = 0.15\,\text{MeV}\,, \\ m_{charm} &= 251.64\,\text{MeV} \quad m_{strange} = 30.47\,\text{MeV}\,, \\ m_{top} &= 79.99\,\text{GeV} \quad m_{bottom} = 1.14\,\text{GeV}\,. \end{aligned} \tag{2.337}$$

While for the charged lepton masses we have the values:

$$\begin{aligned} m_e &= 0.48\,\text{MeV}\,, \\ m_\mu &= 102.07\,\text{MeV}\,, \\ m_\tau &= 1750.62\,\text{MeV}\,. \end{aligned} \tag{2.338}$$

All these masses are within the 3σ range from the experimental values (2.182). For the light neutrinos, instead, we obtain the ratio:

$$r = 1.5 \cdot 10^{-5}\,, \tag{2.339}$$

that is still too small.

2.9.5 Numerical results with mixing

To generalize this extended model to the case with fermion inter-family mixing we can follow exactly what discussed in Sec. 2.8; the only difference is that now we have two more Yukawa couplings, k_Λ and k_T, to promote to general 3×3 matrices. In order to keep at a handleable level for MultiNest the number of parameters in the mixing-case we let the program scan only the values for the Yukawa matrices and the sterile neutrino masses. For the parameters related to the Higgs masses and mixings, instead, we use the values found in the previous examples, without inter-family mixing. In this case, the program has to maximize the likelihood of the parameters (which for the following examples results Log-likelihood ~ -100), not only with respect to the experimental values of the quark masses (2.182) and the mass ratio (2.186) for the light neutrinos, but also with respect to the CKM elements in (2.247). We tried to insert also the PMNS mixings in the fit but the program remained stalled. We remember that, since we use real Yukawa couplings, we will obtain real mixings for the CKM and PMNS matrices.

Example 1

The first results we show are obtained with the Higgs sector given by the values from (2.300) to (2.312); while for the Yukawa couplings MultiNest selects the following values collected in the matricial forms:

$$k_L = \begin{pmatrix} -3.47 & 4.47 & 3.55 \\ 0.98 & 1.62 & -1.18 \\ -0.02 & 0.89 & -0.30 \end{pmatrix} ; \tag{2.340}$$

$$k_R = \begin{pmatrix} -0.56 & 2.04 & 2.62 \\ -0.01 & -0.98 & -1.69 \\ 0.62 & -2.76 & -3.38 \end{pmatrix} ; \tag{2.341}$$

$$k_\Lambda = \begin{pmatrix} 0.68 & -1.72 & -0.98 \\ 0.74 & 0.02 & 1.33 \\ 0.76 & -0.10 & 0.64 \end{pmatrix} ; \tag{2.342}$$

$$k_T = \begin{pmatrix} -0.82 & 1.25 & 1.17 \\ -0.38 & 0.90 & -0.95 \\ 1.04 & -1.24 & -1.39 \end{pmatrix} ; \tag{2.343}$$

and then, to complete the parameter set, we have the sterile neutrino masses:

$$M_0 = \begin{pmatrix} -0.54 & 0 & 0 \\ 0 & 1.20 & 0 \\ 0 & 0 & -2.10 \cdot 10^{-2} \end{pmatrix} \cdot 10^{13}\,\text{GeV}\,. \tag{2.344}$$

With this choice for the parameter set we obtain as eigenvalues from the 3×3 mass matrices for the up-type and down-type quarks the following results:

$$\begin{aligned} m_{up} &= 0.87\,\text{MeV}\,, \quad m_{down} = 1.38\,\text{MeV}\,, \\ m_{charm} &= 225.36\,\text{MeV}\,, \quad m_{strange} = 18.67\,\text{MeV}\,, \\ m_{top} &= 1.11\,\text{GeV}\,, \quad m_{bottom} = 72.33\,\text{GeV}\,, \end{aligned} \tag{2.345}$$

values that are all inside 2.2σ from the experimental values. Beyond that, for the quark sector, we can also evaluate the CKM matrix:

$$V_{CKM} = \begin{pmatrix} -0.943 & -\mathbf{0.331} & \mathbf{0.008} \\ 0.330 & -0.938 & \mathbf{0.102} \\ -0.026 & 0.099 & 0.995 \end{pmatrix}\,, \tag{2.346}$$

where we can see that, although the diagonal elements are of the order $O(1)$ as we expect, the non-diagonal elements in bold are distant from the experimental (2.247). From the charged lepton mass matrix, instead, we get the eigenvalues:

$$\begin{aligned} m_e &= 0.46\,\text{MeV}\,; \\ m_\mu &= 99.27\,\text{MeV}\,; \\ m_\tau &= 1927.06\,\text{MeV}\,; \end{aligned} \tag{2.347}$$

all masses are within the 2σ range from the experimental values. While for the light neutrino masses we reach the ratio:

$$r = 0.026\,, \tag{2.348}$$

a value that is inside the 5σ from the refence one (2.186).

Before showing also the resulting PMNS matrix, we have to mentioned another difference with the simple model. In fact, because in this extended model we have also a generally non-diagonal charged lepton mass matrix, the PMNS matrix is effectively given by the (2.250), where U_l is different from the identity matrix. So in the normal hierarchy we have:

$$U_{PMNS} = \begin{pmatrix} -0.648 & 0.760 & 0.042 \\ 0.323 & 0.224 & 0.920 \\ -0.690 & -0.610 & 0.390 \end{pmatrix}\,, \tag{2.349}$$

from which we obtain the mixings angles:

$$\sin^2\theta_{13} = 0.002\,, \quad \sin^2\theta_{12} = 0.579 \quad \text{and} \quad \sin^2\theta_{23} = 0.847\,, \tag{2.350}$$

where the best result is given by the mixing $\sin^2\theta_{13}$ which is inside the 9σ range from the experimental value (2.256), while the other two mixing are inside more than the 15σ ranges. In the inverted hierarchy, instead, the PMNS matrix is given by:

$$U_{PMNS} = \begin{pmatrix} 0.760 & 0.042 & -0.648 \\ 0.224 & 0.920 & 0.323 \\ -0.610 & 0.390 & -0.690 \end{pmatrix}\,, \tag{2.351}$$

with

$$\sin^2\theta_{13} = 0.420\,, \quad \sin^2\theta_{12} = 0.003 \quad \text{and} \quad \sin^2\theta_{23} = 0.180\,, \tag{2.352}$$

where, this time, we have the best result with the mixing $\sin^2\theta_{23}$ which is inside the 10σ range from the experimental value, while $\sin^2\theta_{12}$ is inside the 18σ range and $\sin^2\theta_{13}$ is even more distant.

Example 2

The second set of results that we show is obtained using the values from (2.321) to (2.331) for the scalar sector, while for the four Yukawa couplings and the sterile neutrino masses the code found respectively the values:

$$k_L = \begin{pmatrix} -0.43 & -0.85 & 0.68 \\ -4.60 & 0.84 & 0.48 \\ -0.26 & -0.57 & 0.72 \end{pmatrix}\,; \tag{2.353}$$

$$k_R = \begin{pmatrix} 3.59 & -1.94 & -0.86 \\ 3.41 & -1.89 & -0.64 \\ 0.28 & -0.40 & 0.19 \end{pmatrix}\,; \tag{2.354}$$

$$k_\Lambda = \begin{pmatrix} -0.45 & -0.20 & 1.43 \\ -0.02 & 3.35 & -0.42 \\ -0.39 & 0.07 & 0.93 \end{pmatrix}\,; \tag{2.355}$$

$$k_T = \begin{pmatrix} -2.42 & 1.17 & 1.49 \\ -2.26 & 2.10 & 0.89 \\ 0.11 & 1.57 & 0.26 \end{pmatrix} ; \tag{2.356}$$

$$M_0 = \begin{pmatrix} 3.56 & 0 & 0 \\ 0 & -2.31 & 0 \\ 0 & 0 & -0.23 \end{pmatrix} \cdot 10^{13}\,\mathrm{GeV}\,. \tag{2.357}$$

This second set of parameters leads to the quark masses:

$$\begin{aligned} &m_{up} = 0.43\,\mathrm{MeV}\,, \quad m_{down} = 1.14\,\mathrm{MeV}\,, \\ &m_{charm} = 246.73\,\mathrm{MeV}\,, \quad m_{strange} = 23.48\,\mathrm{MeV}\,, \\ &m_{top} = 74.67\,\mathrm{GeV}\,, \quad m_{bottom} = 1.16\,\mathrm{GeV}\,, \end{aligned} \tag{2.358}$$

all values within the 1σ range from the experimental values in (2.182), apart from the bottom which is inside the 2σ. While for the CKM matrix we obtain:

$$V_{CKM} = \begin{pmatrix} 0.973 & \mathbf{0.229} & \mathbf{0.015} \\ -0.228 & 0.961 & \mathbf{0.156} \\ 0.022 & -0.156 & 0.988 \end{pmatrix} , \tag{2.359}$$

where we note that, although the diagonal elements are again of order $O(1)$, the absolute value of the element $|V_{us}|$ is within the 6σ range from the reference value in (2.247), while $|V_{cb}|$ and $|V_{ub}|$ are even more distant.
Also for the charged lepton masses we obtain results within the 1σ range from the expected values in (2.182):

$$\begin{aligned} &m_e = 0.50\,\mathrm{MeV}\,, \\ &m_\mu = 101.88\,\mathrm{MeV}\,, \\ &m_\tau = 1726.27\,\mathrm{MeV}\,. \end{aligned} \tag{2.360}$$

While for the light neutrinos we get the ratio:

$$r = 0.006\,, \tag{2.361}$$

that if far at least 25σ from the reference result in (2.186). We conclude this second example with the results for the PMNS matrix in the normal hierarchy:

$$U_{PMNS} = \begin{pmatrix} -0.411 & 0.891 & 0.195 \\ 0.906 & 0.376 & 0.192 \\ 0.098 & 0.256 & -0.962 \end{pmatrix}, \tag{2.362}$$

which leads to the mixing angles:

$$\sin^2\theta_{13} = 0.038\,, \quad \sin^2\theta_{12} = 0.825 \quad \text{and} \quad \sin^2\theta_{23} = 0.038\,, \tag{2.363}$$

where the best result is for the mixing $\sin^2\theta_{13}$ which is within the 6σ range from the experimental value, while $\sin^2\theta_{12}$ and $\sin^2\theta_{23}$ are inside the 31σ and 16σ ranges, respectively. In the inverted hierarchy, instead, we have:

$$U_{PMNS} = \begin{pmatrix} 0.891 & 0.195 & -0.411 \\ 0.376 & 0.192 & 0.906 \\ 0.256 & -0.962 & 0.098 \end{pmatrix}, \tag{2.364}$$

from which we obtain:

$$\sin^2\theta_{13} = 0.169\,, \quad \sin^2\theta_{12} = 0.046 \quad \text{and} \quad \sin^2\theta_{23} = 0.988\,, \tag{2.365}$$

which are all inside ranges bigger than 16σ.

Although with the extended model we are now able to generate the correct mass spectrum of all the quarks and the charged leptons, within the range of 2σ from the experimental values, the model seems to fail in reproducing the mixings, of the CKM and PMNS matrices, and the light neutrino masses, which give a ratio r still too small.

Chapter 3

Model 2

The second model that we study was developed by R. R. Volkas [3] and it differs from the previous one only in the scalar fields content. In fact, under the gauge group $SU(4) \otimes SU(2)_L \otimes SU(2)_R$, the scalar content of the model 2 is given by the Higgs fields in the representations:

$$R^{i\alpha} \sim (4,1,2) \qquad \text{and} \qquad \Phi^I_i \sim (1,2,2) \tag{3.1}$$

where $\alpha = 1,2,3,4$ is the index of $SU(4)$, while $i = 1,2$ and $I = 1,2$ are the $SU(2)_R$ and $SU(2)_L$ indices, respectively. So we do not have any left-handed Higgs multiplets, as L and Λ in the model 1, but only a right-handed Higgs multiplets R, responsible for the $SU(4)$ symmetry-breaking, and a bi-doublet Φ [29, 30, 31], responsible for the electroweak symmetry-breaking. Under gauge symmetry these two multiplets transform as:

$$R \to U_R R V^T \qquad \text{and} \qquad \Phi \to U_L \Phi U_R^\dagger \tag{3.2}$$

with $U_{L,R} \in SU(2)_{L,R}$ and $V \in SU(4)$. Let us recall that for the complex conjugate fields the notation is given by: $(R^{i\alpha})^* = R_{i\alpha}$ and $(\Phi^i_I)^* = \Phi^I_i$. Furthermore we can note that it is possible to build up another bi-doublet $\tilde{\Phi}$, using Φ and the Pauli's matrix σ_2:

$$\Phi^I_i = \begin{pmatrix} \phi^1_1 & \phi^1_2 \\ \phi^2_1 & \phi^2_2 \end{pmatrix} \quad \Longrightarrow \quad \tilde{\Phi}^I_i = \left(\sigma_{2L}\Phi^*\sigma_{2R}\right)^I_i = \begin{pmatrix} (\phi^2_2)^* & -(\phi^2_1)^* \\ -(\phi^1_2)^* & (\phi^1_1)^* \end{pmatrix}, \tag{3.3}$$

which transforms as Φ under gauge symmetry:

$$\tilde{\Phi} \to U_L \tilde{\Phi} U_R^\dagger\,. \tag{3.4}$$

3.1 Scalar potential

Since the only difference with the model 1 is the scalar content, the Lagrangian of the model 2 is practically similar to (2.11):

$$\mathcal{L} = \sum_{\mathbf{f}=1}^{3}\Bigg\{ i\bar{\Psi}_L^{(\mathbf{f})}\gamma^\mu D_\mu \Psi_L^{(\mathbf{f})} + i\bar{\Psi}_R^{(\mathbf{f})}\gamma^\mu D_\mu \Psi_R^{(\mathbf{f})} + i\bar{s_0}^{(\mathbf{f})}\gamma^\mu \partial_\mu s_0^{(\mathbf{f})} + i\bar{s_0^c}^{(\mathbf{f})}\gamma^\mu \partial_\mu s_0^{c\,(\mathbf{f})} - \\ - \Big[\bar{s_0^c}^{(\mathbf{f})} m_0^{(\mathbf{f})} s_0^{(\mathbf{f})} + h.c.\Big] + \mathcal{L}_{Yukawa}^{(\mathbf{f})} \Bigg\} - \mathbf{V}(R,\Phi) + \mathcal{L}_{gauge\,fixing} + \\ + D_\mu \Phi_i^I D^\mu \Phi_I^i + D_\mu R_{\alpha i} D^\mu R^{\alpha i} - \frac{1}{2}\mathbf{Tr}\Big[\mathbb{G}_{\mu\nu}\mathbb{G}^{\mu\nu} + \mathbb{W}_{L\mu\nu}\mathbb{W}_L^{\mu\nu} + \mathbb{W}_{R\mu\nu}\mathbb{W}_R^{\mu\nu}\Big]\,, \tag{3.5}$$

apart for the terms including the Higgs fields, that are the scalar potential, the scalar kinetic terms and the Yukawa interactions, from which we get a completely different phenomenology compared to the model 1.

Let us analyze the Higgs potential for the scalar multiplets R and Φ. We start looking for all possible singlets that can be generated with the representations $(4,1,2)$ and $(1,2,2)$. For what concerns the multiplet R we skip the analysis which has already been done in Sec. 2.1, where we obtained the structure of the terms containing only R (see (2.28)) and we know that its VEV is given by $\langle R^{14}\rangle = v \neq 0$. For the gauge invariant terms involving the bi-doublet, instead, we start from the quadratic terms given by the following tensor products:

$$(1,2,2)\otimes(1,\bar{2},\bar{2}) = (1, 3\oplus 1, 3\oplus 1) \ni (1,1,1) \sim \Phi_i^I \Phi_I^i\,, \tag{3.6}$$

where we have shown only the structure of the resulting singlet, and

$$(1,2,2)\otimes(1,2,2) = (1, 3\oplus 1, 3\oplus 1) \ni (1,1,1) \sim \Phi_i^I \epsilon^{ij}\epsilon_{IJ}\Phi_j^J = -\Phi_i^I \epsilon^{ij}\Phi_j^J \epsilon_{JI} = \\ = -\Phi_i^I (i\sigma_{2R})^{ij}\Phi_j^J (i\sigma_{2L})_{JI} = \Phi_i^I (\sigma_{2R})^{ij}(\Phi_J^j)^*(\sigma_{2L})_{JI} = \Phi_i^I \tilde{\Phi}_I^i\,, \tag{3.7}$$

with the related hermitian conjugate:

$$(1,\bar{2},\bar{2})\otimes(1,\bar{2},\bar{2}) \ni (1,1,1) \sim \tilde{\Phi}_i^I \Phi_I^i \tag{3.8}$$

(for more details about the tensor products in $SU(2)$ see Appendix B).

Since we cannot generate singlets with three representations $(1,2,2)$, we pass to the quartic terms in Φ and we find four possible singlets:

$$
\begin{aligned}
&(1,2,2)\otimes(1,2,2)\otimes(1,2,2)\otimes(1,2,2)=(1,3\oplus 1,3\oplus 1)\otimes(1,3\oplus 1,3\oplus 1)\ni\\
&\ni\begin{cases}
(1,1,1)\otimes(1,1,1)\sim \Phi^I_i\tilde{\Phi}^i_I\Phi^J_j\tilde{\Phi}^j_J\\
(1,3,1)\otimes(1,3,1)\ni(1,1,1)\sim H^{IJ}H^{MN}(\epsilon_{IM}\epsilon_{JN}+\epsilon_{IN}\epsilon_{JM})\\
(1,1,3)\otimes(1,1,3)\ni(1,1,1)\sim H_{ij}H_{mn}(\epsilon^{im}\epsilon^{jn}+\epsilon^{in}\epsilon^{jm})\\
(1,3,3)\otimes(1,3,3)\ni(1,1,1)\sim H^{IJ}_{ij}H^{MN}_{mn}(\epsilon^{im}\epsilon^{jn}+\epsilon^{in}\epsilon^{jm})(\epsilon_{IM}\epsilon_{JN}+\epsilon_{IN}\epsilon_{JM})
\end{cases}
\end{aligned}
\tag{3.9}
$$

where we used for the symmetric tensors the notation:

$$
\begin{cases}
(1,3,3)\sim H^{IJ}_{ij}=\frac{1}{2}\big(\Phi^I_i\Phi^J_j+\Phi^J_i\Phi^I_j\big)\\
(1,1,3)\sim H_{ij}=H^{IJ}_{ij}\epsilon_{IJ}\\
(1,3,1)\sim H^{IJ}=H^{IJ}_{ij}\epsilon^{ij}
\end{cases}
. \tag{3.10}
$$

All possible combinations of bi-doublets, giving rise to singlets, are listed in the following gauge-invariant trace relations:

$$\Phi^I_i\Phi^i_I=Tr[\Phi\Phi^\dagger]=Tr[\Phi^\dagger\Phi]\,; \tag{3.11a}$$

$$Tr[\Phi\Phi^\dagger]=Tr[\tilde{\Phi}\tilde{\Phi}^\dagger]\,; \tag{3.11b}$$

$$Tr[\Phi^\dagger\tilde{\Phi}]\cdot Tr[\tilde{\Phi}^\dagger\Phi]=2\cdot Tr[\Phi^\dagger\tilde{\Phi}\tilde{\Phi}^\dagger\Phi]=2\cdot Tr[\Phi\tilde{\Phi}^\dagger\tilde{\Phi}\Phi^\dagger]\,; \tag{3.11c}$$

$$Tr[\Phi^\dagger\Phi]\cdot Tr[\tilde{\Phi}^\dagger\Phi]=2\cdot Tr[\tilde{\Phi}^\dagger\Phi\Phi^\dagger\Phi]\,; \tag{3.11d}$$

$$Tr[\Phi^\dagger\Phi\Phi^\dagger\Phi]=\Big(Tr[\Phi^\dagger\Phi]\Big)^2-Tr[\tilde{\Phi}^\dagger\tilde{\Phi}\Phi^\dagger\Phi]\,; \tag{3.11e}$$

$$Tr[\tilde{\Phi}^\dagger\tilde{\Phi}\tilde{\Phi}^\dagger\Phi]=Tr[\tilde{\Phi}^\dagger\Phi\Phi^\dagger\Phi]\,. \tag{3.11f}$$

It remains to consider the mixing between R and Φ. It is easy to understand that this kind of terms have to be quartic in the fields in order to produce a singlet; in particular, in order to have the contraction of all the indices we need two multiplets R, the only multiplet with the index of $SU(4)$, and two bi-doublets Φ, the only multiplet with the $SU(2)_L$ index. An example of this type of quartic term is:

$$
\begin{aligned}
&(4,1,2)\otimes(\bar{4},1,2)\otimes(1,2,2)\otimes(1,2,2)=(1,1,3\oplus 1)\otimes(1,1,3\oplus 1,3\oplus 1)\ni\\
&\ni\begin{cases}
(1,1,1)\otimes(1,1,1)\sim R^{i\alpha}\epsilon_{ik}R^k_\alpha\Phi^j_J\tilde{\Phi}^J_j\\
(1,1,3)\otimes(1,1,3)\ni(1,1,1)\sim K^{ij}H^{mn}(\epsilon_{im}\epsilon_{jn}+\epsilon_{in}\epsilon_{jm})
\end{cases}
\end{aligned}
\tag{3.12}
$$

where $R^i_\alpha = \epsilon^{ij} R_{j\alpha}$ and $K^{ij} = \frac{1}{2}\big(R^{i\alpha}\epsilon^{jk}R_{k\alpha} + R^{j\alpha}\epsilon^{ik}R_{k\alpha}\big)$ is a symmetric tensor. The complete list of gauge-invariant traces with R-Φ mixing is given by:

$$Tr[RR^\dagger]\cdot Tr[\Phi\Phi^\dagger]\,; \tag{3.13a}$$

$$Tr[RR^\dagger\Phi^\dagger\Phi]\,; \tag{3.13b}$$

$$Tr[RR^\dagger]\Big(Tr[\Phi^\dagger\tilde{\Phi}] + h.c.\Big)\,; \tag{3.13c}$$

$$Tr[RR^\dagger\Phi^\dagger\tilde{\Phi}] + h.c.\,; \tag{3.13d}$$

$$Tr[RR^\dagger\tilde{\Phi}^\dagger\tilde{\Phi}]\,. \tag{3.13e}$$

In this way we arrive to the final result for the most general gauge-invariant scalar potential that has the form:

$$\begin{aligned}
\mathbf{V}(R,\Phi) = &-\mu_1^2 R_{\alpha i}R^{\alpha i} + \eta_1 R_{\alpha i}R^{\alpha i}R_{\beta j}R^{\beta j} + \eta_2 R_{\alpha i}R^{\beta i}R_{\beta j}R^{\alpha j} \\
&-\mu_2^2 \Phi^i_I\Phi^I_i + \eta_3\Phi^i_I\Phi^I_i\Phi^j_J\Phi^J_j + \eta_4\Phi^i_I\Phi^I_j\Phi^j_J\Phi^J_i \\
&-\mu_3^2\Big(\Phi^i_I\epsilon^{IJ}\epsilon_{ij}\Phi^j_J + h.c.\Big) + \eta_5\Big(\Phi^i_I\epsilon^{IJ}\epsilon_{ij}\Phi^j_J\Phi^l_L\epsilon^{LR}\epsilon_{lr}\Phi^r_R + h.c.\Big) \\
&+\eta_6\Big(\Phi^I_i\Phi^j_I\Phi^K_j\epsilon_{KL}\epsilon^{il}\Phi^L_l + h.c.\Big) \\
&+\xi_1 R_{\alpha i}R^{\alpha i}\Phi^j_J\Phi^J_j + \xi_2 R_{\alpha i}R^{\alpha j}\Phi^i_I\Phi^I_j + \xi_3 R_{\alpha i}R^{\alpha i}\Big(\Phi^i_I\epsilon^{IJ}\epsilon_{ij}\Phi^j_J + h.c.\Big) \\
&+\xi_4\Big(R_{\alpha i}R^{\alpha j}\Phi^I_j\epsilon_{IJ}\epsilon^{ik}\Phi^J_k + h.c.\Big) + \xi_5 R_{\alpha i}R^{\alpha j}\epsilon^{IL}\epsilon_{jl}\Phi^l_L\epsilon_{IR}\epsilon^{ir}\Phi^R_r\,.
\end{aligned} \tag{3.14}$$

What remains to understand is how the symmetry breaking to SM realizes, therefore which is the vacuum expectation value for the bi-doublet Φ.

3.1.1 VEV of the bi-doublet

In order to evaluate the structure of the VEV of Φ, we start to consider the part of the potential (3.14) involving Φ only:

$$\begin{aligned}
\mathbf{V}(\Phi) = &-\mu_2^2\, Tr[\Phi^\dagger\Phi] + \eta_3\Big(Tr[\Phi^\dagger\Phi]\Big)^2 + \eta_4\, Tr[\Phi^\dagger\Phi\Phi^\dagger\Phi] \\
&-\mu_3^2\Big[Tr[\Phi^\dagger\tilde{\Phi}] + h.c.\Big] + \eta_5\Big[\Big(Tr[\Phi^\dagger\tilde{\Phi}]\Big) + h.c.\Big] \\
&+\eta_6\Big[Tr[\Phi^\dagger\Phi\Phi^\dagger\tilde{\Phi}] + h.c.\Big]\,,
\end{aligned} \tag{3.15}$$

and we look for the minimum of this potential calculating its first derivative:

$$\begin{aligned}\frac{\partial\mathbf{V}(\Phi)}{\partial\Phi^{Ii}} =& \Big[-\mu_2^2+2\big(\eta_3+\eta_4\big)|\Phi|^2+\eta_6\big(\det[\Phi]+\det[\Phi^\dagger]\big)\Big]\Phi_{Ii}\\ &+\Big[-2\mu_3^2+\eta_6|\Phi|^2+2\big(4\eta_5\det[\Phi]-\eta_4\det[\Phi^\dagger]\big)\Big]\epsilon_{IJ}\epsilon_{ij}\Phi^{Jj}\,.\end{aligned} \tag{3.16}$$

Asking that (3.16) is equal to zero when Φ gets VEV, we find that the most general cases with $\langle\Phi\rangle\neq 0$ are given by the following two possibilities:

$$\langle\Phi^{Ii}\rangle = \begin{cases}\begin{pmatrix}\langle\Phi^{11}\rangle & 0\\ 0 & \langle\Phi^{22}\rangle\end{pmatrix} & \text{with}\quad \langle\Phi^{11}\rangle\,,\langle\Phi^{22}\rangle\neq 0\\ \\ \begin{pmatrix}0 & \langle\Phi^{12}\rangle\\ \langle\Phi^{21}\rangle & 0\end{pmatrix} & \text{with}\quad \langle\Phi^{12}\rangle\,,\langle\Phi^{21}\rangle\neq 0\end{cases}\,. \tag{3.17}$$

In order to discriminate between these two cases we use another request, that is we ask $\langle\Phi\rangle$ to preserve the electric charge; so using the definition in (2.14) we find:

$$Q\langle\Phi^{Ii}\rangle = \begin{pmatrix}\langle\Phi^{11}\rangle & 0\\ 0 & -\langle\Phi^{22}\rangle\end{pmatrix} = 0 \quad\Longrightarrow\quad \langle\Phi^{Ii}\rangle = \begin{pmatrix}0 & u_1\\ -u_2 & 0\end{pmatrix}\,, \tag{3.18}$$

which is consistent with the second of (3.17). We assume real VEV, so $u_1, u_2 \in \mathbb{R}$, as in the previous model 1. At this point, inserting this VEV in (3.16) we find the relations:

$$\Big[\frac{\partial\mathbf{V}(\Phi)}{\partial\Phi^{Ii}}\Big]_{VEV} = 0 \qquad\Longrightarrow\qquad \begin{cases}|\Phi|^2 = u_1^2+u_2^2 = \frac{(4\eta_5-\eta_4)\mu_2^2-2\eta_6\mu_3^2}{2(\eta_3+\eta_4)(4\eta_5-\eta_4)-\eta_6^2}\\ \det[\Phi] = u_1u_2 = \frac{4(\eta_3+\eta_4)\mu_3^2-\eta_6\mu_2^2}{4(\eta_3+\eta_4)(4\eta_5-\eta_4)-2\eta_6^2}\end{cases}\,. \tag{3.19}$$

Moving the indices of the bi-doublet, we define the VEV as:

$$\langle\Phi^I_i\rangle = \epsilon_{ij}\langle\Phi^{Ij}\rangle = \begin{pmatrix}u_1 & 0\\ 0 & u_2\end{pmatrix}\,, \tag{3.20}$$

with

$$\sqrt{|u_1|^2+|u_2|^2} = \frac{v_L}{\sqrt{2}} = 174\,\text{GeV}\,. \tag{3.21}$$

To complete the analysis of the bi-doublet, we show its composition in terms of complex fields:

$$\Phi^{Ii} = \begin{pmatrix} (\Phi^{11})^+ & (\Phi^{12})^0 \\ (\Phi^{21})^0 & (\Phi^{22})^- \end{pmatrix} \implies \Phi_i^I = \begin{pmatrix} \phi_1^0 & \phi_2^+ \\ \phi_1^- & \phi_2^0 \end{pmatrix} \quad \text{and} \quad \tilde{\Phi}_i^I = \begin{pmatrix} \phi_2^{0*} & -\phi_1^+ \\ -\phi_2^- & \phi_1^{0*} \end{pmatrix}. \tag{3.22}$$

Here we have to pay attention to the way we move the indices, in particular we have that:

$$\epsilon_{IJ}\epsilon_{ij}\Phi^{Jj} = \hat{\Phi}_{Ii} \neq \Phi_{Ij} = (\Phi^{Ii})^* \qquad \text{and} \qquad \Phi_i^I = \epsilon_{ij}\Phi^{Ij} \neq \epsilon^{IJ}\Phi_{Ji} = (\Phi^*)_i^I\,. \tag{3.23}$$

3.1.2 Minimum of the potential and Higgs masses

To get the minimum of the complete potential (3.14) we evaluate, as usual, the zeros of its first derivatives when the Higgs fields get VEVs; so from the equations:

$$\Big[\frac{\partial \mathbf{V}(R,\Phi)}{\partial \Phi_1^1}\Big]_{VEV} = 0\,, \quad \Big[\frac{\partial \mathbf{V}(R,\Phi)}{\partial \Phi_2^2}\Big]_{VEV} = 0 \quad \text{and} \quad \Big[\frac{\partial \mathbf{V}(R,\Phi)}{\partial R^{14}}\Big]_{VEV} = 0 \tag{3.24}$$

we obtain the conditions:

$$\implies \begin{cases} \mu_1^2 = 2(\eta_1+\eta_2)v^2 + (\xi_1+\xi_2)u_1^2 + (\xi_1+\xi_5)u_2^2 + 2(2\xi_3+\xi_4)u_1u_2 \\ \mu_2^2 = 2(\eta_3+\eta_4)(u_1^2+u_2^2) + 2\eta_6 u_1u_2 + (\xi_1+\xi_2)\frac{v^2u_1^2}{u_1^2-u_2^2} - (\xi_1+\xi_5)\frac{v^2u_2^2}{u_1^2-u_2^2} \\ \mu_3^2 = (2\eta_5-\eta_4)u_1u_2 + \eta_6\frac{u_1^2+u_2^2}{2} + (2\xi_3+\xi_4)\frac{v^2}{2} - (\xi_2-\xi_5)\frac{u_1u_2}{u_1^2-u_2^2}\frac{v^2}{2} \end{cases} \tag{3.25}$$

where we assumed $u_1 \neq u_2$. Inserting the constraints (3.25) in the potential (3.14) we obtain the minimum:

$$\begin{aligned} \mathbf{V}_{min} = &-(\eta_1+\eta_2)v^4 - (\eta_3+\eta_4)(u_1^4+u_2^4) - 2\eta_3u_1^2u_2^2 - 2\eta_6u_1u_2(u_1^2+u_2^2) \\ &-(\xi_1+\xi_2)v^2u_1^2 - (\xi_1+\xi_5)v^2u_2^2 - 2(2\xi_3+\xi_4)v^2u_1u_2\,. \end{aligned} \tag{3.26}$$

Concerning the Higgs masses instead, since we need to evaluate the terms proportional to $H^\dagger M_H^2 H$, we have to consider the second derivatives of the potential (3.14):

$$\Big[\frac{\partial^2 \mathbf{V}(R,\Phi)}{\partial \Phi_i^I \partial \Phi_J^j}\Big]_{VEV}, \quad \Big[\frac{\partial^2 \mathbf{V}(R,\Phi)}{\partial R^{i\alpha}\partial R_{j\beta}}\Big]_{VEV} \quad \text{and} \quad \Big[\frac{\partial^2 \mathbf{V}(R,\Phi)}{\partial \Phi_i^I \partial R_{j\beta}}\Big]_{VEV}. \tag{3.27}$$

For the Higgs R_u^a (with $a = 1,2,3$ the colour index of $SU(3)_C$), related to the up-type quark sector, we find $M_{R_u}^2 = 0$; so, since we have three copies of R_u^a and these are complex fields, they can be interpreted as the six Goldstone bosons which give mass to the three leptoquarks $X^{\pm}_{i\mu}$, as we will see in Sec. 3.3. For the Higgs fields R^a_d (with $a = 1, 2, 3$), related to the down-type quark sector, instead we find the squared mass:

$$M^2_{R_d} = (\xi_5 - \xi_2)(u_1^2 - u_2^2) - 2\eta_2 v^2 \,. \tag{3.28}$$

Another scalar field decoupled from the others is the charged Higgs field Φ_2^+, for which we find the mass:

$$M^2_{\phi_2^\pm} = (\xi_5 - \xi_2)\frac{v^2 u_1^2}{u_1^2 - u_2^2} \,. \tag{3.29}$$

At this point, since the partial-unification scale has to be higer than the electroweak scale we assume $v \gg u_1 > u_2$ and asking to have positive squared Higgs masses, we find the constraints:

$$\begin{cases} \xi_5 - \xi_2 > 0 \\ \eta_2 < 0 \end{cases} . \tag{3.30}$$

The last two charged Higgs fields, Φ_1^- and R_e, mix each other in the squared mass matrix:

$$M^2_{H_{\pm e}} = (\xi_5 - \xi_2)\begin{pmatrix} \frac{v^2 u_2^2}{u_1^2 - u_2^2} & -vu_2 \\ -vu_2 & u_1^2 - u_2^2 \end{pmatrix} , \tag{3.31}$$

in the basis (ϕ_1^-, R_e). The matrix $M^2_{H_{\pm e}}$ can be diagonalized using the orthogonal matrix:

$$O_{H\pm} = \frac{1}{\sqrt{|v^2 u_2^2 - (u_1^2 - u_2^2)^2|}}\begin{pmatrix} vu_2 & -(u_2^2 - u_1^2) \\ u_2^2 - u_1^2 & vu_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_{H\pm} & -\sin\theta_{H\pm} \\ \sin\theta_{H\pm} & \cos\theta_{H\pm} \end{pmatrix} , \tag{3.32}$$

obtaining:

$$(M^2_{H_{\pm e}})^{diag} = O^T_{H\pm} M^2_{H_{\pm e}} O_{H\pm} = \begin{pmatrix} M^2_{H_1^\pm} & 0 \\ 0 & M^2_{H_2^\pm} \end{pmatrix} = (\xi_5 - \xi_2)\begin{pmatrix} u_1^2 - u_2^2 + \frac{v^2 u_2^2}{u_1^2 - u_2^2} & 0 \\ 0 & 0 \end{pmatrix} \tag{3.33}$$

where the zero eigenvalues are related to the Goldstone bosons that give mass to the gauge bosons $W^{\pm}_{\mu R}$. Instead, for the Goldstone bosons needed to give mass to the vector bosons $W^{\pm}_{\mu L}$ we can use a combination of the imaginary parts of the fields: $\mathbf{Im}[\phi_2^-]$ and $\mathbf{Im}[\phi_1^- \cos\theta_{H\pm} - R_e \sin\theta_{H\pm}]$, which can be considered massless, while their real parts remain massive.

Last we have the squared mass matrix for the neutral Higgs, in the basis $(\phi_1^0, \phi_2^0, R_\nu)$, which is given by:

$$M^2_{H_{0\nu}} = \begin{pmatrix} A & 2\eta_3 u_1 u_2 + \eta_6(u_1^2+u_2^2) & D \\ 2\eta_3 u_1 u_2 + \eta_6(u_1^2+u_2^2) & B & C \\ D & C & 2(\eta_1+\eta_2)v^2 \end{pmatrix} \tag{3.34}$$

with

$$\begin{cases} A = 2(\eta_3+\eta_4)u_1^2 - 2\eta_4 u_2^2 + 2\eta_6 u_1 u_2 + (\xi_5-\xi_2)\frac{v^2u_2^2}{u_1^2-u_2^2} \\ B = 2(\eta_3+\eta_4)u_2^2 - 2\eta_4 u_1^2 + 2\eta_6 u_1 u_2 + (\xi_5-\xi_2)\frac{v^2u_1^2}{u_1^2-u_2^2} \\ C = (\xi_1+\xi_5)vu_2 + (2\xi_3+\xi_4)vu_1 \\ D = (\xi_1+\xi_2)vu_1 + (2\xi_3+\xi_4)vu_2 \end{cases} . \tag{3.35}$$

In order to simplify the analytical computation of the eigenvalues of $M^2_{H_{0\nu}}$, we limit our attention to the following particular case:

$$\eta_3 = \eta_4 = \eta_6 = 2\xi_3 + \xi_4 = \xi_1 + \xi_2 = 0\,; \tag{3.36}$$

as we can see this choice does not involve the masses for the charged Higgs found above. With these constraints we obtain, for the neutral Higgs, the following squared mass matrix:

$$M^2_{H_{0\nu}} \to (\xi_5-\xi_2)v^2 \begin{pmatrix} \frac{u_2^2}{u_1^2-u_2^2} & 0 & 0 \\ 0 & \frac{u_1^2}{u_1^2-u_2^2} & \frac{u_2}{v} \\ 0 & \frac{u_2}{v} & 2\frac{\eta_1+\eta_2}{\xi_5-\xi_2} \end{pmatrix}, \tag{3.37}$$

which gives the eigenvalue:

$$M^2_{H0} = (\xi_5-\xi_2)\frac{v^2u_2^2}{u_1^2-u_2^2}\,, \tag{3.38}$$

and, from the 2×2 submatrix of (3.37), the other two eigenvalues:

$$M^2_{H\pm} = \frac{v^2}{2(u_1^2-u_2^2)}\Big[2(\eta_1+\eta_2)(u_1^2-u_2^2) + (\xi_5-\xi_2)u_1^2 \pm \frac{\Delta}{v^2}\Big] \tag{3.39}$$

with

$$\Delta^2 = v^4\Big[2(\eta_1+\eta_2)(u_1^2-u_2^2) - (\xi_5-\xi_2)u_1^2\Big]^2 + v^2\Big[2(\xi_5-\xi_2)(u_1^2-u_2^2)u_2\Big]^2. \tag{3.40}$$

Assuming again $v \gg u_1 > u_2$, we find for the eigenvalues in (3.39):

$$\begin{aligned} M_{H+}^2 &\approx 2(\eta_1+\eta_2)v^2 + \frac{(\xi_5-\xi_2)^2(u_1^2-u_2^2)u_2^2}{2(\eta_1+\eta_2)(u_1^2-u_2^2)-(\xi_5-\xi_2)u_1^2} \\ M_{H-}^2 &\approx \frac{(\xi_5-\xi_2)v^2u_1^2}{u_1^2-u_2^2} - \frac{(\xi_5-\xi_2)^2(u_1^2-u_2^2)u_2^2}{2(\eta_1+\eta_2)(u_1^2-u_2^2)-(\xi_5-\xi_2)u_1^2} \end{aligned} \quad , \tag{3.41}$$

while the orthogonal matrix is given by:

$$\begin{aligned} O_{H0} &= \frac{E'}{(\xi_5-\xi_2)u_2v}\begin{pmatrix} (\xi_5-\xi_2)u_2v & m_+^2 - 2(\eta_1+\eta_2)v^2 \\ m_-^2 - \frac{(\xi_5-\xi_2)u_1^2v^2}{u_1^2-u_2^2} & (\xi_5-\xi_2)u_2v \end{pmatrix} \\ &\approx \frac{E}{(\xi_5-\xi_2)u_2v}\begin{pmatrix} (\xi_5-\xi_2)u_2v & -\frac{(\xi_5-\xi_2)(u_1^2-u_2^2)u_2^2}{(\xi_5-\xi_2)u_1^2-2(\eta_1+\eta_2)(u_1^2-u_2^2)} \\ \frac{(\xi_5-\xi_2)(u_1^2-u_2^2)u_2^2}{(\xi_5-\xi_2)u_1^2-2(\eta_1+\eta_2)(u_1^2-u_2^2)} & (\xi_5-\xi_2)u_2v \end{pmatrix} \end{aligned} \tag{3.42}$$

with

$$E = \frac{(\xi_5-\xi_2)u_1^2v - 2(\eta_1+\eta_2)(u_1^2-u_2^2)v}{\sqrt{v^2\Big[(\xi_5-\xi_2)u_1^2 - 2(\eta_1+\eta_2)(u_1^2-u_2^2)\Big]^2 + (\xi_5-\xi_2)^2(u_1^2-u_2^2)u_1^2u_2^2}} \, . \tag{3.43}$$

We note that using the assumption $v \gg u_1 > u_2$ directly on (3.34) we would have found:

$$M_{H_{0\nu}}^2 \approx (\xi_5-\xi_2)v^2 \begin{pmatrix} \frac{u_2^2}{u_1^2-u_2^2} & 0 & 0 \\ 0 & \frac{u_1^2}{u_1^2-u_2^2} & 0 \\ 0 & 0 & 2\frac{\eta_1+\eta_2}{\xi_5-\xi_2} \end{pmatrix} \tag{3.44}$$

which are quite the same values of the eigenvalues obtained from (3.37), with the fields ϕ_1^0, ϕ_2^0 and R_ν all decoupled. As a last consideration we note that, under the assumption $v \gg u_1 > u_2$, we have to interpret M_{H0}^2 as the squared mass of the Standard Model Higgs, while the other two masses, M_{H+}^2 and M_{H-}^2, have to be at an higher energy scale.

Interlude on the case: $u_1 = u_2 = u$

In the particular case where the two VEVs of the bi-doublet are equal, $\langle\phi_1^0\rangle = \langle\phi_2^0\rangle$, from the derivatives in (3.24) we obtain a different set of constraints given by:

$$\begin{cases} \xi_2 = \xi_5 \\ \mu_1^2 = 2v^2(\eta_1 + \eta_2) + 2u^2(\xi_1 + 2\xi_3 + \xi_4 + \xi_5) \\ \mu_2^2 = -2\mu_3^2 + 2u^2(2\eta_3 + \eta_4 + 2\eta_5 + 2\eta_6) + v^2(\xi_1 + 2\xi_3 + \xi_4 + \xi_5) \end{cases} , \tag{3.45}$$

while μ_3 remains a free parameter whose natural scale will be the Pati-Salam scale, so $\mu_3 \sim v$. Consequently they produce a new minimum of the potential (3.14):

$$\mathbf{V'}_{min} = -(\eta_1 + \eta_2)v^4 - 2(2\eta_3 + \eta_4 + 2\eta_6)u^4 - 2(\xi_1 + 2\xi_3 + \xi_4 + \xi_5)u^2v^2 \,. \tag{3.46}$$

The same happens for the Higgs masses. In fact, even if for the Higgs R_u^a we continue to have $M^2_{R_u} = 0$, the masses of the Higgs R_d^a are now given by:

$$M^2_{R_d} = -2\eta_2 v^2 \,. \tag{3.47}$$

The rest of the charged Higgs are, in this case, totally decoupled from each other; in particular we have that, since $M^2_{R_e} = 0$, the complex Higgs field R_e, related to the charged lepton sector, can be divided in two Goldtone bosons: $\mathbf{Re}[R_e]$ and $\mathbf{Im}[R_e]$. While the other two charged Higgs, $\phi_1^\pm$ and $\phi_2^\pm$, get the same squared mass value:

$$M^2_{\phi_1^\pm} = M^2_{\phi_2^\pm} = 2(\eta_4 - 2\eta_5 - \eta_6)u^2 + 2\mu_3^2 - (2\xi_3 + \xi_4)v^2 \,. \tag{3.48}$$

Finally, for the neutral Higgs we get, always in the basis $(\phi_1^0, \phi_2^0, R_\nu)$, the squared mass matrix:

$$M^2_{H_{0\nu}} = \begin{pmatrix} F & 2u^2(\eta_3 + \eta_6) & uv(\xi_1 + 2\xi_3 + \xi_4 + \xi_5) \\ 2u^2(\eta_3 + \eta_6) & F & uv(\xi_1 + 2\xi_3 + \xi_4 + \xi_5) \\ uv(\xi_1 + 2\xi_3 + \xi_4 + \xi_5) & uv(\xi_1 + 2\xi_3 + \xi_4 + \xi_5) & 2v^2(\eta_1 + \eta_2) \end{pmatrix} , \tag{3.49}$$

with

$$F = 2u^2(\eta_3 + \eta_4 - 2\eta_5) + 2\mu_3^2 - v^2(2\xi_3 + \xi_4) \,, \tag{3.50}$$

from which we obtain the following eigenvalues:

$$\begin{cases} M^2_{H1} = 2\mu_3^2 + 2(\eta_4 - 2\eta_5 - 2\eta_6)u^2 - (2\xi_3 + \xi_4)v^2 \\ M^2_{H2} = 2\mu_3^2 - (2\xi_3 + \xi_4)v^2 + o(u^2) \\ M^2_{H3} = 2(\eta_1 + \eta_2)v^2 + o(u^2) \end{cases} . \tag{3.51}$$

We note that, while the eigenvalue M^2_{H3} is of order $O(v^2)$, in order to generate a squared mass for the Standard Model Higgs of order $O(u^2)$ we have to demand:

$$\mu_3^2 = (2\xi_3 + \xi_4)\frac{v^2}{2} \, . \tag{3.52}$$

However in this case we obtain two eigenvalues proportional to the electro-weak scale: $M^2_{H1}, M^2_{H2} \sim u^2$. So we have to make a further choice on the scalar parameters (in this case on η_i and ξ_i), in order to send to zero one of the two eigenvalues, M^2_{H1} or M^2_{H2}, producing two Goldstone bosons (because the eigenvalues are related to complex scalar fields) which will be eaten to give mass to the vector bosons Z_μ and Z'_μ, while the remaining non-zero eigenvalue will give the SM Higgs squared mass.

3.2 RGE for the gauge coupling constants

In order to compare the model 2 with the model 1, we assume again that the coupling constants of the $SU(2)_L$ and $SU(2)_R$ groups are the same at the partial unification scale M_U, so $g_L(M_U) = g_R(M_U) = g_2(M_U)$. Through the symmetry breaking from the Pati-Salam group to the SM one the bi-doublet switches in two doublets in the following way:

$$\begin{aligned} SU(4) \otimes SU(2)_L \otimes SU(2)_R &\xrightarrow{\langle R_\nu \rangle} SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \\ (2,2,1) \sim \Phi, \tilde{\Phi} &\xrightarrow{\langle R_\nu \rangle} \begin{pmatrix} \phi_1^0 \\ \phi_1^- \end{pmatrix}, \begin{pmatrix} \phi_2^{0*} \\ -\phi_2^- \end{pmatrix} \sim (1,2,-1) \, . \end{aligned} \tag{3.53}$$

This is a situation similar to a Two Higgs Doublet Model (2HDM) [32], in which the scalar potential has the form:

$$\begin{aligned} V_{2HDM} =& \zeta_1^2 \Phi_1^\dagger \Phi_1 + \zeta_1^2 \Phi_1^\dagger \Phi_1 - \zeta_3^2 \left(\Phi_1^\dagger \Phi_2 + h.c. \right) + \frac{\lambda_1}{2} \left(\Phi_1^\dagger \Phi_1 \right)^2 + \frac{\lambda_2}{2} \left(\Phi_2^\dagger \Phi_2 \right)^2 \\ &+ \lambda_3 \Phi_1^\dagger \Phi_1 \Phi_2^\dagger \Phi_2 + \lambda_4 \Phi_1^\dagger \Phi_2 \Phi_2^\dagger \Phi_1 + \frac{\lambda_5}{2} \left[\left(\Phi_1^\dagger \Phi_2 \right)^2 + h.c. \right] \end{aligned} \tag{3.54}$$

where the VEVs are given by:

$$\langle \Phi_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 \end{pmatrix} \qquad \text{and} \qquad \langle \Phi_2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \tag{3.55}$$

with

$$\sqrt{v_1^2 + v_2^2} = 246\,\text{GeV}\,. \tag{3.56}$$

We suppose therefore that the evolution of the coupling constants of the Standard Model from the electro-weak energy scale, $\mu = M_Z$, to the Pati-Salam scale M_U is given in a 2HDM, without intermediate scales; just a note, we can see in Fig. 3.1 that the difference between the running of the gauge coupling constants in the Standard Model or in the Two Higgs Doublet Model is very small.

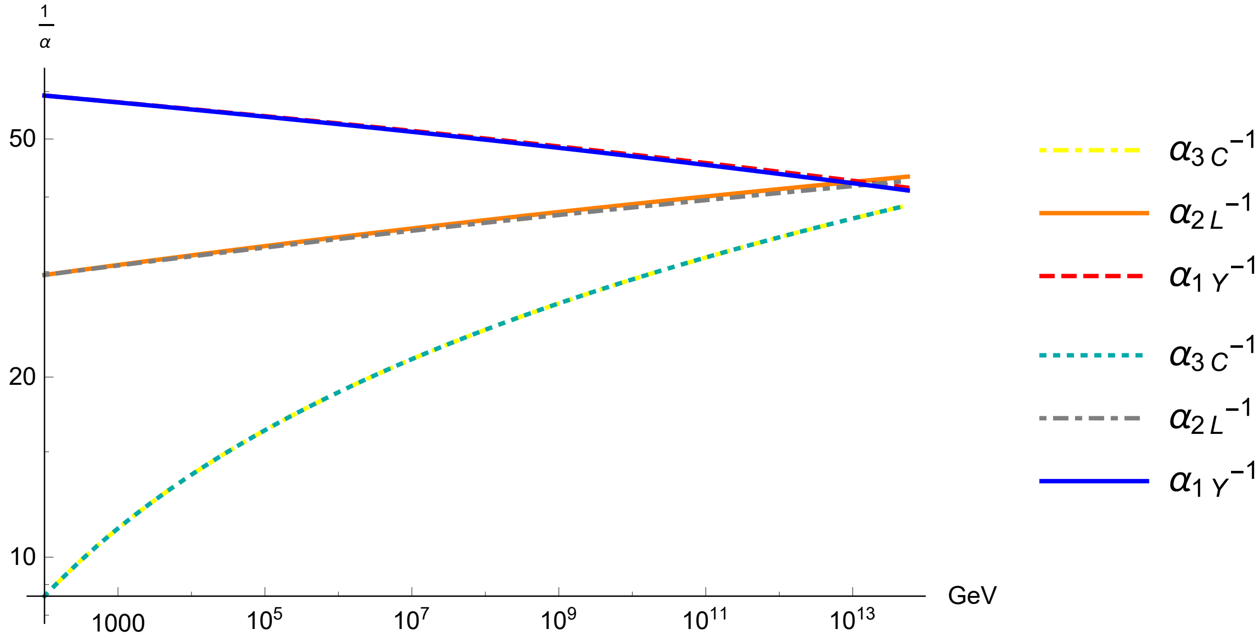


Figure 3.1: *Running of the gauge coupling constants for the $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$ groups in the Standard Model, in yellow dot-dashed, orange solid and red dashed lines, respectively, and the 2 Higgs Doublet Model, in cyan dotted, grey dot-dashed and blue solid lines, respectively.*

In this case we have that the coefficients (2.111) of the running coupling constants (2.110) take the values:

$$a_{3C} = -7\,, \quad a_{2L} = -3 \quad \text{and} \quad a_{1Y} = \frac{21}{5}\,; \tag{3.57}$$

while the matching conditions with the Pati-Salam gauge coupling constants are given by:

$$\begin{cases} \alpha_{3C}^{-1}(M_U) = \alpha_{3C}^{-1}(M_Z) + \frac{7}{2\pi}\ln\frac{M_U}{M_Z} = \alpha_{4C}^{-1}(M_U) \\ \alpha_{2L}(M_U) = \alpha_{2L}^{-1}(M_Z) + \frac{3}{2\pi}\ln\frac{M_U}{M_Z} = \alpha_{2R}^{-1}(M_U) \\ \alpha_{1Y}^{-1}(M_U) = \alpha_{1Y}^{-1}(M_Z) - \frac{21}{10\pi}\ln\frac{M_U}{M_Z} = \frac{3}{5}\alpha_{2R}^{-1}(M_U) + \frac{2}{5}\alpha_{4C}^{-1}(M_U) \end{cases} \tag{3.58}$$

where the value of the α_i at the electro-weak scale are listed in (2.118). From the system (3.58) we can evaluate the value of the partial unification scale $M_U \approx 5.61 \cdot 10^{13}$ GeV, practically the same order as for the model 1. Therefore we assign to the Higgs multiplet $R^{i\alpha}$ the same VEV used in the model 1, that is:

$$\langle R_\nu \rangle = v = \frac{v_R}{\sqrt{2}} = \frac{10^{14}}{\sqrt{2}}\,\text{GeV}\,. \tag{3.59}$$

At the partial-unification scale the values of the gauge couplings are:

$$g_4(M_U) = \sqrt{4\pi\alpha_{4C}(M_U)} \approx 0.5695 \quad \text{and} \quad g_2(M_U) = \sqrt{4\pi\alpha_{2L(R)}(M_U)} \approx 0.5435\,. \quad (3.60)$$

For the running above the Pati-Salam scale, shown in Fig. 3.2, we have the coefficients:

$$a_{4C} = -\frac{31}{3}\,, \quad a_{2R} = -\frac{7}{3} \quad \text{and} \quad a_{2L} = -3\,. \quad (3.61)$$

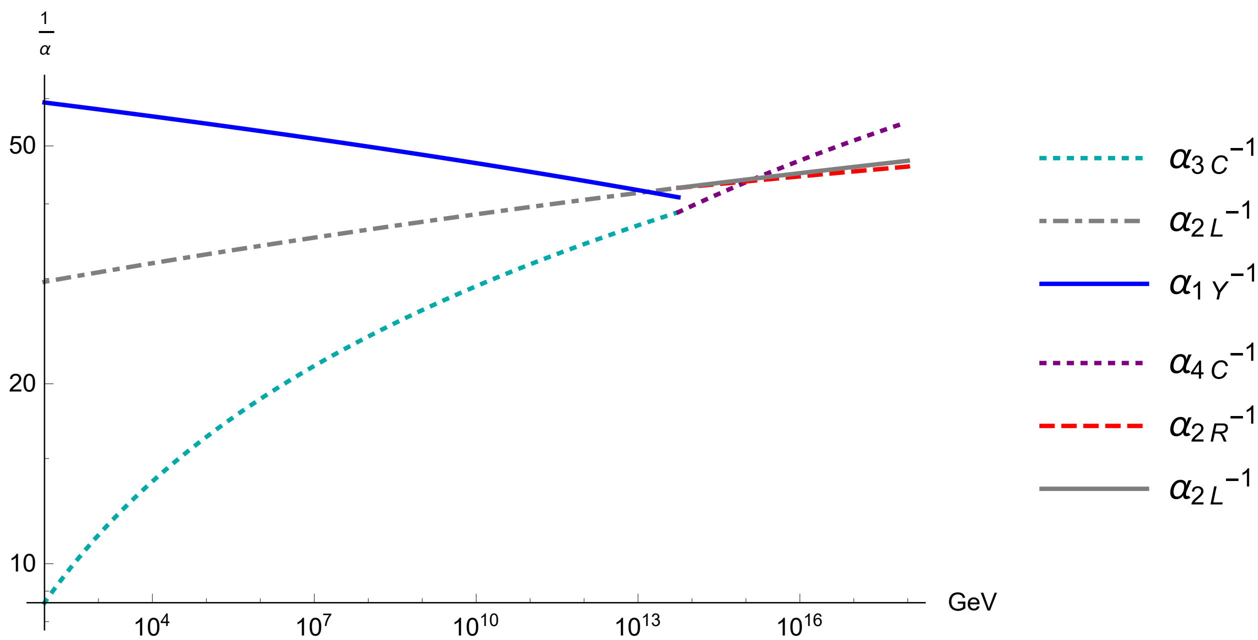


Figure 3.2: *Running of the gauge coupling constants in 2HDM and Pati-Salam model through the partial-unification scale. The cyan dotted, grey dot-dashed and blue solid lines represent the evolutions of the coupling constants for the groups $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$, respectively; the purple dotted, red dashed and grey solid lines represent the evolutions of the coupling constants for the groups $SU(4)$, $SU(2)_R$ and $SU(2)_L$, respectively*

3.2.1 Evaluation of $\tan\beta$

It remains to find the value of the ratio between the two VEVs of the bi-doublet:

$$0 \leq \tan\beta = |\frac{u_2}{u_1}| < 1\,. \quad (3.62)$$

In order to evaluate it we have two constraints: the first one is given by the experimental SM Higgs mass ($m_h^{exp} \approx 125$ GeV). In fact, since we want to associate the mass M_{H0} in (3.38) to the SM Higgs mass, if we make a simple calculation on orders of magnitude, assuming $\xi_5 - \xi_2 \sim O(10)$, we can extrapolate for $\tan\beta$ the value:

$$M_{H0} = \sqrt{\frac{(\xi_5 - \xi_2)v^2 \tan^2\beta}{1 - \tan^2\beta}} \sim O(m_h^{exp}) \quad \Longrightarrow \quad \tan\beta \sim O(10^{-12})\,. \tag{3.63}$$

The second costraint to take into account, instead, is given by the experimental limit for a charged Higgs [33]. The non-zero eigenvalue of the charged Higgs squared mass matrix (3.33) has to verify the relation:

$$M_{H_1^\pm} = \sqrt{(\xi_5 - \xi_2)v^2 \Big[\frac{u_1^2}{v^2}(1 - \tan^2\beta) + \frac{\tan^2\beta}{1 - \tan^2\beta}\Big]} > 650\,\text{GeV}\,, \tag{3.64}$$

from which, assuming a very small $\tan\beta$, we get the condition:

$$\tan\beta \to 0 \quad \Longrightarrow \quad u_2 \to 0 \quad \text{and} \quad u_1 \to \frac{246}{\sqrt{2}}\,\text{GeV} \quad \Longrightarrow \quad \xi_5 - \xi_2 > 14 \tag{3.65}$$

that is consistent with the order of magnitude used above for the scalar parameters.

3.3 Gauge boson masses

From the kinetic terms of the Higgs sector in (3.5), written using the covariant derivative:

$$\begin{aligned} D_\mu \Phi_i^I D^\mu \Phi_I^i \; &= \; \left(\delta_i^j \delta_J^I \partial_\mu - i g_R \delta_J^I W_{\mu R i}^j + i g_L \delta_i^j W_{\mu L J}^I\right) \Phi_j^J \times \\ & \quad\; \left(\delta_k^i \delta_k^K \partial^\mu - i g_L \delta_k^i W_{\mu L I}^K + i g_R \delta_I^K W_{Rk}^{\mu i}\right) \Phi_K^k \end{aligned} \tag{3.66}$$

$$\begin{aligned} D_\mu R_{i\alpha} D^\mu R^{i\alpha} \; &= \; \left(\delta_i^j \delta_\alpha^\beta \partial_\mu + i g_4 \delta_i^j G_{\mu\alpha}^\beta + i g_R \delta_\alpha^\beta W_{\mu R i}^j\right) R_{j\beta} \times \\ & \quad\; \left(\delta_k^i \delta_\gamma^\alpha \partial^\mu - i g_4 \delta_k^i G_\gamma^{\mu\alpha} - i g_R \delta_\gamma^\alpha W_{Rk}^{\mu i}\right) R^{k\gamma} \end{aligned} \tag{3.67}$$

we obtain, as usual, the masses for the gauge bosons. In particular, for the leptoquarks $X_{\mu a}^\pm$ (with $a = 1, 2, 3$) we find, at the scale M_U, the mass value:

$$M_X^2 = g_4^2 \frac{v_R^2}{4} \approx 8.1 \cdot 10^{26}\,\text{GeV}^2\,; \tag{3.68}$$

while the rest of the charged vector bosons mix with a squared mass matrix given by:

$$\begin{pmatrix} W^+_{\mu L} & W^+_{\mu R} \end{pmatrix} M_W^2 \begin{pmatrix} W_L^{\mu -} \\ W_R^{\mu -} \end{pmatrix} = \begin{pmatrix} W^+_{\mu L} \\ W^+_{\mu R} \end{pmatrix}^T \begin{pmatrix} g_L^2 \frac{v_L^2}{4} & -g_L g_R u_1 u_2 \\ -g_L g_R u_1 u_2 & g_R^2 \frac{v_R^2 + v_L^2}{4} \end{pmatrix} \begin{pmatrix} W_L^{\mu -} \\ W_R^{\mu -} \end{pmatrix} . \tag{3.69}$$

From M_W^2 we calculate the two eigenvalues:

$$p_\pm = \frac{1}{2}\left[(g_R^2 + g_L^2)\frac{v_L^2}{4} + g_R^2 \frac{v_R^2}{4} \right] \pm \frac{\Delta}{2} \tag{3.70}$$

with

$$\Delta^2 = \left[(g_R^2 - g_L^2)\frac{v_L^2}{4} + g_R^2 \frac{v_R^2}{4} \right]^2 + 4 g_L^2 g_R^2 u_1^2 u_2^2 \quad , \tag{3.71}$$

and the orthogonal matrix:

$$\begin{aligned} O_\pm &= \frac{C}{2 g_L g_R u_1 u_2} \begin{pmatrix} 2 g_L g_R u_1 u_2 & 2\left[g_R^2 \frac{v_R^2 + v_L^2}{4} - p_+ \right] \\ 2\left[g_L^2 \frac{v_L^2}{4} - p_- \right] & 2 g_L g_R u_1 u_2 \end{pmatrix} = \\ &= \frac{C}{2 g_L g_R u_1 u_2} \begin{pmatrix} 2 g_L g_R u_1 u_2 & -\left[(g_L^2 - g_R^2)\frac{v_L^2}{4} - g_R^2 \frac{v_R^2}{4} + \Delta \right] \\ (g_L^2 - g_R^2)\frac{v_L^2}{4} - g_R^2 \frac{v_R^2}{4} + \Delta & 2 g_L g_R u_1 u_2 \end{pmatrix} \end{aligned} \tag{3.72}$$

with

$$C^2 = \frac{2 g_L^2 g_R^2 u_1^2 u_2^2}{\Delta\left[\Delta - \sqrt{\Delta^2 - 4 g_L^2 g_R^2 u_1^2 u_2^2} \right]} \quad . \tag{3.73}$$

Since we have that $v_R \gg v_L$, we can expand the eigenvalues and the related eigenvectors finding the values at the partial-unification scale M_U:

$$\begin{cases} M_{W_2}^2 = p_+ \approx g_R^2 \frac{v_R^2 + v_L^2}{4} \approx 7.40 \cdot 10^{26}\,\mathrm{GeV}^2 \\ M_{W_1}^2 = p_- \approx g_L^2 \frac{v_L^2}{4} \approx 4.47 \cdot 10^{3}\,\mathrm{GeV}^2 \end{cases} \quad \text{and} \quad O_\pm \approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \tag{3.74}$$

this means that, in first approximation, the gauge bosons $W^\pm_{\mu L}$ and $W^\pm_{\mu R}$ do not mix with each other:

$$O_\pm^T M_W^2 O_\pm = \begin{pmatrix} p_- & 0 \\ 0 & p_+ \end{pmatrix} \approx \begin{pmatrix} M_{W_L}^2 & 0 \\ 0 & M_{W_R}^2 \end{pmatrix} , \tag{3.75}$$

so their contribution to the radiative fermion mass generation will be negligible.

For the neutral vector bosons, instead, using the basis $A_\mu^0 = (W_{\mu L}^0, W_{\mu R}^0, B_\mu)$, we find the squared mass matrix:

$$A^0_\mu\, M_0^2\, A^{0T}_\mu = \frac{1}{2} A^0_\mu \begin{pmatrix} g_L^2 \frac{v_L^2}{4} & -g_L g_R \frac{v_L^2}{4} & 0 \\ -g_L g_R \frac{v_L^2}{4} & g_R^2 \frac{v_R^2+v_L^2}{4} & -\sqrt{\frac{3}{2}} g_L g_R \frac{v_R^2}{4} \\ 0 & -\sqrt{\frac{3}{2}} g_L g_R \frac{v_R^2}{4} & \frac{3}{4} g_4^2 \frac{v_R^2}{2} \end{pmatrix} A^{0T}_\mu\,, \tag{3.76}$$

from which we obtain the three eigenvalues: $q_0 = 0$, related to the photon, and

$$\begin{aligned} q_\pm = {} & \frac{1}{16}\Big[\Big(\frac{3}{2}g_4^2 + g_R^2\Big)v_R^2 + 2(g_L^2+g_R^2)v_L^2\Big] \pm \\ & \pm \frac{1}{16}\sqrt{\Big[\Big(\frac{3}{2}g_4^2 + g_R^2\Big)v_R^2 - 2(g_L^2+g_R^2)v_L^2\Big]^2 + 8 g_R^4 v_R^2 v_L^2}\,, \end{aligned} \tag{3.77}$$

related to the vector bosons Z_μ and Z'_μ. The orthogonal matrix, which diagonalizes this squared mass matrix:

$$O_0^T M_0^2 O_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & M_Z^2 & 0 \\ 0 & 0 & M_{Z'}^2 \end{pmatrix}\,, \tag{3.78}$$

is builded up, as usual, with the normalized eigenvectors of M_0^2:

$$O_0 = \Big(|q_0\rangle^T, |q_-\rangle^T, |q_+\rangle^T\Big)\,, \tag{3.79}$$

where the eigenstates of Z_μ and Z'_μ are given by:

$$|q_\pm\rangle = C_\pm\Big(\frac{g_L g_R v_L^2}{g_L^2 v_L^2 - 4q_\pm}, 1, \sqrt{\frac{3}{2}}\frac{g_4 g_R v_R^2}{(3/2) g_4^2 v_R^2 - 4q_\pm}\Big)\,, \tag{3.80}$$

with $C_\pm$ the normalization constants, while for the photon eigenstate we have:

$$|q_0\rangle = \sqrt{\frac{3 g_4^2 g_L^2}{3 g_4^2 (g_R^2 + g_L^2) + 2 g_R^2 g_L^2}}\Big(\frac{g_R}{g_L}, 1, \sqrt{\frac{2}{3}}\frac{g_R}{g_4}\Big)\,. \tag{3.81}$$

From $|q_0\rangle$, similarly to what seen in Sec. 2.3, using the charge definition (2.14) we find the relation:

$$e = \sqrt{\frac{3 g_4^2 g_L^2 g_R^2}{3 g_4^2 (g_R^2 + g_L^2) + 2 g_R^2 g_L^2}}\,. \tag{3.82}$$

Again, using the fact that $v_R \gg v_L$, we can expand the eigenvalues and the related eigenvectors finding in first approximation, at the scale M_U, the values:

$$\begin{cases} M^2_{Z'} = q_+ \approx \left(\frac{3}{2}g_4^2 + g_R^2\right)\frac{v_R^2}{4} + \frac{2g_R^4}{3g_4^2+2g_R^2}\frac{v_L^2}{4} \approx 1.9547 \cdot 10^{27}\,\text{GeV}^2 \\ M^2_{Z} = q_- \approx \left(g_L^2 + \frac{3g_4^2 g_R^2}{3g_4^2+2g_R^2}\right)\frac{v_L^2}{4} \approx 7.2496 \cdot 10^{3}\,\text{GeV}^2 \end{cases}$$

$$\text{and} \qquad O_0 \approx \begin{pmatrix} 0.6193 & -0.7851 & 3.4379\cdot 10^{-17} \\ 0.6193 & 0.4885 & 0.6146 \\ 0.4826 & 0.3807 & -0.7888 \end{pmatrix}. \tag{3.83}$$

3.4 Fermion masses

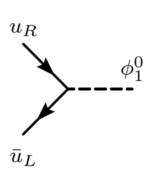
In the case without inter-family mixing, we have for each fermion family ($\mathbf{f} = 1, 2, 3$) the Yukawa interactions that, with the particular scalar content of the model 2, takes the form:

$$\mathcal{L}^{(\mathbf{f})}_{Yukawa} = -\left[k_R^{(\mathbf{f})}\, \bar{s}_0^{c\,(\mathbf{f})}\, R_{\alpha i} + \lambda_1^{(\mathbf{f})} \Psi^{(\mathbf{f})}_{L\,\alpha I}\, \Phi^I_i + \lambda_2^{(\mathbf{f})} \Psi^{(\mathbf{f})}_{L\,\alpha I}\, \tilde{\Phi}^I_i \right] \Psi_R^{\alpha i (\mathbf{f})} + h.c. \tag{3.84}$$

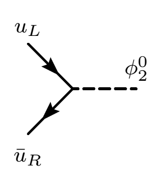
where the Yukawa couplings $k_R^{(\mathbf{f})}$, $\lambda_1^{(\mathbf{f})}$ and $\lambda_2^{(\mathbf{f})}$are real number, $s_0^{(\mathbf{f})}$ is the sterile neutrino field, while $\Psi_L^{\alpha I(\mathbf{f})}$ and $\Psi_R^{\alpha i(\mathbf{f})}$ represent the fermion multiplets $(4, 2, 1)$ and $(4, 1, 2)$ under the Pati-Salam group, respectively. In this model, unlike the model 1, the Yukawa terms are responsible for tree level mass generations for all the fermions; in fact, once the Higgs fields get VEVs:

$$\langle R \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ v_R & 0 \end{pmatrix}, \quad \langle \Phi \rangle = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \quad \text{and} \quad \langle \tilde{\Phi} \rangle = \begin{pmatrix} u_2 & 0 \\ 0 & u_1 \end{pmatrix}, \tag{3.85}$$

from the interaction vertices:



$= -i\lambda_1$ and

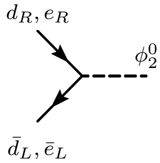


$= -i\lambda_2\,,$ (3.86)

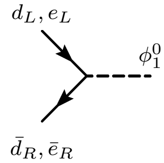
we obtain for the up-type quarks a mass of the form:

$$m_u = \lambda_1 u_1 + \lambda_2 u_2 = \lambda_1 u_1 \left(1 + \frac{\lambda_2}{\lambda_1}\tan\beta\right); \tag{3.87}$$

while from the Yukawa vertices:



$= -i\lambda_1$ and



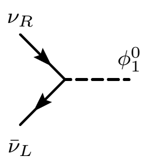
$= -i\lambda_2\,,$ (3.88)

we obtain for the down-type quarks and the charged leptons, of the same fermion family, the same mass of the form:

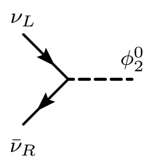
$$m_d = m_e = \lambda_2 u_1 + \lambda_1 u_2 = \lambda_2 u_1 \Big(1 + \frac{\lambda_1}{\lambda_2} \tan\beta\Big)\,. \tag{3.89}$$

3.4.1 Neutrino masses

Also the neutrinos get a tree level mass, but there are more contributions to take into account. In addition to the Yukawa vertices:

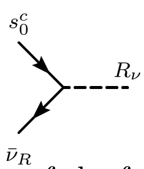


$= -i\lambda_1$ and



$= -i\lambda_2\,,$ (3.90)

analogous to those for the up-type quarks sector in (3.86), we have also the interaction vertex:



$= -ik_R\,,$ (3.91)

and the sterile neutrino mass term of the form as in (2.121). Using the same notation of Sec. 2.5, we can collect all these mass contributions to the neutrinos in the Majorana mass matrix:

$$M'_\nu = \begin{pmatrix} 0 & \frac{m_u}{2} & 0 \\ \frac{m_u}{2} & 0 & \frac{k_R v_R}{2\sqrt{2}} \\ 0 & \frac{k_R v_R}{2\sqrt{2}} & m_0 \end{pmatrix}\,. \tag{3.92}$$

Calculating the eigenvalues of M'_ν we get the cubic equation:

$$\rho^3 - \rho^2 m_0 - \rho\Big(\frac{k_R^2 v_R^2}{8} + \frac{m_u^2}{4}\Big) + \frac{m_u^2}{4} m_0 = 0\,, \tag{3.93}$$

which can be solved making the following three sobstitutions:

$$\rho = x + \frac{m_0}{3} \quad \Longrightarrow \quad x^3 + ax + b = 0 \quad \text{with} \quad \begin{cases} a = -\frac{m_0}{3} - \frac{k_R^2 v_R^2}{8} - \frac{m_u^2}{4} \\ b = -\frac{2m_0^3}{27} - \frac{m_0}{3}\Big(\frac{k_R^2 v_R^2}{8} - \frac{m_u^2}{2}\Big) \end{cases} ; \tag{3.94}$$

$$x = y - \frac{a}{3y} \quad \Longrightarrow \quad y^6 + by^3 - \frac{a^3}{27} = 0\,; \tag{3.95}$$

$$z = y^3 \quad \Longrightarrow \quad z = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + \frac{a^3}{27}} \quad \text{with} \quad \frac{b^2}{4} + \frac{a^3}{27} < 0\,. \tag{3.96}$$

At this point we can come back to the explicit solution for the neutrino mass eigenvalues:

$$z = |z| \exp\Big[\pm i(\vartheta + 2\kappa\pi)\Big] \quad \text{with} \quad \begin{cases} |z| = (-\frac{a}{3})^{\frac{3}{2}} \\ \cos\vartheta = -\frac{b}{2}(-\frac{a}{3})^{-\frac{3}{2}} \end{cases} \tag{3.97a}$$

$$\Longrightarrow \quad y = |z|^{\frac{1}{3}} \exp\Big[\pm i \frac{\vartheta + 2\kappa\pi}{3}\Big] \tag{3.97b}$$

$$\Longrightarrow \quad x = |z|^{\frac{1}{3}} \exp\Big[\pm i \frac{\vartheta + 2\kappa\pi}{3}\Big] - \frac{a \exp\Big[\mp i \frac{\vartheta + 2\kappa\pi}{3}\Big]}{3|z|^{\frac{1}{3}}} \tag{3.97c}$$

$$\Longrightarrow \quad \rho_\kappa = \frac{m_0}{3} + 2\cos\Big(\frac{\vartheta + 2\kappa\pi}{3}\Big)\sqrt{\frac{k_R^2 v_R^2}{24} + \frac{m_0^2}{9} + \frac{m_u^2}{12}} \quad \text{with} \quad \kappa = 0, 1, 2\,, \tag{3.97d}$$

and write the physical neutrino masses: $m_\kappa = |\rho_\kappa|$. While the orthogonal matrix is given by:

$$O'_\nu = \begin{pmatrix} A_0 \frac{m_u}{2\rho_0} & A_1 \frac{m_u}{2\rho_1} & A_2 \frac{m_u}{2\rho_2} \\ A_0 & A_1 & A_2 \\ A_0 \frac{k_R v_R}{2\sqrt{2}(\rho_0 - m_0)} & A_1 \frac{k_R v_R}{2\sqrt{2}(\rho_1 - m_0)} & A_2 \frac{k_R v_R}{2\sqrt{2}(\rho_2 - m_0)} \end{pmatrix} \tag{3.98}$$

where A_κ are the normalization constants.

Interlude on a particular case: $m_0 = 0$

As we discussed in Sec. 2.5, we consider for the sterile neutrino mass the natural scale $m_0 \sim v_R$; however if we introduce a global symmetry $s_0 \to e^{i\varphi} s_0$, demanding for an invariance under this symmetry we have to neglect the sterile neutrino mass and in this case the neutrino mass matrix takes the form:

$$M'_\nu = \begin{pmatrix} 0 & \frac{m_u}{2} & 0 \\ \frac{m_u}{2} & 0 & \frac{k_R v_R}{2\sqrt{2}} \\ 0 & \frac{k_R v_R}{2\sqrt{2}} & 0 \end{pmatrix} \tag{3.99}$$

which leads to the eigenvalues:

$$\rho_0 = 0 \quad \text{and} \quad \rho_\pm = \pm\sqrt{\frac{k_R^2 v_R^2}{8} + \frac{m_u^2}{4}} = \pm m_S\,. \tag{3.100}$$

If we consider the eigenvalues of M'_ν as Majorana masses then the orthogonal matrix is given by:

$$O'_{\nu M} = \frac{1}{\sqrt{2}m_S}\begin{pmatrix} \frac{m_u}{2} & -\frac{m_u}{2} & \frac{k_R v_R}{2} \\ m_S & m_S & 0 \\ \frac{k_R v_R}{2\sqrt{2}} & -\frac{k_R v_R}{2\sqrt{2}} & -\frac{m_u}{2\sqrt{2}} \end{pmatrix} \Rightarrow \quad O'^T_{\nu M} M'_\nu O'_{\nu M} = \begin{pmatrix} m_S & 0 & 0 \\ 0 & -m_S & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.101}$$

with the mass eigenstates defined as:

$$O'^T_{\nu M}\begin{pmatrix} \nu_L^c \\ \nu_R \\ s_0 \end{pmatrix} = \begin{pmatrix} \nu_+ \\ \nu_- \\ \nu_0 \end{pmatrix}_R \quad ; \tag{3.102}$$

otherwise, if we consider the eigenvalues as Dirac masses then the orthogonal matrix has the form:

$$O'_{\nu D} = \cos\theta_\nu \begin{pmatrix} \tan\theta_\nu & 0 & 1 \\ 0 & \frac{1}{\cos\theta_\nu} & 0 \\ 1 & 0 & -\tan\theta_\nu \end{pmatrix}, \quad \text{with} \quad \tan\theta_\nu = \frac{\sqrt{2}m_u}{k_R v_R}\,, \tag{3.103}$$

which produces a non-diagonal mass matrix:

$$O'^T_{\nu D} M'_\nu O'_{\nu D} = \begin{pmatrix} 0 & s_\nu \frac{m_u}{2} + c_\nu \frac{k_R v_R}{2\sqrt{2}} & 0 \\ s_\nu \frac{m_u}{2} + c_\nu \frac{k_R v_R}{2\sqrt{2}} & 0 & c_\nu \frac{m_u}{2} - s_\nu \frac{k_R v_R}{2\sqrt{2}} \\ 0 & c_\nu \frac{m_u}{2} - s_\nu \frac{k_R v_R}{2\sqrt{2}} & 0 \end{pmatrix} = \\ = \begin{pmatrix} 0 & m_S & 0 \\ m_S & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.104}$$

with the mass eigenstates defined as:

$$O'^T_{\nu D} \begin{pmatrix} \nu^c_L \\ \nu_R \\ s_0 \end{pmatrix} = \begin{pmatrix} S^c_L \\ S_R \\ \nu_{0R} \end{pmatrix}. \tag{3.105}$$

Interlude on a particular case: $|m_0| \ll v_R$

As we said the natural scale of the sterile neutrino mass is of the order of the Pati-Salam scale; as an exercise however let us consider a small m_0, for example of the order of the tree level mass m_u. In this case we have from (3.97a) that:

$$\vartheta = \arccos\left[\frac{\sqrt{2}m_0}{3v_R}\left(\frac{k_R^2}{8} + \frac{2m_0^2}{9v_R^2} - \frac{m_u^2}{2v_R^2}\right)\left(\frac{k_R^2}{12} + \frac{2m_0^2}{9v_R^2} - \frac{m_u^2}{6v_R^2}\right)^{-\frac{3}{2}}\right] \to \frac{\pi}{2}, \tag{3.106}$$

which leads to the eigenvalues:

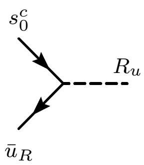
$$\lambda_\kappa \approx \frac{m_0}{3} + \frac{k_R v_R}{\sqrt{6}} \cos\left(\frac{4\kappa+1}{6}\pi\right) \quad \Rightarrow \quad \begin{cases} \lambda_0 \approx \frac{m_0}{3} + \frac{k_R v_R}{2\sqrt{2}} \\ \lambda_1 \approx \frac{m_0}{3} - \frac{k_R v_R}{2\sqrt{2}} \\ \lambda_2 \approx \frac{m_0}{3} \end{cases}. \tag{3.107}$$

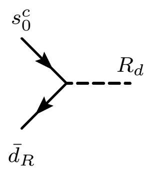
3.5 Loops

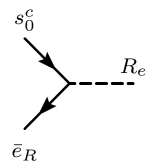
The one-loop corrections, to the tree level fermion mass terms, that we are going to analyze in this section are responsible for the distinction between the masses for the charged leptons and the down-type quarks. Furthermore we will have one-loop contributions to the radiative generation of Majorana masses for the neutrinos. Again, as in the model 1, the loop diagrams which contribute to the radiative mass corrections are the scalar loops, where we have Higgs boson mass eigenstates exchange, and the gauge loop, where we have a gauge boson exchange. All the following calculations are done in Feynman gauge ($\xi = 1$). We work in the Minimal Subtraction (MS) scheme, so we remove only the pole $1/\epsilon$, by means of field and parameter redefinitions, maintaining all the finite parts.

3.5.1 Scalar loop

From the Yukawa interactions, in addition to (3.86), (3.88), (3.90) and (3.91), we find also the Higgs-fermions vertices:

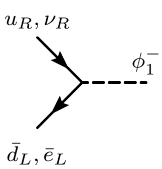


$= -ik_R$,


$= -ik_R$ and


$= -ik_R\,,$ (3.108)

coming from the interactions between fermions and the Higgs multiplet R; while from the Yukawa interactions with the bi-doublet we obtain the vertices:



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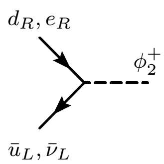
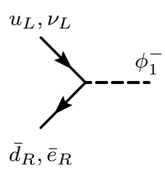
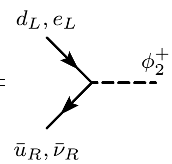


$= -i\lambda_1$ and


$=$


$= i\lambda_2\,.$ (3.109)

With these interactions we can build up one-loop diagrams of the type in Fig. 3.3, which contribute to the corrections of the tree level masses in (3.87), (3.89) and (3.97d).

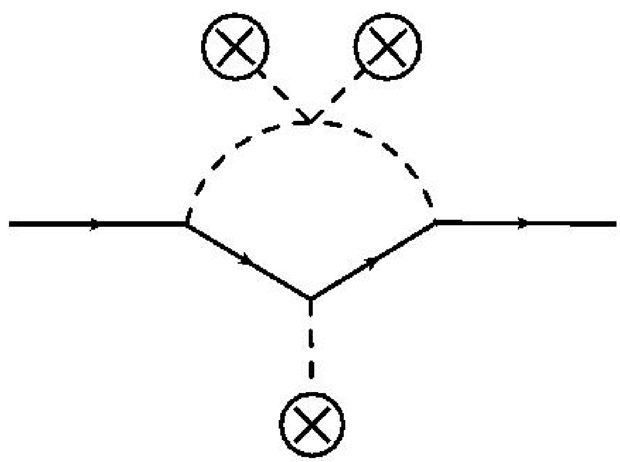
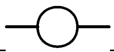


Figure 3.3: *One-loop diagram for the correction to the fermion mass term $m\bar{\psi}_R\psi_L$ with an Higgs boson exchange; each circled cross represents a VEV insertion.*

In dimensional regularization ($D = 4-\epsilon$) and using the Weyl propagators (Appendix G) with zero external momentum, the resulting correction to the tree level mass m_ψ (with $\psi = u, d$) coming from this type of diagrams is given by:

$$m_H^{(\psi)} = i\Big[$$  $$\Big]_H^{(\psi)} = -i\lambda_A\lambda_B \int \frac{d^4q}{(2\pi)^4}\frac{im_\chi}{q^2-m_\chi^2}\frac{i}{q^2-M_H^2} =$$

$$\begin{aligned}
&= -m_\chi\lambda_A\lambda_B\mu^\epsilon \int \frac{d^Dq}{(2\pi)^D}\frac{1}{(q^2+m_\chi^2)(q^2+M_H^2)} = \\
&= -m_\chi\lambda_A\lambda_B\mu^\epsilon \int_0^1 dx \frac{\Gamma(\frac{D}{2})\Gamma(2-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}\Gamma(2)\Gamma(\frac{D}{2})}\Big[m_\chi^2 x + M_H^2(1-x)\Big]^{\frac{D}{2}-2} = \\
&= -\frac{m_\chi\lambda_A\lambda_B}{(4\pi)^{2-\frac{\epsilon}{2}}}\left(\frac{\mu^2}{m_\chi^2-M_H^2}\right)\Gamma(\epsilon/2)\left(\frac{2}{2-\epsilon}\right)\left[\left(\frac{m_\chi^2}{\mu^2}\right)^{1-\frac{\epsilon}{2}} - \left(\frac{M_H^2}{\mu^2}\right)^{1-\frac{\epsilon}{2}}\right] = \\
&= -\frac{m_\chi\lambda_A\lambda_B}{(4\pi)^2}\left[\frac{2}{\epsilon} + \ln(4\pi) - \gamma_E - \ln\left(\frac{m_\chi^2}{\mu^2}\right) - \frac{M_H^2}{m_\chi^2-M_H^2}\ln\left(\frac{m_\chi^2}{M_H^2}\right)\right] + \\
&\quad + O(\epsilon)
\end{aligned} \tag{3.110}$$

where μ is the energy scale. In particular, looking at the vertices we can note that the only Higgs bosons able to link in a loop a left handed fermions with a right handed one are: the charged Higgs $\phi_1^\pm$ and $\phi_2^\pm$; and the mixing between the neutral Higgs ϕ_1^0 and ϕ_2^0. However, using the mass matrix (3.37), we can neglect the mixing between ϕ_1^0 and ϕ_2^0. This means that the only scalar contribution comes from the loops with the charged Higgs exchanges. For the up-type quarks ($\psi = u$) we have that:

$$\begin{aligned}
m_H^{(u)} = &\frac{m_d\lambda_1\lambda_2}{(4\pi)^2}\left[\ln(4\pi) - \gamma_E - \ln\left(\frac{m_d^2}{\mu^2}\right) - \frac{M^2_{\phi_2^\pm}}{m_d^2 - M^2_{\phi_2^\pm}}\ln\left(\frac{m_d^2}{M^2_{\phi_2^\pm}}\right)\right] + \\
&+ \frac{m_d\lambda_1\lambda_2}{(4\pi)^2}(O_{H\pm})^2_{11}\left[\ln(4\pi) - \gamma_E - \ln\left(\frac{m_d^2}{\mu^2}\right) - \frac{M^2_{H_1^\pm}}{m_d^2 - M^2_{H_1^\pm}}\ln\left(\frac{m_d^2}{M^2_{H_1^\pm}}\right)\right] + \\
&+ \frac{m_d\lambda_1\lambda_2}{(4\pi)^2}(O_{H\pm})^2_{12}\left[\ln(4\pi) - \gamma_E - \ln\left(\frac{m_d^2}{\mu^2}\right) - \frac{M^2_{W_R}}{m_d^2 - M^2_{W_R}}\ln\left(\frac{m_d^2}{M^2_{W_R}}\right)\right],
\end{aligned} \tag{3.111}$$

where we used the mass $\sqrt{\xi}M_{W_R}$ for the propagating Goldstone boson $H_2^\pm$, and we have the tree level mass m_d because is a down-type quark which propagates in the loop. Similarly for the down-type quarks we have:

$$m_H^{(d)} = \frac{m_u\lambda_1\lambda_2}{(4\pi)^2}\left[\ln(4\pi) - \gamma_E - \ln\left(\frac{m_u^2}{\mu^2}\right) - \frac{M^2_{\phi_2^\pm}}{m_u^2 - M^2_{\phi_2^\pm}}\ln\left(\frac{m_u^2}{M^2_{\phi_2^\pm}}\right)\right] + \\ + \frac{m_u\lambda_1\lambda_2}{(4\pi)^2}(O_{H\pm})^2_{11}\left[\ln(4\pi) - \gamma_E - \ln\left(\frac{m_u^2}{\mu^2}\right) - \frac{M^2_{H_1^\pm}}{m_u^2 - M^2_{H_1^\pm}}\ln\left(\frac{m_u^2}{M^2_{H_1^\pm}}\right)\right] + \\ + \frac{m_u\lambda_1\lambda_2}{(4\pi)^2}(O_{H\pm})^2_{12}\left[\ln(4\pi) - \gamma_E - \ln\left(\frac{m_u^2}{\mu^2}\right) - \frac{M^2_{W_R}}{m_u^2 - M^2_{W_R}}\ln\left(\frac{m_u^2}{M^2_{W_R}}\right)\right], \tag{3.112}$$

where the only difference with $m_H^{(u)}$ is that now it is an up-type quark which propagates in the loop. For the charged leptons, instead, the fermions which propagates in the loop are the physical neutrinos, with the masses $m_\kappa = |\rho_\kappa|$ given in (3.97d), so we have that the scalar loop correction to the tree level mass $m_e = m_d$ is given by:

$$m_H^{(e)} = \sum_{\kappa=0}^{2} \frac{m_\kappa\lambda_1\lambda_2}{(4\pi)^2}(O'_\nu)_{1,\,\kappa+1}(O'_\nu)_{2,\,\kappa+1}\times \\ \times\left\{\left[\ln(4\pi) - \gamma_E - \ln\left(\frac{m_\kappa^2}{\mu^2}\right) - \frac{M^2_{\phi_2^\pm}}{m_\kappa^2 - M^2_{\phi_2^\pm}}\ln\left(\frac{m_\kappa^2}{M^2_{\phi_2^\pm}}\right)\right] + \right. \\ + (O_{H\pm})^2_{11}\left[\ln(4\pi) - \gamma_E - \ln\left(\frac{m_\kappa^2}{\mu^2}\right) - \frac{M^2_{H_1^\pm}}{m_\kappa^2 - M^2_{H_1^\pm}}\ln\left(\frac{m_\kappa^2}{M^2_{H_1^\pm}}\right)\right] + \\ \left. + (O_{H\pm})^2_{12}\left[\ln(4\pi) - \gamma_E - \ln\left(\frac{m_\kappa^2}{\mu^2}\right) - \frac{M^2_{W_R}}{m_\kappa^2 - M^2_{W_R}}\ln\left(\frac{m_\kappa^2}{M^2_{W_R}}\right)\right]\right\}. \tag{3.113}$$

We conclude with the contribution $m_H^{(\nu)}$ to the neutrinos, which is identical to $m_H^{(u)}$, in fact it is of the form:

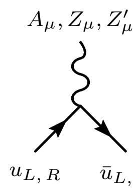
$$m_H^{(\nu)} = \frac{m_d\lambda_1\lambda_2}{(4\pi)^2}\left[\ln(4\pi) - \gamma_E - \ln\left(\frac{m_d^2}{\mu^2}\right) - \frac{M^2_{\phi_2^\pm}}{m_d^2 - M^2_{\phi_2^\pm}}\ln\left(\frac{m_d^2}{M^2_{\phi_2^\pm}}\right)\right] + \\ + \frac{m_d\lambda_1\lambda_2}{(4\pi)^2}(O_{H\pm})^2_{11}\left[\ln(4\pi) - \gamma_E - \ln\left(\frac{m_d^2}{\mu^2}\right) - \frac{M^2_{H_1^\pm}}{m_d^2 - M^2_{H_1^\pm}}\ln\left(\frac{m_d^2}{M^2_{H_1^\pm}}\right)\right] + \\ + \frac{m_d\lambda_1\lambda_2}{(4\pi)^2}(O_{H\pm})^2_{12}\left[\ln(4\pi) - \gamma_E - \ln\left(\frac{m_d^2}{\mu^2}\right) - \frac{M^2_{W_R}}{m_d^2 - M^2_{W_R}}\ln\left(\frac{m_d^2}{M^2_{W_R}}\right)\right]. \tag{3.114}$$

We have to take in account that this correction, due to the scalar loop, is only for the elements of the neutrino mass matrix M'_ν, related to the Dirac mass; to be more precise, considering only this type of corrections, the neutrino mass matrix at one-loop becomes of the form:

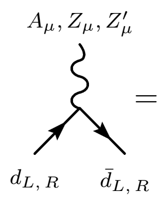
$$M'^{\dagger}_\nu \to \begin{pmatrix} 0 & \frac{(m_u+m_H^{(\nu)})^T}{2} & 0 \\ \frac{m_u+m_H^{(\nu)}}{2} & 0 & \frac{k_R v_R}{2\sqrt{2}} \\ 0 & \frac{k_R v_R}{2\sqrt{2}} & m_0 \end{pmatrix} . \tag{3.115}$$

3.5.2 Gauge loops

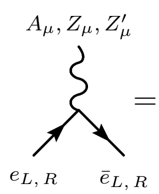
We pass now to analyze the mass corrections generated by the loops with a gauge boson exchange. First of all, let us have a look at the vertices coming from the kinetic term $i\bar{\Psi}_{L,R}\gamma^\mu D_\mu \Psi_{L,R}$. Since we are investigating the fermion masses at the partial-unification scale, for simplicity we define $g_2(M_U) = g_L(M_U) = g_R$ in the following formulae. So the fermion vertices with the neutral gauge bosons are:



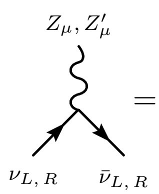
$$= -\frac{i}{2}\left[\frac{g_4}{\sqrt{6}}(O_0)_{3k} + g_2(O_0)_{pk}\right]\gamma^\mu ; \tag{3.116}$$



$$= -\frac{i}{2}\left[\frac{g_4}{\sqrt{6}}(O_0)_{3k} - g_2(O_0)_{pk}\right]\gamma^\mu ; \tag{3.117}$$



$$= \frac{i}{2}\left[g_4\sqrt{\frac{3}{2}}(O_0)_{3k} + g_2(O_0)_{pk}\right]\gamma^\mu ; \tag{3.118}$$



$$= \frac{i}{2}\left[g_4\sqrt{\frac{3}{2}}(O_0)_{3k} - g_2(O_0)_{pk}\right]\gamma^\mu ; \tag{3.119}$$

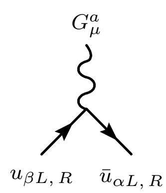
where, for the indices of the mixing matrices O_0 defined in Sec. 3.3, we used the notation:

$$k = \begin{cases} 1 & \text{for} \quad A_\mu \\ 2 & \text{for} \quad Z_\mu \\ 3 & \text{for} \quad Z'_\mu \end{cases} \quad \text{and} \quad p = \begin{cases} 1 & \text{for Left part} \\ 2 & \text{for Right part} \end{cases} . \tag{3.120}$$

Just as a simple proof of consistency of this model we can see that no interactions between neutrinos and photon exist, in fact we have that:

$$\begin{cases} g_4\sqrt{\frac{3}{2}}(O_0)_{31} - g_L(O_0)_{11} = \sqrt{\frac{3g_4^2 g_L^2}{3g_4^2(g_R^2+g_L^2)+2g_R^2 g_L^2}}\left[g_4\sqrt{\frac{3}{2}}\left(\sqrt{\frac{2}{3}}\frac{g_2}{g_4}\right) - g_L\left(\frac{g_R}{g_L}\right)\right] = 0 \\ g_4\sqrt{\frac{3}{2}}(O_0)_{31} - g_R(O_0)_{21} = \sqrt{\frac{3g_4^2 g_L^2}{3g_4^2(g_R^2+g_L^2)+2g_R^2 g_L^2}}\left[g_4\sqrt{\frac{3}{2}}\left(\sqrt{\frac{2}{3}}\frac{g_2}{g_4}\right) - g_R\right] = 0 \end{cases} . \tag{3.121}$$

The quark interactions with the gluons are instead given by:



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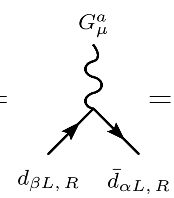


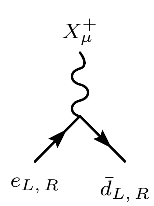
$$= -\frac{ig_4}{2}\gamma^\mu (T^a)_{\alpha\beta}\,, \tag{3.122}$$

where $T^a = \frac{\lambda_a}{2}$ (with $a = 1\dots 8$) are the Gell-Mann matrices of $SU(3)_C$, which verify the relation:

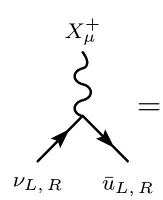
$$\sum_{a=1}^{8}(T^a)^2 = \frac{N^2-1}{2N}\mathbb{I}\,, \tag{3.123}$$

with N the degree of the group $SU(N)$ and $\mathbb{I}$ the $N\times N$ identity matrix, in this case $N = 3$ is the number of colours.

We continue the list considering the fermion vertices containing the charged gauge bosons, starting from the interaction vertices with the leptoquarks:

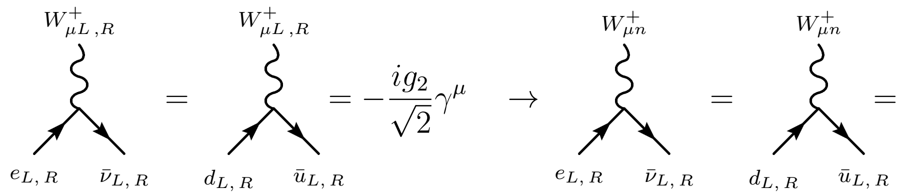


$=$



$$= -\frac{ig_4}{\sqrt{2}}\gamma^\mu\,. \tag{3.124}$$

Then we have the interaction vertices with the vector bosons $W^\pm_{\mu L}$ and $W^\pm_{\mu R}$, which can be turned in their mass eigenstates $W^{\pm}_{\mu 1}$ and $W^{\pm}_{\mu 2}$:



$$ [\text{vertex } e\bar{\nu}W] = [\text{vertex } d\bar{u}W] = -\frac{ig_2}{\sqrt{2}}\gamma^{\mu} \;\rightarrow\; [\text{vertex } e\bar{\nu}W_n] = [\text{vertex } d\bar{u}W_n] = -\frac{ig_2}{\sqrt{2}}(O_{\pm})_{pn}\gamma^{\mu}\,; \quad (3.125) $$

however these last vertices can be neglected because, as we have seen previously, considering that $v_R \gg v_L$ we can use the approximation in (3.75), which gives us no-mixing between $W^{\pm}_{\mu L}$ and $W_{\mu R}\pm$, so we cannot connect in a loop an incoming left-handed fermion to an outcoming right-handed fermion.

Using these vertices, the gauge loop for the correction to the fermion masses are given by the diagram shown in Fig. 3.4.

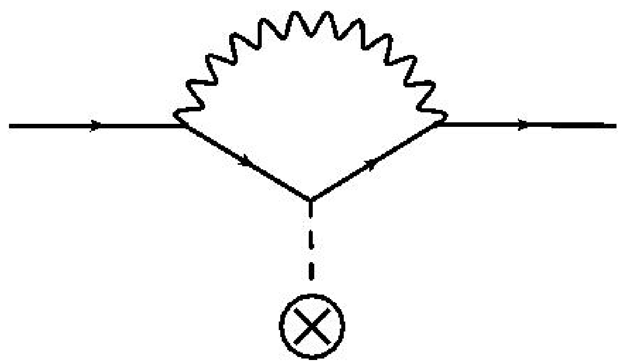
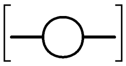


Figure 3.4: *One-loop diagram for the correction to the fermion mass term $m\bar{\psi}_R\psi_L$ with a gauge boson exchange, where the circled cross represents a VEV insertion.*

The case with a propagating massless gauge boson (photon and gluon) leads to a different correction to the fermion masses, which is given by:

$$
\begin{aligned}
m^{(\psi)}_{G,M_G=0} = i\Big[\!-\!\bigcirc\!-\!\Big]^{(\psi)}_{G,M_G=0} &= -ig_A g_B \int \frac{d^4q}{(2\pi)^4}\gamma^{\mu}\frac{im_{\chi}}{q^2-m_{\chi}^2}\gamma^{\nu}\frac{-ig_{\mu\nu}}{q^2} = \\
&= m_{\chi} g_A g_B \mu^{\epsilon}\int_0^1 dx \int \frac{d^Dq}{(2\pi)^D}\frac{D}{(q^2+m_{\chi}^2 x)^2} = \\
&= \frac{4m_{\chi}g_A g_B}{(4\pi)^2}\left[\frac{2}{\epsilon}-\frac{1}{2}+\ln(4\pi)-\gamma_E-\ln\left(\frac{m_{\chi}^2}{\mu^2}\right)\right]+O(\epsilon)\,;
\end{aligned}
\quad (3.126)
$$

while for a propagating massive gauge boson we obtain the result:

$$m^{(\psi)}_{G,M_G\neq 0} = i\Big[$$

$$\Big]^{(\psi)}_{G,M_G\neq 0} = -ig_A g_B \int \frac{d^4q}{(2\pi)^4}\gamma^\mu \frac{im_\chi}{q^2-m_\chi^2}\gamma^\nu \frac{-ig_{\mu\nu}}{q^2-M_G^2} =$$

$$\begin{aligned}
&= m_\chi g_A g_B \mu^\epsilon \int \frac{d^Dq}{(2\pi)^D}\frac{D}{(q^2+m_\chi^2)(q^2+M_G^2)} = \\
&= m_\chi g_A g_B D\mu^\epsilon \int_0^1 dx \frac{\Gamma(\frac{D}{2})\Gamma(2-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}\Gamma(2)\Gamma(\frac{D}{2})}\Big[m_\chi^2 x + M_G^2(1-x)\Big]^{\frac{D}{2}-2} = \\
&= \frac{m_\chi g_A g_B}{(4\pi)^{2-\frac{\epsilon}{2}}}\left(\frac{(4-\epsilon)\mu^2}{m_\chi^2-M_G^2}\right)\Gamma(\epsilon/2)\left(\frac{2}{2-\epsilon}\right)\left[\left(\frac{m_\chi^2}{\mu^2}\right)^{1-\frac{\epsilon}{2}} - \left(\frac{M_G^2}{\mu^2}\right)^{1-\frac{\epsilon}{2}}\right] = \\
&= \frac{4m_\chi g_A g_B}{(4\pi)^2}\Bigg[\frac{2}{\epsilon} - \frac{1}{2} + \ln(4\pi) - \gamma_E - \ln\left(\frac{m_\chi^2}{\mu^2}\right) - \\
&\quad - \frac{M_G^2}{m_\chi^2 - M_G^2}\ln\left(\frac{m_\chi^2}{M_G^2}\right)\Bigg] + O(\epsilon)\,.
\end{aligned} \tag{3.127}$$

Let us point out again that, using the MS scheme, in the formulae for the corrections of the tree level fermion masses all the finite parts, in the limit $\epsilon \to 0$, of (3.126) and (3.127) are present. In particular, the corrections to the down-type quark masses, coming from the gauge loop, are of the form:

$$\begin{aligned}
m_G^{(d)} =& \frac{m_d}{(4\pi)^2}\left(\frac{g_4}{\sqrt{6}}(O_0)_{32} - g_2(O_0)_{12}\right)\left(\frac{g_4}{\sqrt{6}}(O_0)_{32} - g_2(O_0)_{22}\right)\times \\
&\times\left[-\frac{1}{2} + \ln(4\pi) - \gamma_E - \ln\left(\frac{m_d^2}{\mu^2}\right) - \frac{M_Z^2}{m_d^2 - M_Z^2}\ln\left(\frac{m_d^2}{M_Z^2}\right)\right] + \\
&+ \frac{m_d}{(4\pi)^2}\left(\frac{g_4}{\sqrt{6}}(O_0)_{33} - g_2(O_0)_{13}\right)\left(\frac{g_4}{\sqrt{6}}(O_0)_{33} - g_2(O_0)_{23}\right)\times \\
&\times\left[-\frac{1}{2} + \ln(4\pi) - \gamma_E - \ln\left(\frac{m_d^2}{\mu^2}\right) - \frac{M_{Z'}^2}{m_d^2 - M_{Z'}^2}\ln\left(\frac{m_d^2}{M_{Z'}^2}\right)\right] + \\
&+ \frac{2m_d g_4^2}{(4\pi)^2}\left[-\frac{1}{2} + \ln(4\pi) - \gamma_E - \ln\left(\frac{m_d^2}{\mu^2}\right) - \frac{M_X^2}{m_d^2 - M_X^2}\ln\left(\frac{m_d^2}{M_X^2}\right)\right] + \\
&+ \frac{m_d}{(4\pi)^2}\left[g_4^2\frac{4}{3} + \left(\frac{g_4}{\sqrt{6}}(O_0)_{31} - g_2(O_0)_{11}\right)\left(\frac{g_4}{\sqrt{6}}(O_0)_{31} - g_2(O_0)_{21}\right)\right]\times \\
&\times\left[-\frac{1}{2} + \ln(4\pi) - \gamma_E - \ln\left(\frac{m_d^2}{\mu^2}\right)\right],
\end{aligned} \tag{3.128}$$

where we can see that the first two contributions come from the Z_μ and Z'_μ boson exchange, then we have the part due to the leptoquark exchange, with the charged leptons that propagate in the loop, and finally the contribution from gluon and photon exhange, respectively. The correction for the up-type quarks is similar, apart from the signs in front of the gauge coupling g_2 and the neutrino mass eigenstates which in this case substitute the charged leptons in the loop containing the leptoquark:

$$\begin{aligned}
m_G^{(u)} =& \frac{m_u}{(4\pi)^2}\left(\frac{g_4}{\sqrt{6}}(O_0)_{32}+g_2(O_0)_{12}\right)\left(\frac{g_4}{\sqrt{6}}(O_0)_{32}+g_2(O_0)_{22}\right)\times \\
&\times\left[-\frac{1}{2}+\ln(4\pi)-\gamma_E-\ln\left(\frac{m_u^2}{\mu^2}\right)-\frac{M_Z^2}{m_u^2-M_Z^2}\ln\left(\frac{m_u^2}{M_Z^2}\right)\right]+ \\
&+\frac{m_u}{(4\pi)^2}\left(\frac{g_4}{\sqrt{6}}(O_0)_{33}+g_2(O_0)_{13}\right)\left(\frac{g_4}{\sqrt{6}}(O_0)_{33}+g_2(O_0)_{23}\right)\times \\
&\times\left[-\frac{1}{2}+\ln(4\pi)-\gamma_E-\ln\left(\frac{m_u^2}{\mu^2}\right)-\frac{M_{Z'}^2}{m_u^2-M_{Z'}^2}\ln\left(\frac{m_u^2}{M_{Z'}^2}\right)\right]+ \\
&+\sum_{\kappa=0}^{2}\frac{2m_\kappa g_4^2}{(4\pi)^2}(O'_\nu)_{1,\kappa+1}(O'_\nu)_{2,\kappa+2}\times \\
&\times\left[-\frac{1}{2}+\ln(4\pi)-\gamma_E-\ln\left(\frac{m_\kappa^2}{\mu^2}\right)-\frac{M_X^2}{m_\kappa^2-M_X^2}\ln\left(\frac{m_\kappa^2}{M_X^2}\right)\right]+ \\
&+\frac{m_u}{(4\pi)^2}\left[g_4^2\frac{4}{3}+\left(\frac{g_4}{\sqrt{6}}(O_0)_{31}+g_2(O_0)_{11}\right)\left(\frac{g_4}{\sqrt{6}}(O_0)_{31}+g_2(O_0)_{21}\right)\right]\times \\
&\times\left[-\frac{1}{2}+\ln(4\pi)-\gamma_E-\ln\left(\frac{m_u^2}{\mu^2}\right)\right].
\end{aligned} \tag{3.129}$$

Compared to $m_G^{(d)}$ in (3.128) instead, for the charged leptons we have to change, in addition to the signs in front of the gauge coupling g_2, the coefficient related to the gauge coupling g_4 and obviously we have to eliminate the part related to the gluon exchange. Then, using for the mass of the propagating fermion the down quark tree level mass (since

$m_d = m_e$), we obtain the formula:

$$\begin{aligned}
m_G^{(e)} =& \frac{m_d}{(4\pi)^2}\left(g_4\sqrt{\frac{3}{2}}(O_0)_{32} + g_2(O_0)_{12}\right)\left(g_4\sqrt{\frac{3}{2}}(O_0)_{32} + g_2(O_0)_{22}\right)\times \\
& \times\left[-\frac{1}{2} + \ln(4\pi) - \gamma_E - \ln\left(\frac{m_d^2}{\mu^2}\right) - \frac{M_Z^2}{m_d^2 - M_Z^2}\ln\left(\frac{m_d^2}{M_Z^2}\right)\right] + \\
& + \frac{m_d}{(4\pi)^2}\left(g_4\sqrt{\frac{3}{2}}(O_0)_{33} + g_2(O_0)_{13}\right)\left(g_4\sqrt{\frac{3}{2}}(O_0)_{33} + g_2(O_0)_{23}\right)\times \\
& \times\left[-\frac{1}{2} + \ln(4\pi) - \gamma_E - \ln\left(\frac{m_d^2}{\mu^2}\right) - \frac{M_{Z'}^2}{m_d^2 - M_{Z'}^2}\ln\left(\frac{m_d^2}{M_{Z'}^2}\right)\right] + \\
& + \frac{2m_d g_4^2}{(4\pi)^2}\left[-\frac{1}{2} + \ln(4\pi) - \gamma_E - \ln\left(\frac{m_d^2}{\mu^2}\right) - \frac{M_X^2}{m_d^2 - M_X^2}\ln\left(\frac{m_d^2}{M_X^2}\right)\right] + \\
& + \frac{m_d}{(4\pi)^2}\left(g_4\sqrt{\frac{3}{2}}(O_0)_{31} + g_2(O_0)_{11}\right)\left(g_4\sqrt{\frac{3}{2}}(O_0)_{31} + g_2(O_0)_{21}\right)\times \\
& \times\left[-\frac{1}{2} + \ln(4\pi) - \gamma_E - \ln\left(\frac{m_d^2}{\mu^2}\right)\right].
\end{aligned} \tag{3.130}$$

To conclude we have to evaluate the gauge loop corrections for the neutrino masses. The first we show is the one related to the neutrino Dirac mass, which is only due to the neutral gauge bosons and leptoquarks exchanges:

$$\begin{aligned}
m_G^{(\nu)} =& \sum_{\kappa=0}^{2}\frac{m_\kappa}{(4\pi)^2}\left(g_4\sqrt{\frac{3}{2}}(O_0)_{32} - g_2(O_0)_{12}\right)\left(g_4\sqrt{\frac{3}{2}}(O_0)_{32} - g_2(O_0)_{22}\right)\times \\
& (O'_\nu)_{1,\kappa+1}(O'_\nu)_{2,\kappa+1}\left[\ln(4\pi) - \frac{1}{2} - \gamma_E - \ln\left(\frac{m_\kappa^2}{\mu^2}\right) - \frac{M_Z^2}{m_\kappa^2 - M_Z^2}\ln\left(\frac{m_\kappa^2}{M_Z^2}\right)\right] + \\
& + \sum_{\kappa=0}^{2}\frac{m_\kappa}{(4\pi)^2}\left(g_4\sqrt{\frac{3}{2}}(O_0)_{33} - g_2(O_0)_{13}\right)\left(g_4\sqrt{\frac{3}{2}}(O_0)_{33} - g_2(O_0)_{23}\right)\times \\
& (O'_\nu)_{1,\kappa+1}(O'_\nu)_{2,\kappa+1}\left[\ln(4\pi) - \frac{1}{2} - \gamma_E - \ln\left(\frac{m_\kappa^2}{\mu^2}\right) - \frac{M_{Z'}^2}{m_\kappa^2 - M_{Z'}^2}\ln\left(\frac{m_\kappa^2}{M_{Z'}^2}\right)\right] + \\
& + \frac{2m_u g_4^2}{(4\pi)^2}\left[\ln(4\pi) - \frac{1}{2} - \gamma_E - \ln\left(\frac{m_u^2}{\mu^2}\right) - \frac{M_X^2}{m_u^2 - M_X^2}\ln\left(\frac{m_u^2}{M_X^2}\right)\right].
\end{aligned} \tag{3.131}$$

Besides this we have also the radiative generation of two Majorana masses: one for the right-handed ν_R and one for the left-handed ν_L neutrinos, whose relevant diagrams are shown in Fig. 3.5 and Fig. 3.6, respectively, both related to a neutral gauge bosons, Z_μ and Z'_μ, exchange.

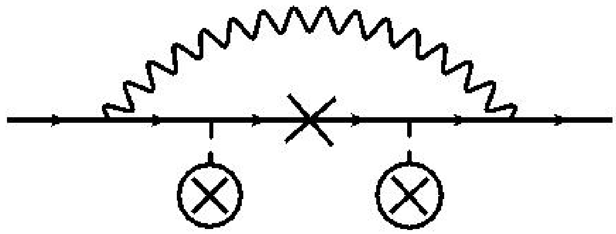


Figure 3.5: *Gauge loop for the Majorana Right mass generation, with insertions of VEV and the sterile neutrino mass represented by circled cross and cross, respectively.*

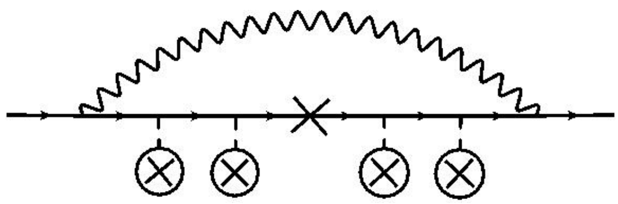


Figure 3.6: *Gauge loop for the Majorana Left mass generation, with insertions of VEV and the sterile neutrino mass represented by circled cross and cross, respectively.*

In particular, for the Majorana Right mass we have the form:

$$\begin{aligned} m^{(\nu)}_{MR} = -i(M'_\nu)_{32}(M'_\nu)_{23}\sum_{\kappa=0}^{2}(O'_\nu)^2_{3,\kappa+1}\int \frac{d^4q}{(2\pi)^4}\gamma^\mu\frac{iq_\alpha\gamma^\alpha}{q^2}\frac{im_\kappa}{q^2-m_\kappa^2}\frac{-iq_\beta\gamma^\beta}{q^2}\gamma^\nu\times \\ \times\left[g_C^2\frac{-ig_{\mu\nu}}{q^2-M_Z^2}+g_D^2\frac{-ig_{\mu\nu}}{q^2-M_{Z'}^2}\right] = \\ = \sum_{\kappa=0}^{2}\frac{m_\kappa}{(4\pi)^2}(M'_\nu)_{32}(M'_\nu)_{23}(O'_\nu)^2_{3,\kappa+1}\left[\frac{4g_C^2}{m_\kappa^2-M_Z^2}\ln\left(\frac{m_\kappa^2}{M_Z^2}\right)+\right. \\ \left.+\frac{4g_D^2}{m_\kappa^2-M_{Z'}^2}\ln\left(\frac{m_\kappa^2}{M_{Z'}^2}\right)\right]+O(\epsilon)\,, \end{aligned} \tag{3.132}$$

where we used the coupling definitions:

$$\begin{cases} g_C = g_4\sqrt{\frac{3}{8}}(O_0)_{32} - g_2(O_0)_{22} \\ g_D = g_4\sqrt{\frac{3}{8}}(O_0)_{33} - g_2(O_0)_{23} \end{cases} . \tag{3.133}$$

While for the Majorana Left mass, using the notation $i, j, k = \kappa + 1$ (with $\kappa = 0, 1, 2$) for the neutrino mass eigenvalues, we get the expression:

$$
\begin{aligned}
m^{(\nu)}_{ML} = & -i(M'_\nu)_{32}(M'_\nu)_{23} \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}(O'_\nu)_{1i}(O'_\nu)_{2i}(O'_\nu)^2_{3j}(O'_\nu)_{1k}(O'_\nu)_{2k}\times \\
& \times \int \frac{d^4q}{(2\pi)^4}\gamma^\mu \frac{im_i}{q^2-m_i^2}\frac{im_j}{q^2-m_j^2}\frac{im_k}{q^2-m_k^2}\gamma^\nu \left[g_E^2\frac{-ig_{\mu\nu}}{q^2-M_Z^2} + g_F^2\frac{-ig_{\mu\nu}}{q^2-M_{Z'}^2}\right] = \\
= & \sum_{i=j=k} 4C_{ijk}g_E^2 m_i^3 \frac{m_i^4 - M_Z^4 + m_i^2M_Z^2\ln\left(\frac{M_Z^4}{m_i^4}\right)}{2m_i^2(M_Z^2-m_i^2)^3} + \\
& + \sum_{i\neq j\neq k} 4C_{ijk}\left[\frac{g_E^2 m_i^2 m_j m_k \ln\left(\frac{m_i^2}{M_Z^2}\right)}{(m_i^2-M_Z^2)(m_i^2-m_j^2)(m_i^2-m_k^2)} + (i\leftrightarrow j) + (i \leftrightarrow k)\right] + \\
& + \sum_{i=j\neq k} 4C_{ijk}g_E^2 m_i^2 m_k \left[\frac{m_i^2 - M_Z^2 - M_Z^2\ln\left(\frac{m_i^2}{M_Z^2}\right)}{(m_i^2-M_Z^2)^2(m_k^2-m_i^2)} - \right. \\
& \left. - \frac{m_i^2\ln\left(\frac{m_i^2}{M_Z^2}\right)}{(m_i^2-M_Z^2)(m_k^2-m_i^2)^2} + \frac{m_k^2\ln\left(\frac{m_k^2}{M_Z^2}\right)}{(m_k^2-M_Z^2)(m_k^2-m_i^2)^2}\right] + \\
& + \sum_{i\neq j=k} 4C_{ijk}g_E^2 m_i m_k^2 \left[\frac{m_k^2 - M_Z^2 - M_Z^2\ln\left(\frac{m_k^2}{M_Z^2}\right)}{(m_k^2-M_Z^2)^2(m_k^2-m_i^2)} - \right. \\
& \left. - \frac{m_k^2\ln\left(\frac{m_k^2}{M_Z^2}\right)}{(m_k^2-M_Z^2)(m_k^2-m_i^2)^2} + \frac{m_i^2\ln\left(\frac{m_i^2}{M_Z^2}\right)}{(m_i^2-M_Z^2)(m_k^2-m_i^2)^2}\right] + \\
& + \sum_{j\neq i=k} 4C_{ijk}g_E^2 m_i^2 m_j \left[\frac{m_i^2 - M_Z^2 - M_Z^2\ln\left(\frac{m_i^2}{M_Z^2}\right)}{(m_i^2-M_Z^2)^2(m_j^2-m_i^2)} - \right. \\
& \left. - \frac{m_k^2\ln\left(\frac{m_i^2}{M_Z^2}\right)}{(m_j^2-M_Z^2)(m_j^2-m_i^2)^2} + \frac{m_j^2\ln\left(\frac{m_j^2}{M_Z^2}\right)}{(m_j^2-M_Z^2)(m_j^2-m_i^2)^2}\right] + \\
& + \left\{(g_E, M_Z) \leftrightarrow (g_F, M_{Z'})\right\} + O(\epsilon)
\end{aligned}
\tag{3.134}
$$

with

$$
C_{ijk} = \frac{1}{(4\pi)^2}(O'_\nu)_{1i}(O'_\nu)_{2i}(M'_\nu)_{23}(O'_\nu)^2_{3j}(M'_\nu)_{32}(O'_\nu)_{2k}(O'_\nu)_{1k} \tag{3.135}
$$

and

$$\begin{cases} g_E = g_4\sqrt{\frac{3}{8}}(O_0)_{32} - g_2(O_0)_{12} \\ g_F = g_4\sqrt{\frac{3}{8}}(O_0)_{33} - g_2(O_0)_{13} \end{cases} . \tag{3.136}$$

So, considering the Dirac and Majorana mass corrections, we obtain the one-loop neutrino mass matrix of the form:

$$(M_\nu'^\dagger)^{(1-loop)} = \begin{pmatrix} m_{ML}^{(\nu)} & \frac{1}{2}(m_u + m_H^{(\nu)} + m_G^{(\nu)})^T & 0 \\ \frac{1}{2}(m_u + m_H^{(\nu)} + m_G^{(\nu)}) & (m_{MR}^{(\nu)})^* & \frac{k_R v_R}{2\sqrt{2}} \\ 0 & \frac{k_R v_R}{2\sqrt{2}} & m_0 \end{pmatrix} . \tag{3.137}$$

3.6 Numerical results without inter-family mixing

For the numerical simulations, we use again MultiNest and the experimental values (2.182) and (2.186), to test the likelihood of the parameters scanned by the code. For the Yukawa couplings k_{Ra}, λ_{1a} and λ_{2a} (with $a = e, \mu, \tau$ the family index) and the parameter m_a, related to sterile neutrino masses by the identity: $m_{0a} = m_a \frac{v_R}{\sqrt{2}}$, we use the interval $[-10, 10]$. For the scalar parameters, considering the case (3.36), we need only the combination $\xi_5 - \xi_2 = \xi > 0$ and a parameter t related to $\tan\beta$ by the identity: $\tan\beta = t \cdot 10^{-12}$; for both the parameters ξ and t we use the interval $(0, 20]$, in agree with what we evaluated in Sec. 3.2.1. In the first example we obtain a Log-likelihood ~ -77.7; while in the second one, instead to use again all the fermion masses as references in the code, we concentrate on the third family using only the bottom quark and tau lepton masses to test the likelihood of the parameters (t, ξ, m_τ, $k_{R\tau}$, $\lambda_{1\tau}$, $\lambda_{2\tau}$), and we obtain a Log-likelihood ~ -25.

3.6.1 Example 1

In this first example we get the following selection for the sterile neutrino masses:

$$\begin{cases} m_{0e} = -3.74 \cdot 10^{14}\,\text{GeV} \\ m_{0\mu} = -1.72 \cdot 10^{14}\,\text{GeV} \\ m_{0\tau} = -6.75 \cdot 10^{14}\,\text{GeV} \end{cases} ; \tag{3.138}$$

the Yukawa couplings:

$$\begin{cases} k_{Re} = -1.72 \\ k_{R\mu} = 0.13 \\ k_{R\tau} = -7.11 \end{cases}, \quad \begin{cases} \lambda_{1e} = -0.26 \cdot 10^{-5} \\ \lambda_{1\mu} = -2 \cdot 10^{-3} \\ \lambda_{1\tau} = -0.39 \end{cases} \quad \text{and} \quad \begin{cases} \lambda_{2e} = -0.35 \cdot 10^{-5} \\ \lambda_{2\mu} = -4 \cdot 10^{-4} \\ \lambda_{2\tau} = 6.7 \cdot 10^{-3} \end{cases}; \tag{3.139}$$

the scalar parameters:

$$\xi_5 - \xi_2 = 19.996 \quad \text{and} \quad \tan\beta = 4.124 \cdot 10^{-13}, \tag{3.140}$$

from which we obtain the Higgs masses:

$$M_{H0} = 130.41\,\text{GeV} \quad \text{and} \quad M_{H_1^\pm} = 788.69\,\text{GeV}, \tag{3.141}$$

where M_{H0} is very close to the SM Higgs mass, while $M_{H_1^\pm}$ can be associated to a BSM charged Higgs respecting the experimental limit.
With this set of parameters, we achieve the following results for the up-type quark masses:

$$\begin{cases} m_{up} = 0.33\,\text{MeV} \\ m_{charm} = 272.61\,\text{MeV} \\ m_{top} = 81.24\,\text{GeV} \end{cases}, \tag{3.142}$$

all values within the 2σ range from the experimental values in (2.182). We obtain, instead, a bad result also for the down-type quarks and the charged leptons:

$$\begin{cases} m_{down} = 0.4\,\text{MeV} \\ m_{strange} = 54.06\,\text{MeV} \\ m_{bottom} = 1.37\,\text{GeV} \end{cases} \quad \text{and} \quad \begin{cases} m_{electron} = 0.48\,\text{MeV} \\ m_{muon} = 54.80\,\text{MeV} \\ m_{tau} = 1.36\,\text{GeV} \end{cases}, \tag{3.143}$$

in fact we can see that there is a virtually zero split between the two sectors, in this way the tree level relation $m_d = m_e$ seems still approximately valid also at one-loop. Finally for the neutrinos, we obtain the ratio:

$$r = 0.0086, \tag{3.144}$$

which is far away from the values in (2.186).

3.6.2 Example 2

In this second example we focus our attention only on the masses of the third fermion family. In particular we test the likelihood of the parameters for the bottom and tau masses only, for which we would expect a mass difference of the order of 500 MeV. So we get the parameters selection:

$$\xi_5 - \xi_2 = 19.999 \quad \text{and} \quad \tan\beta = 4.116 \cdot 10^{-13}\,, \tag{3.145}$$

from which we get the Higgs masses:

$$M_{H0} = 130.14\,\text{GeV} \quad \text{and} \quad M_{H_1^\pm} = 788.72\,\text{GeV}\,; \tag{3.146}$$

we for the tauonic sterile neutrino mass and Yukawa couplings we get:

$$m_{0\tau} = 5.02 \cdot 10^{14}\,\text{GeV}\,, \quad \lambda_{1\tau} = -1.28\,, \quad \lambda_{2\tau} = -0.01\,, \quad k_{R\tau} = 2.06\,. \tag{3.147}$$

With this selection of parametrs we obtain, this time, a mass splitting of $\sim$ 100 MeV for the bottom and the tau:

$$m_{bottom} = 1.65\,\text{GeV} \quad \text{and} \quad m_{tau} = 1.74\,\text{GeV}\,, \tag{3.148}$$

better than in the first example. However, now we have the problem of an unrealistic quark top mass:

$$m_{top} = 181.90\,\text{GeV}\,. \tag{3.149}$$

The model 2, therefore, seems not able to generate the correct mass splitting between the charged leptons and their family-related down-type quarks. Let us look at their one-loop mass difference formula:

$$m_e^{1-loop} - m_d^{1-loop} = m_e + m_G^{(e)} + m_H^{(e)} - m_d - m_G^{(d)} - m_H^{(d)} \qquad \text{with} \qquad m_d = m_e\,, \tag{3.150}$$

where for the difference in the gauge corrections we have:

$$
\begin{aligned}
m_G^{(e)} - m_G^{(d)} = & - \frac{m_d g_4}{(4\pi)^2}\left[g_4 \frac{4}{3}(O_0)_{32}^2 + g_2\sqrt{\frac{2}{3}}\Big((O_0)_{12} + (O_0)_{22}\Big)(O_0)_{32}\right] \times \\
& \times \frac{M_Z^2}{m_d^2 - M_Z^2} \ln\left(\frac{m_d^2}{M_Z^2}\right) - \\
& - \frac{m_d g_4}{(4\pi)^2}\left[g_4 \frac{4}{3}(O_0)_{33}^2 + g_2\sqrt{\frac{2}{3}}\Big((O_0)_{13} + (O_0)_{23}\Big)(O_0)_{33}\right] \times \\
& \times \frac{M_{Z'}^2}{m_d^2 - M_{Z'}^2} \ln\left(\frac{m_d^2}{M_{Z'}^2}\right) ,
\end{aligned}
\tag{3.151}
$$

while the difference in the scalar loop corrections is given by:

$$
\begin{aligned}
m_H^{(e)} - m_H^{(d)} = & \frac{m_u \lambda_1 \lambda_2}{(4\pi)^2}\Bigg\{(O_{H\pm})_{11}^2 \frac{M_{H_1^\pm}^2}{m_u^2 - M_{H_1^\pm}^2} \ln\left(\frac{m_u^2}{M_{H_1^\pm}^2}\right) + \\
& + (O_{H\pm})_{12}^2 \frac{M_{W_R}^2}{m_u^2 - M_{W_R}^2} \ln\left(\frac{m_u^2}{M_{W_R}^2}\right) + \\
& + \left[2\gamma_E - 2\ln(4\pi) + 2\ln\left(\frac{m_u^2}{\mu^2}\right) + \frac{M_{\phi_2^\pm}^2}{m_u^2 - M_{\phi_2^\pm}^2} \ln\left(\frac{m_u^2}{M_{\phi_2^\pm}^2}\right)\right]\Bigg\} - \\
& - \sum_{\kappa=0}^{2} \frac{m_\kappa \lambda_1 \lambda_2}{(4\pi)^2}(O'_\nu)_{1,\kappa+1}(O'_\nu)_{2,\kappa+1} \times \\
& \times \Bigg\{(O_{H\pm})_{11}^2 \frac{M_{H_1^\pm}^2}{m_\kappa^2 - M_{H_1^\pm}^2} \ln\left(\frac{m_\kappa^2}{M_{H_1^\pm}^2}\right) + \\
& + (O_{H\pm})_{12}^2 \frac{M_{W_R}^2}{m_\kappa^2 - M_{W_R}^2} \ln\left(\frac{m_\kappa^2}{M_{W_R}^2}\right) + \\
& + \left[2\gamma_E - 2\ln(4\pi) + 2\ln\left(\frac{m_\kappa^2}{\mu^2}\right) + \frac{M_{\phi_2^\pm}^2}{m_\kappa^2 - M_{\phi_2^\pm}^2} \ln\left(\frac{m_\kappa^2}{M_{\phi_2^\pm}^2}\right)\right]\Bigg\} .
\end{aligned}
\tag{3.152}
$$

We can see that the larger contribution in the mass splitting comes from the scalar corrections, because we have in the numerators the heavy neutrino and the up-type quark masses. So, in order to generate at one-loop a realistic mass difference between tauon and bottom, we have to increase the tree level top mass $m_t = \lambda_{1\tau} u_1 + \lambda_{2\tau} u_2$, but obtaining at the end a too large m_{top}.

Chapter 4

Conclusions

In this thesis we have described two Pati-Salam models, which differ for the scalar content, and their capability to reproduce the experimental values for the fermion masses and mixings, by means of one-loop corrections. Both models contain sterile neutrinos, and furthermore, imposing on the gauge coupling constants the condition $g_R(M_U) = g_L(M_U)$, they share almost the same partial-unification scale of the order of $M_U \sim 10^{14}$ GeV.

For model 1, which does not contain any tree-level mass term for the SM fermions, we have demonstrated that we need two Higgs multiplets in the representation $(4, 2, 1)$ and two $(4, 1, 2)$, under the group $SU(4) \otimes SU(2)_L \otimes SU(2)_R$, in order to generate the correct mass spectra for the quarks and the charged leptons. However, we have also seen, by means of numerical simulations, that it seems not enough to obtain the neutrino squared masses ratio r value close to the experimental one; the same happens for the mixings of the CKM and PMNS matrices.

For model 2, instead, the scalar content given by a bi-doublet $(1, 2, 2)$ and a multiplet $(4, 1, 2)$ generates tree-level mass terms for all fermions; in particular, for each fermion family, the identity $m_e = m_d$ is valid a at tree-level, but it is still approximately valid at one-loop. Therefore the model 2 is not able to generate the correct mass hierarchy by means of the one-loop corrections. In fact, we have shown that it is necessary a too heavy top quark in order to generate a not negligible mass difference between the quark bottom and the tau lepton.

Possible approaches to follow in order to try to save the two models could be:

- to evaluate the two-loops corrections;

- to remove all the approximations applied in both the scalar potentials, in order to simplify the analitycal calculations of Higgs masses and mixings;

- to relax the request $g_R(M_U) = g_L(M_U)$ in order to have the freedom to move, in agree with the experimental limits, the partial-unification scale M_U; a lower scale for M_U could increase for example the Higgs mixings, generally of the order of $O(v_L/v_R)$ and the contributions coming from the logarithm of the type $\ln(m_\psi^2/M_H^2)$, obatainig larger corrections.

Appendix A

Generators of $SU(4)$ and $SU(2)$

Generators $\frac{\lambda_a}{2}$ for the $SU(4)$ gorup:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{A.1}$$
$$\lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$
$$\lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \lambda_{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

Generators $\frac{\sigma_i}{2}$ for the $SU(2)$ group:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.2}$$

Appendix B

Tensor products in $SU(4)$ and $SU(2)$

In $SU(4)$ the product between two fundamental representations is given by:

$$\begin{aligned} 4 \otimes 4 = A_\alpha B_\beta &= \frac{1}{2}(A_\alpha B_\beta + A_\beta B_\alpha) + \frac{1}{2}(A_\alpha B_\beta - A_\beta B_\alpha) = \\ &= S_{\alpha\beta} + \frac{1}{2} A_\mu B_\nu (\delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta) = S_{\alpha\beta} + \frac{1}{2} A_\mu B_\nu \epsilon^{\gamma\rho\mu\nu} \epsilon_{\gamma\rho\alpha\beta} = \\ &= S_{\alpha\beta} + \frac{1}{2} \epsilon_{\alpha\beta\gamma\rho} T^{\gamma\rho} = 10 \oplus 6\,, \end{aligned} \tag{B.1}$$

where the representation 10 is a complex symmetric tensor, while the representation 6 is a real antisymmetric tensor. The product between fundamental (A_α) and anti-fundamental ($A^\alpha = A^*_\alpha$) representations, instead, is:

$$4 \otimes \bar{4} = A_\alpha B^\beta = \left(A_\alpha B^\beta - \frac{1}{2} A_\mu B^\mu \delta^\beta_\alpha \right) + \frac{1}{2} A_\mu B^\mu \delta^\beta_\alpha = 15 \oplus 1\,. \tag{B.2}$$

Here we list some products of irreducible representations:

$$\begin{aligned} &\bar{10} \otimes 10 = 84 \oplus 15 \oplus 1\,; \\ &10 \otimes 6 = 45 \oplus 15\,; \\ &\bar{10} \otimes 6 = \bar{45} \oplus 15\,; \\ &6 \otimes 6 = 20 \oplus 15 \oplus 1\,; \\ &15 \otimes 15 = 84 \oplus 45 \oplus \bar{45} \oplus 20 \oplus 15 \oplus 15 \oplus 1\,. \end{aligned} \tag{B.3}$$

The group $SU(2)$ brings more difficulties to use the tensorial approach because its fundamental representation is pseudo-real ($2 \sim \bar{2}$), in fact we can write:

$$\begin{aligned} 2 \otimes 2 = A_i B_j &= \frac{1}{2}(A_i B_j + A_j B_i) + \frac{1}{2}(A_i B_j - A_j B_i) = \\ &= S_{ij} + \frac{1}{2} A_k B_l (\delta_i^k \delta_j^l - \delta_i^l \delta_j^k) = S_{ij} + \frac{1}{2} A_k B_l \epsilon^{kl} \epsilon_{ij} = \\ &= S_{ij} + \frac{1}{2} \epsilon_{ij} A_k B^k = 3 \oplus 1 \,, \end{aligned} \tag{B.4}$$

where the representation 3 is represented by a symmetric tensor S_{ij}, but we have also:

$$2 \otimes \bar{2} = A_i B^j = \left(A_i B^j - \frac{1}{2} A_k B^k \delta_i^j \right) + \frac{1}{2} A_k B^k \delta_i^j = 3 \oplus 1 \tag{B.5}$$

where the representation 3 is now represented by a traceless tensor. Unlike what happens in a group like $SU(4)$, where we have $A_\alpha B^\alpha \neq A^\alpha B_\alpha$, since for the fundamental representation we have $4 \sim A_\alpha \neq A^*_\alpha = A^\alpha \sim \bar{4}$, in the group $SU(2)$ we can transorm a covariant vector ($A_i \sim 2$) in a controvariant one ($A^i \sim \bar{2}$) using the Levi-Civita tensor ϵ_{ij} (the completely antisymmetric tensor). However we have to pay attention to the fact that for the fundamental representation the relation $\bar{2} \sim \epsilon^{ij} A_j = \hat{A}^i \neq A^i = A^*_i \sim \bar{2}$ is valid, as we can see looking at the components:

$$A_i = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \Longrightarrow \begin{cases} A^i = (A_i)^* = \begin{pmatrix} a_1^* & a_2^* \end{pmatrix} \\ \hat{A}^i = \epsilon^{ij} A_j = \begin{pmatrix} a_2 & -a_1 \end{pmatrix} \end{cases} , \tag{B.6}$$

from which it is simple to verify the identities:

$$\begin{cases} A^i A_i = \hat{A}^i \hat{A}_i = |a_1|^2 + |a_2|^2 \\ A_i \hat{A}^i = A_i \epsilon^{ij} A_j = a_1 a_2 - a_2 a_1 = 0 \end{cases} . \tag{B.7}$$

Using the inverse relation $A_i = -\epsilon_{ij} \hat{A}^j$, instead, we can rewrite the product $\bar{2} \otimes 2$ as:

$$A^i B_i = \begin{cases} (-\epsilon^{ij} \hat{A}_j)(-\epsilon_{ik} \hat{B}^k) = (\delta_i^i \delta_k^j - \delta_k^i \delta_i^j) \hat{A}_j \hat{B}^k = \delta_k^j \hat{A}_j \hat{B}^k \\ (-\epsilon^{ij} \hat{A}_j) B_i = \hat{A}_j \epsilon^{ji} B_i = \hat{A}_i \hat{B}^i \end{cases} . \tag{B.8}$$

To conclude this chapter we show the product $3 \otimes 3$, which can be written as a product of two symmetric tensor $H_{ij} = H_{ji}$ of $SU(2)$:

$$\begin{aligned} 3\otimes 3 = H_{ij}H_{kl} = \frac{1}{3}(H_{ij}H_{kl} + H_{ik}H_{jl} + H_{il}H_{jk}) + \frac{1}{3}\Big[(H_{ij}H_{kl} - H_{ik}H_{jl}) + \\ + (H_{ij}H_{lk} - H_{il}H_{jk})\Big] = S_{ijkl} + \frac{1}{6}(\epsilon_{il}\epsilon_{jk} + \epsilon_{ik}\epsilon_{jl})\epsilon^{mq}\epsilon^{np}H_{mn}H_{pq} = 5\oplus 3\oplus 1\,, \end{aligned} \tag{B.9}$$

or as a product of two traceless tensor in the form:

$$\begin{aligned} 3\otimes 3 &= \Big(A_iA^j - \frac{1}{2}A_kA^k\delta_i^j\Big)\Big(A_xA^y - \frac{1}{2}A_zA^z\delta_x^y\Big) = \\ &= T_{ix}^{jy} + \frac{1}{2}A_kA^k\Big(\frac{1}{2}A_zA^z\delta_i^j\delta_x^y - \delta_i^jA_xA^y - \delta_x^yA_iA^j\Big) = 5\oplus 3\oplus 1\,. \end{aligned} \tag{B.10}$$

Appendix C

SM Higgs potential and Goldstone bosons

In the Standard Model, the Higgs field φ is composed by two complex scalar fields electrically charged, $\varphi_1 = \varphi^+$ and $\varphi_2 = \varphi^0$, that form two doublets of $SU(2)_L$:

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \qquad \text{and} \qquad \tilde{\varphi} = i\sigma_2 \varphi^* = \begin{pmatrix} \varphi_2^* \\ -\varphi_1^* \end{pmatrix}, \tag{C.1}$$

where the VEV $\langle\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & v_L \end{pmatrix}^T$ gives mass to the charged leptons and the down-type quarks, while $\langle\tilde{\varphi}\rangle$ generates mass for the up-type quarks. Because both these doublets transform, under gauge transformation, as:

$$\varphi \to U_L \varphi \qquad \text{and} \qquad \tilde{\varphi} \to U_L \tilde{\varphi} \tag{C.2}$$

with $U_L \in SU(2)_L$, we have that the most general gauge invariant scalar potential can be written as:

$$V(\varphi) = -\mu^2 \varphi^\dagger \varphi + \lambda \big(\varphi^\dagger \varphi\big)^2, \tag{C.3}$$

where the doublet $\tilde{\varphi}$ does not appeare explicitly because, by means of the following relations:

$$\begin{cases} \tilde{\varphi}^\dagger \tilde{\varphi} = \varphi^\dagger \varphi \\ \varphi^\dagger \tilde{\varphi} = 0 \end{cases}, \tag{C.4}$$

all the gauge invariant terms containing $\tilde{\varphi}$ can be absorbed in the terms in (C.3). Furthermore, if we collect the two doublets φ and $\tilde{\varphi}$ in a bi-doublet H, in the following way:

$$H = \begin{pmatrix} \tilde{\varphi} & \varphi \end{pmatrix} = \begin{pmatrix} \varphi_2^* & \varphi_1 \\ -\varphi_1^* & \varphi_2 \end{pmatrix} , \tag{C.5}$$

we can rewrite the scalar potential (C.3) as:

$$V(\varphi) = -\frac{\mu^2}{2} Tr[H^\dagger H] + \frac{\lambda}{4}\Big(Tr[H^\dagger H]\Big)^2 , \tag{C.6}$$

where we can see that there exists an extra $SU(2)_X$ symmetry for the Higgs potential (the custodial symmetry), due to the gauge transformation of the bi-doublet:

$$H \to U_L H U_X^\dagger \tag{C.7}$$

with $U_L \in SU(2)_L$ and $U_X \in SU(2)_X$.

Writing the potential of the Higgs doublet $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ in components, we obtain the form:

$$V(\varphi) = -\mu^2 \varphi^\dagger \varphi + \lambda(\varphi^\dagger \varphi)^2 = -\mu^2(\varphi_1^* \varphi_1 + \varphi_2^* \varphi_2) + \lambda(\varphi_1^* \varphi_1 + \varphi_2^* \varphi_2)^2 , \tag{C.8}$$

from which we compute the minimum calculating the first derivative:

$$\left.\frac{\partial V(\varphi)}{\partial \varphi_i}\right|_{\langle\varphi\rangle} = \Big[-\mu^2 + 2\lambda(\mid \varphi_1 \mid^2 + \mid \varphi_2 \mid^2)\Big]\varphi_i^*\Big|_{\langle\varphi\rangle} = 0 , \tag{C.9}$$

where the solutions are given by:

$$\langle\varphi_1\rangle = \langle\varphi_2\rangle = 0 ; \tag{C.10}$$

$$\mid \langle\varphi_1\rangle \mid^2 + \mid \langle\varphi_2\rangle \mid^2 = \frac{\mu^2}{2\lambda} = v^2 ; \tag{C.11}$$

which bring to two possibilities:

$$\text{no symmetry-breaking:} \qquad \langle\varphi_1\rangle = \langle\varphi_2\rangle\,; \tag{C.12}$$

$$\text{symmetry-breaking:} \qquad \langle\varphi\rangle = \begin{pmatrix} 0 \\ v\,e^{i\alpha} \end{pmatrix}. \tag{C.13}$$

If now we calculate the second derivatives:

$$\begin{aligned} &\frac{\partial^2 V(\varphi)}{\partial\varphi_i^*\partial\varphi_i} = \Big[-\mu^2 + 2\lambda(\mid\varphi_1\mid^2 + \mid\varphi_2\mid^2)\Big] + 2\lambda\mid\varphi_i\mid^2\,, \\ &\frac{\partial^2 V(\varphi)}{\partial\varphi_2^*\partial\varphi_1} = 2\lambda\varphi_2\varphi_1^* \quad \text{and} \quad \frac{\partial^2 V(\varphi)}{\partial\varphi_1^*\partial\varphi_2} = 2\lambda\varphi_1\varphi_2^*\,, \end{aligned} \tag{C.14}$$

we find the mass term:

$$M_\varphi^2\varphi^\dagger\varphi = \varphi^\dagger\begin{pmatrix} 0 & 0 \\ 0 & 2\lambda v^2 \end{pmatrix}\varphi \quad \Rightarrow \begin{cases} m_1^2\Big(\mid\Re(\varphi_1)\mid^2 + \mid\Im(\varphi_1)\mid^2\Big) = 0 \\ m_2^2\Big(\mid\Re(\varphi_2)\mid^2 + \mid\Im(\varphi_2)\mid^2\Big) = 2\lambda v^2\varphi_2^*\varphi_2 \end{cases}, \tag{C.15}$$

from which we find two Goldstone bosons; while the last ones, which must give mass to the third vector boson of $SU(2)_L$, must be a combination of the real scalar fields ξ_2 and η_2 that compose $\varphi_2 = \xi_2 + i\eta_2$; in fact we have:

$$\varphi_2^*(2\lambda v^2)\varphi_2 = \begin{pmatrix} \xi_2 & \eta_2 \end{pmatrix}\begin{pmatrix} 2\lambda v^2 & i2\lambda v^2 \\ -i2\lambda v^2 & 2\lambda v^2 \end{pmatrix}\begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix}, \tag{C.16}$$

from which we obtain the mass eigenvalues:

$$\det\begin{pmatrix} 2\lambda v^2 - p & i2\lambda v^2 \\ -i2\lambda v^2 & 2\lambda v^2 - p \end{pmatrix} = p(p - 4\lambda v^2) = 0 \quad \Rightarrow \quad m_3^2 = 4\lambda v^2 \quad \text{and} \quad m_4^2 = 0\,. \tag{C.17}$$

Therefore the last Goldstone boson will be a linear combination of the real fields ξ_2 and η_2. It can also be visualized writing the Higgs field as $\varphi = \begin{pmatrix} \xi_1 + i\eta_1 \\ \xi_2 + i\eta_2 \end{pmatrix}$, with $\xi_{1,2}$ and $\eta_{1,2}$ real scalar fields, and repeating the procedure above. Using the VEV: $\begin{pmatrix} \langle\xi_1\rangle + i\langle\eta_1\rangle \\ \langle\xi_2\rangle + i\langle\eta_2\rangle \end{pmatrix} =$

$\begin{pmatrix} 0 \\ v\, e^{i\alpha} \end{pmatrix}$, and the basis of real fields $h^T = (\xi_1, \eta_1, \xi_2, \eta_2)$, we arrive at the mass term:

$$\frac{1}{2} h_i^T \mathbf{M}^2_{ij} h_j = \frac{1}{2} \begin{pmatrix} \xi_1 & \eta_1 & \xi_2 & \eta_2 \end{pmatrix} 4\lambda v^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cos^2\alpha & \cos\alpha\sin\alpha \\ 0 & 0 & \cos\alpha\sin\alpha & \sin^2\alpha \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \\ \xi_2 \\ \eta_2 \end{pmatrix} \tag{C.18}$$

where the first two Goldstone bosons are those related to the complex field φ_1 (the same found before), while the third, and last, Goldstone boson is given, similarly to (C.17), by a linear combination of the real fields ξ_2 and η_2. Instead, in the case we choose a real VEV ($\alpha = 0$) we would obtain:

$$\frac{1}{2} h_i^T \mathbf{M}^2_{ij} h_j = \frac{1}{2} \begin{pmatrix} \xi_1 & \eta_1 & \xi_2 & \eta_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4\lambda v^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \\ \xi_2 \\ \eta_2 \end{pmatrix} , \tag{C.19}$$

where the real field η_2 represents directly the last Goldstone boson.

Appendix D

Normalization of the Hypercharge

In order to normalize the hypercharge generator of $U(1)_Y$ in a way consistent to the other diagonal generators of the Standard Model group [20], we start considering the matter fields of the SM:

$$\begin{pmatrix} u \\ d \end{pmatrix}_L , \quad \begin{pmatrix} \nu \\ e \end{pmatrix}_L , \quad \bar{u}_R \, , \quad \bar{d}_R \, , \quad \text{and} \quad \bar{e}_R \tag{D.1}$$

(all these taken in $n_g = 3$ different generations). The doublets, under $SU(2)_L$, are related to the eigenvalues of the diagonal generator $\frac{\sigma_3}{2}$ in this way: ν_L and u_L^a (with $a = 1, 2, 3$ the colour index) are linked to the eigenvalue $\frac{1}{2}$; e_L and d_L^a linked to the eigenvalues $-\frac{1}{2}$; while the singlets $\bar{e}_R$, $\bar{u}_R$ and $\bar{d}_R$ correspond to the eigenvalues 0. Therefore we have the trace:

$$Tr\left[\left(g_{2L}\frac{\sigma_3}{2}\right)^2\right] = n_g\,(3+1)\left[\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2\right] g_{2L}^2 = 2n_g g_{2L}^2 \, . \tag{D.2}$$

The extension of σ_3 in the group $SU(3)_C$ is given by:

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{D.3}$$

and, as above, we can relate the fermionic fields to the eigenvalues of the generators $\frac{\lambda_3}{2}$ in this way: $(u,d)_L^a$, $\bar{u^b}_R$ and $\bar{d^b}_R$ are linked to the eigenvalue $\frac{1}{2}$ if the colour numbers are $a = 1$ and $b = 2$; to the eigenvalue $-\frac{1}{2}$ if $a = 2$ and $b = 1$; while all the others fermions are related to the eigenvalue 0. Then the trace of the squared generator is given by:

$$Tr\left[\left(g_{3C}\frac{\lambda_3}{2}\right)^2\right] = n_g\,(2+1+1)\left[\left(\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2\right]g_{3C}^2 = 2n_g g_{3C}^2\,, \tag{D.4}$$

where we can note that (D.2) and (D.4) have the same constant in front of the squared gauge couplings. What we want is to mantain this constant also for the group $U(1)_Y$, whose generator is given by $\frac{Y}{2} = Q - \frac{\sigma_3}{2}$. Again we can relate the eigenvalues of this generator to the matter fields as follows: the eigenvalue $-\frac{1}{2}$ to the fields $(\nu, e)_L$; the eigenvalue 1 to the field $\bar{e}_R$; the eigenvalue $\frac{1}{6}$ to the fields $(u,d)_L^a$; the eigenvalues $-\frac{2}{3}$ and $\frac{1}{3}$ to the fields $\bar{u^a}_R$ and $\bar{d^a}_R$ (with $a = 1,2,3$), respectively. So we have that:

$$\begin{aligned} Tr\left[\left(g_{1Y}\frac{Y}{2}\right)^2\right] &= n_g\left[2\left(-\frac{1}{2}\right)^2+1+3\cdot 2\left(\frac{1}{6}\right)^2+3\left(-\frac{2}{3}\right)^2+3\left(\frac{1}{3}\right)^2\right]g_{1Y}^2 = \\ &= \frac{10}{3}n_g g_{1Y}^2\,. \end{aligned} \tag{D.5}$$

Thus, using the redefinitions:

$$g_{1Y} = \sqrt{\frac{3}{5}}g_Y \quad \text{and} \quad Y = \sqrt{\frac{5}{3}}Y'\,, \tag{D.6}$$

we find the correct value for the trace:

$$Tr\left[\left(g_{1Y}\frac{Y}{2}\right)^2\right] = Tr\left[\left(g_Y\frac{Y'}{2}\right)^2\right] = 2n_g g_Y^2\,. \tag{D.7}$$

Appendix E

Matching conditions for coupling constants

Let us consider the breaking of the type $SU(4) \otimes SU(2)_R \rightarrow SU(3)_C \otimes U(1)_Y$. What are the matching conditions between the couplings of the conserved $U(1)_Y$ and those of the broken groups? We find the answer following the procedure in [21]. We know from (2.14), and using the normalization redefinition in (D.6), that the hypercharge generator $Y/2$ can be written as:

$$\frac{Y}{2} = \frac{\sigma_{3R}}{2} + \sqrt{\frac{2}{3}}\frac{\lambda_{15}}{2} \dashrightarrow \text{Normalization:} \quad \frac{Y'}{2} = \sqrt{\frac{3}{5}}\frac{Y}{2} = \sqrt{\frac{3}{5}}\frac{\sigma_{3R}}{2} + \sqrt{\frac{2}{5}}\frac{\lambda_{15}}{2} \,. \tag{E.1}$$

Therefore, in general, we write the generator $Y/2$ (from here we sobstitute $Y' \rightarrow Y$ in the notation) as a combination of generators T^a of the higher groups:

$$\frac{Y}{2} = p_a T^a \quad \text{with} \quad \sum_a p_a^2 = 1\,; \tag{E.2}$$

moreover, since the symmetry under $U(1)_Y$ is conserved, we must have that:

$$Y \cdot \langle R \rangle = 0\,, \tag{E.3}$$

where $\langle R \rangle$ is the VEV of the Higgs boson multiplet responsible for the symmetry-breaking from the Pati-Salam group to the SM. Furthermore, the Higgs mechanism, using the covariant derivative:

$$D_\mu R = \partial_\mu R + i\left(g_4 G_\mu + g_2 W_{\mu R}\right) R = \partial_\mu R + i g_a T^a A^a_\mu R\,, \tag{E.4}$$

generates for the vector bosons the squared mass:

$$M^2_{ab} = \langle R\rangle^\dagger g_a g_b T^a T^b \langle R\rangle\,, \tag{E.5}$$

so that we have the mass term:

$$A^a_\mu M^2_{ab} A^{b\mu} = A^a_\mu O_{ai}\left[O^T M^2 O\right]_{ij}(O^T)_{jb}A^{b\mu} = B^i_\mu (M^2)^{diag}_{ij} B^{j\mu}\,, \tag{E.6}$$

where the orthogonal matrix O (with $O_{ik}O_{jk} = O_{ki}O_{kj} = \delta_{ij}$) transforms the gauge eigenvectors A^a_μ to the mass eigenstates B^i_μ. At this point, let us call B^Y_μ the massless vector bosons related to the $U(1)_Y$ symmetry:

$$B^Y_\mu = (O^T)_{Ya} A^a_\mu = q_a A^a_\mu \quad \text{with} \quad \delta_{YY} = (O_{kY})^2 = \sum_k q_k^2 = 1\,. \tag{E.7}$$

To look for the form of the vectors q_i, let us start writing:

$$0 = B^x_\mu (M^2)^{diag}_{xY} B^{Y\mu} = A^a_\mu O_{ax}\left[O^T M^2 O\right]_{xY} O_{bY} A^{b\mu} \quad \Rightarrow \quad O_{ix} M^2_{ij} O_{jY} = 0 \tag{E.8}$$

$$\Rightarrow O_{kx}\left[O_{ix} M^2_{ij} O_{jY}\right] = \delta_{ki} M^2_{ij} O_{jY} = M^2_{kj} q_j = 0\,; \tag{E.9}$$

from this, with (E.3) and (E.2), we obtain:

$$\begin{aligned} M^2_{ab} q_b &= \left[\langle R\rangle^\dagger g_a T^a\right]\left[g_b T^b \langle R\rangle q_b\right] = 0 = \left[\langle R\rangle^\dagger g_a T^a\right]\left[Y \cdot \langle R\rangle\right] \\ &\Longrightarrow q_a = N \frac{p_a}{g_a}\,, \end{aligned} \tag{E.10}$$

where N is the normalization factor:

$$\sum_a q_a^2 = N^2 \sum_a \left(\frac{p_a}{g_a}\right)^2 = 1 \quad \Longrightarrow \quad N = \sqrt{\sum_k \left(\frac{p_k}{g_k}\right)^2}\,. \tag{E.11}$$

To conclude we have to find the form of the gauge coupling costant g_Y related to the group $U(1)_Y$, so we look at the interaction term between the gauge bosons and the fermions:

$$g_a A^a_\mu T^a \bar{\psi}\gamma^\mu\psi = g_a \left[q_a B^Y_\mu + O_{ak} B^{k\neq Y}_\mu\right] T^a \bar{\psi}\gamma^\mu\psi \tag{E.12}$$

$$\Longrightarrow g_a q_a T^a = g_Y \frac{Y}{2} = g_Y p_a T^a \quad \Longrightarrow \quad q_a = g_Y \frac{p_a}{g_a}\,, \tag{E.13}$$

so $g_Y = N$. Finally, for the matching conditions we have:

$$\alpha_Y^{-1} = \frac{4\pi}{g_Y^2} = 4\pi \sum_a \left(\frac{p_a}{g_a}\right)^2 = \sum_a p_a^2 \alpha_a^{-1}\,, \tag{E.14}$$

and therefore for the breaking from the Pati-Salam model to SM the matching conditions are given by:

$$\alpha_Y^{-1} = \frac{3}{5}\alpha_{2R}^{-1} + \frac{2}{5}\alpha_4^{-1}\,. \tag{E.15}$$

Appendix F

Dirac and Majorana mass

The fermionic field can be written in components as:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \quad \text{and} \quad \bar{\Psi} = \Psi^\dagger \gamma_0 = \begin{pmatrix} \psi_1^* & \psi_2^* & \psi_3^* & \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix} \,, \tag{F.1}$$

where we have used the chiral representations:

$$\psi_L = \frac{1-\gamma_5}{2} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_R = \frac{1+\gamma_5}{2} = \begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ \psi_4 \end{pmatrix} \,, \tag{F.2}$$

and the the γ-matrix definitions:

$$\gamma_5 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \,, \quad \gamma_0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \quad \text{and} \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad \text{with } i = 1,2,3 \quad , \tag{F.3}$$

so that the Dirac mass term takes the form:

$$m_D \bar{\Psi}\Psi = m_D(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R) = m_D(\psi_3^* \psi_1 + \psi_4^* \psi_2 + \psi_1^* \psi_3 + \psi_2^* \psi_4) \,. \tag{F.4}$$

Defining, instead, the operation of charge conjugation as:

$$\Psi^C = i\gamma_2\gamma_0\bar{\Psi}^T = \begin{pmatrix} \psi_4^* \\ -\psi_3^* \\ -\psi_2^* \\ \psi_1^* \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}^C = \begin{pmatrix} (\Psi^C)_L \\ (\Psi^C)_R \end{pmatrix} = \begin{pmatrix} (\psi_R)^C \\ (\psi_L)^C \end{pmatrix}, \tag{F.5}$$

where

$$\begin{cases} (\psi_R)^C = i\sigma_2\psi_R^* = \begin{pmatrix} \psi_4^* \\ -\psi_3^* \end{pmatrix} = \frac{1-\gamma_5}{2}\Psi^C = (\Psi^C)_L \\ \\ (\psi_L)^C = -i\sigma_2\psi_L^* = \begin{pmatrix} -\psi_2^* \\ \psi_1^* \end{pmatrix} = \frac{1+\gamma_5}{2}\Psi^C = (\Psi^C)_R \end{cases}, \tag{F.6}$$

if a fermion realizes the identity:

$$\Psi = \Psi^C \Rightarrow \begin{cases} \psi_1 = \psi_4^* \\ \psi_2 = -\psi_3^* \end{cases} \Rightarrow \Psi = \begin{pmatrix} \psi_L \\ (\psi_L)^C \end{pmatrix} = \begin{pmatrix} (\psi_R)^C \\ \psi_R \end{pmatrix}, \tag{F.7}$$

it is called a Majorana fermion, whose mass term, the Majorana mass term, takes the form:

$$\begin{aligned} \frac{m_M}{2}\bar{\Psi}\Psi &= \frac{m_M}{2}\left[(\psi_L)^{C\dagger}\psi_L + \psi_L^\dagger(\psi_L)^C\right] = \\ &= \frac{m_M}{2}\left[(\psi_L)^T(-i\sigma_2)^\dagger\psi_L + \psi_L^\dagger(-i\sigma_2)(\psi_L^\dagger)^T\right] = m_M(\psi_1\psi_2 + \psi_1^*\psi_2^*)\,. \end{aligned} \tag{F.8}$$

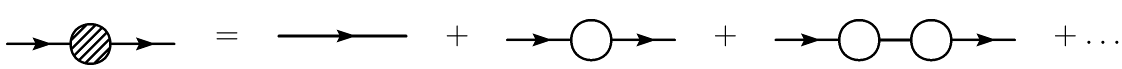
Appendix G

Mass shift and Weyl propagators

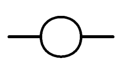
Let us consider the Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m_0)\psi - \frac{1}{4}(F_{\mu\nu})^2 - g\bar{\psi}\gamma^\mu G_\mu \psi \,, \tag{G.1}$$

the two-points function, for the fermions, can be written as:



$$\begin{aligned} &= \frac{i}{\widehat{p} - m_0 + i\epsilon} + \frac{i}{\widehat{p} - m_0 + i\epsilon}\Big(-i\Sigma(\widehat{p})\Big)\frac{i}{\widehat{p} - m_0 + i\epsilon} + \dots \\ &= \frac{i}{\widehat{p} - m_0 - \Sigma(\widehat{p})} \,, \end{aligned} \tag{G.2}$$

where $\widehat{p} = p_\mu \gamma^\mu$ and  $= -i\Sigma(\widehat{p})$ is the sum of one-particle irreducible diagrams.

In general, if we consider also chiral interactions, the function $\Sigma(\widehat{p})$ is of the form:

$$\begin{aligned} \Sigma(\widehat{p}) &= A\widehat{p} + Bm_0 + C\gamma_5\widehat{p} + D\widehat{p}\gamma_5 + Em_0\gamma_5 = \\ &= A\Big(\widehat{p} - m_0\Big) + (A+B)m_0 + C\gamma_5\Big(\widehat{p} - m_0\Big) + D\Big(\widehat{p} - m_0\Big)\gamma_5 - \\ &\quad - (E + C + D)m_0 \frac{(\widehat{p} - m_0)\gamma_5 + \gamma_5(\widehat{p} - m_0)}{2m_0} = \\ &= A\Big(\widehat{p} - m_0\Big) + (A+B)m_0 + \frac{C - E - D}{2}\gamma_5\Big(\widehat{p} - m_0\Big) + \\ &\quad + \frac{D - E - C}{2}\Big(\widehat{p} - m_0\Big)\gamma_5 = \\ &= A'\Big(\widehat{p} - m_0\Big) + B'm_0 + C'\gamma_5\Big(\widehat{p} - m_0\Big) + D'\Big(\widehat{p} - m_0\Big)\gamma_5 \end{aligned} \tag{G.3}$$

where A, B, C, D, E are functions of the coupling constant g and the cut-off Λ.

The physical mass is defined as the pole of the full propagator (G.2):

$$\left[\widehat{p} - m_0 - \Sigma(\widehat{p})\right]_{\widehat{p}=m_{ph}} = 0 \quad \Longrightarrow \quad \Sigma(\widehat{p} = m_{ph}) = \delta m \,, \tag{G.4}$$

where $\delta m = m_{ph} - m_0$ is called the mass shift. At the three-level, $o(g^0)$, we obtain:

$$\left[\widehat{p} - m_0\right]_{\widehat{p}=m_{ph}} = 0 \quad \Longrightarrow \quad m_0 = m_{ph}\big(1 + o(g^2)\big) \,; \tag{G.5}$$

therefore at 1-loop, $o(g^2)$, we have:

$$\Sigma(\widehat{p}) \sim A(g^2, \Lambda)\Big(\widehat{p} - m_0\Big) + B(g^2, \Lambda)m_0 = A(g^2, \Lambda)\Big(\widehat{p} - m_{ph}\Big) + \delta m + o(g^4) \tag{G.6}$$

where, because the $\Sigma(\widehat{p})$ starts from $o(g^2)$, we replaced the m_0 with his order-zero value to highlight the one-loop contribution to the mass shift.

In case we have a theory with a massless fermion, we can write the tree-level relation $m_0 = 0 + o(g^2)$, that we put in the definition (G.6) obtaining:

$$\Sigma(\widehat{p}) \sim A(g^2, \Lambda)\widehat{p} + \delta m + o(g^4) \,, \tag{G.7}$$

therefore $\Sigma(\widehat{p} = 0)$ gives the one-loop contribution to the mass shift, and it will correspond to the generation of a mass $m_{ph} = \delta m$.

An alternative for defining the fermion mass is to use the Weyl representation [23]. Let us start with a free fermionic Lagrangian of the type:

$$\begin{aligned} \mathcal{L} &= i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi = \\ &= i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L + i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R - m\psi_L^\dagger\psi_R - m\psi_R^\dagger\psi_L \,, \end{aligned} \tag{G.8}$$

where we have used the notation:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \text{with} \qquad \sigma^\mu = (1, \sigma_i) \quad \text{and} \quad \bar{\sigma}^\mu = (1, -\sigma_i) \,. \tag{G.9}$$

From (G.8) we can obtain the propagators:

$$\underset{\Psi\qquad\qquad\bar{\Psi}}{\longrightarrow} = \langle \Psi\bar{\Psi}\rangle = \langle \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \begin{pmatrix} \psi_L^\dagger & \psi_R^\dagger \end{pmatrix} \rangle = \langle \begin{pmatrix} \psi_L\psi_R^\dagger & \psi_L\psi_L^\dagger \\ \psi_R\psi_R^\dagger & \psi_R\psi_L^\dagger \end{pmatrix} \rangle =$$
$$= \frac{i(\gamma^\mu p_\mu + m)}{p^2 - m^2} = \frac{i}{p^2 - m^2} \begin{pmatrix} m & \sigma^\mu p_\mu \\ \bar{\sigma}^\mu p_\mu & m \end{pmatrix}, \tag{G.10}$$

where we can see:

$$\underset{\psi_L\qquad\qquad\bar{\psi}_L}{\longrightarrow} = \frac{ip_\mu\sigma^\mu}{p^2 - m_i^2} \tag{G.11}$$

$$\underset{\psi_R\qquad\qquad\bar{\psi}_R}{\longrightarrow} = \frac{ip_\mu\bar{\sigma}^\mu}{p^2 - m_i^2} \tag{G.12}$$

$$\underset{\psi_L\qquad\qquad\bar{\psi}_R}{\longrightarrow} = \underset{\psi_R\qquad\qquad\bar{\psi}_L}{\longrightarrow} = \frac{im}{p^2 - m_i^2}\,. \tag{G.13}$$

In the same way we can define the propagators for the charge conjugated fields:

$$\langle \Psi^c\bar{\Psi}^c\rangle = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \langle \begin{pmatrix} \psi_L\psi_R^\dagger & \psi_L\psi_L^\dagger \\ \psi_R\psi_R^\dagger & \psi_R\psi_L^\dagger \end{pmatrix} \rangle^* \begin{pmatrix} i\sigma_2^* & 0 \\ 0 & -i\sigma_2^* \end{pmatrix} =$$
$$= \langle \begin{pmatrix} (\psi_L)^c(\psi_L)^{c\dagger} & (\psi_L)^c(\psi_R)^{c\dagger} \\ (\psi_R)^c(\psi_L)^{c\dagger} & (\psi_R)^c(\psi_R)^{c\dagger} \end{pmatrix} \rangle =$$
$$= \frac{-i}{p^2 - m^2} \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} m & \sigma^{\mu *} p_\mu \\ \bar{\sigma}^{\mu *} p_\mu & m \end{pmatrix} \begin{pmatrix} i\sigma_2^* & 0 \\ 0 & -i\sigma_2^* \end{pmatrix} =$$
$$= \frac{-i}{p^2 - m^2} \begin{pmatrix} \sigma^\mu p_\mu & -m \\ -m & \bar{\sigma}^\mu p_\mu \end{pmatrix} = \langle \begin{pmatrix} (\psi^c)_R(\psi^c)_R^\dagger & (\psi^c)_R(\psi^c)_L^\dagger \\ (\psi^c)_L(\psi^c)_R^\dagger & (\psi^c)_L(\psi^c)_L^\dagger \end{pmatrix} \rangle, \tag{G.14}$$

Or equivalently, we can consider, as a first approximation, massless propagators for left- and right-handed fermions, and the mass term as couplings between a left and a right state:

$$\underset{\psi_{R,L}\qquad\qquad\bar{\psi}_{R,L}}{\longrightarrow} = \frac{\pm ip_\mu\sigma^\mu}{p^2 - m^2} \qquad \text{and} \qquad \underset{\psi_{R,L}\qquad\qquad\bar{\psi}_{L,R}}{-\!\!-\!\!\times\!\!-\!\!-} = -im\,. \tag{G.15}$$

However it is possible to recover the propagators seen above making infinite sums of the type:

$$\underset{\psi_L\qquad\qquad\bar{\psi}_R}{\longrightarrow} = -i\frac{p_\mu\sigma^\mu}{p^2}\,[-im]\,i\frac{p_\mu\sigma^\mu}{p^2} + \ldots \text{(odd number of mass insertions)} \tag{G.16}$$

$$\underset{\psi_R}{}\!\longrightarrow\!\underset{\bar{\psi}_R}{} = i\frac{p_\mu \sigma^\mu}{p^2} + i\frac{p_\mu \sigma^\mu}{p^2}\,[-im]\,(-i)\frac{p_\mu \sigma^\mu}{p^2}\,[-im]\,i\frac{p_\mu \sigma^\mu}{p^2} + \\ + \ldots \text{(even number of mass insertions)}\,. \tag{G.17}$$

At this point to find the value of the mass, instead of using the normalization condition (G.4), we use the one related to the coupling constants, in particular for the mass insertion in (G.15); so this tells us that the value of the mass can be identified with the value of the two-point function (between a left and a right state) in the limit of zero external momentum:

$$m = \lim_{k\to 0} i\left[-\!\!\bigcirc\!\!- \right]. \tag{G.18}$$

Appendix H

Extended model 1 - Higgs potential

Using $V_{LR} = V(L,R)$ given by (2.28), for what seen in Sec. 2.9.2 the part of the scalar potential with less than three fields $\Lambda_{i\alpha}$ and $T_{i\alpha}$ takes the form:

$$\begin{aligned}
\tilde{\mathbf{V}} = & V_{LR} - 2\mu_Y^2 \Lambda^{i\alpha}\Lambda_{i\alpha} - 2\mu_B^2 T^{i\alpha}T_{i\alpha} - \frac{1}{2}\mu_Z^2(L^{i\alpha}\Lambda_{i\alpha} + h.c.) - \frac{1}{2}\mu_C^2(R^{i\alpha}T_{i\alpha} + h.c.) \\
& + \lambda_{XY1} L^{i\alpha}L_{i\alpha}\Lambda^{j\beta}\Lambda_{j\beta} + \lambda_{XY2} L^{i\alpha}L_{j\alpha}\Lambda^{j\beta}\Lambda_{i\beta} + \lambda_{ZZ} L^{i\alpha}\Lambda_{i\alpha}\Lambda^{j\beta}L_{j\beta} \\
& + \lambda_{ZZ1}(L^{i\alpha}\Lambda_{i\alpha}L^{j\beta}\Lambda_{j\beta} + h.c.) + \lambda_{ZZ2}(L^{i\alpha}\Lambda_{j\alpha}L^{j\beta}\Lambda_{i\beta} + h.c.) \\
& + \lambda_{ZZ3}(L^{i\alpha}\Lambda_{j\alpha}\Lambda^{j\beta}L_{i\beta} + h.c.) \\
& + \lambda_{AB1} R^{i\alpha}R_{i\alpha}T^{j\beta}T_{j\beta} + \lambda_{AB2} R^{i\alpha}R_{j\alpha}T^{j\beta}T_{i\beta} + \lambda_{CC} R^{i\alpha}T_{i\alpha}T^{j\beta}R_{j\beta} \\
& + \lambda_{CC1}(R^{i\alpha}T_{i\alpha}R^{j\beta}T_{j\beta} + h.c.) + \lambda_{CC2}(R^{i\alpha}T_{j\alpha}R^{j\beta}T_{i\beta} + h.c.) \\
& + \lambda_{CC3}(R^{i\alpha}T_{j\alpha}T^{j\beta}R_{i\beta} + h.c.) \\
& + \lambda_{XZ1}(L^{i\alpha}L_{i\alpha}L_{j\beta}\Lambda^{j\beta} + h.c.) + \lambda_{XZ2}(L^{i\alpha}L_{i\beta}L_{j\alpha}\Lambda^{j\beta} + h.c.) \\
& + \lambda_{AC1}(R^{i\alpha}R_{i\alpha}R_{j\beta}T^{j\beta} + h.c.) + \lambda_{AC2}(R^{i\alpha}R_{i\beta}R_{j\alpha}T^{j\beta} + h.c.) \\
& + \lambda_{II1} L_{i\alpha}L^{i\alpha}T_{j\beta}T^{j\beta} + \lambda_{II2} L_{i\alpha}T^{j\alpha}L^{i\beta}T_{j\beta} + \lambda_{II3}(L_{i\alpha}T^{j\alpha}L^i_\beta T^\beta_j + h.c.) \\
& + \lambda_{KK1} R_{i\alpha}R^{i\alpha}\Lambda_{j\beta}\Lambda^{j\beta} + \lambda_{KK2} R_{i\alpha}\Lambda^{j\alpha}R^{i\beta}\Lambda_{j\beta} + \lambda_{KK3}(R_{i\alpha}\Lambda^{j\alpha}R^i_\beta\Lambda^\beta_j + h.c.) \\
& + \lambda_{HI1}(L^{i\alpha}L_{i\alpha}R^{j\beta}T_{j\beta} + h.c.) + \lambda_{HI2}(L^{i\alpha}R_{j\alpha}L_{i\beta}T^{j\beta} + h.c.) \\
& + \lambda_{HI3}(L^{i\alpha}L^\beta_i R_{j\alpha}T^j_\beta + h.c.) \\
& + \lambda_{HK1}(R^{i\alpha}R_{i\alpha}L^{j\beta}\Lambda_{j\beta} + h.c.) + \lambda_{HK2}(R^{i\alpha}L_{j\alpha}R_{i\beta}\Lambda^{j\beta} + h.c.) \\
& + \lambda_{HK3}(R^{i\alpha}R^\beta_i L_{j\alpha}\Lambda^j_\beta + h.c.) \\
& + \lambda_{HJ1}(L^{i\alpha}\Lambda_{i\alpha}R^{j\beta}T_{j\beta} + h.c.) + \lambda_{HJ2}(L^{i\alpha}R_{j\alpha}T^{j\beta}\Lambda_{i\beta} + h.c.) \\
& + \lambda_{HJ3}(L^{i\alpha}R_{j\alpha}\Lambda^\beta_i T^j_\beta + h.c.) \\
& + \lambda_{IK1}(R^{i\alpha}T_{i\alpha}L^{j\beta}\Lambda_{j\beta} + h.c.) + \lambda_{IK2}(R^{i\alpha}L_{j\alpha}\Lambda^{j\beta}T_{i\beta} + h.c.) \\
& + \lambda_{IK3}(R^{i\alpha}L_{j\alpha}T^\beta_i\Lambda^j_\beta + h.c.)\,.
\end{aligned} \tag{H.1}$$

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