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Moduli spaces of Algebraic and Tropical Curves with Hassett Stability Conditions

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Introduction

The purpose of this work is to develop the theory of moduli spaces of tropical curves with generalized stability conditions, collecting original results from different works in the area, in particular my works [KLSY22] (joint with Siddarth Kannan, Shiyue Li, and Claudia He Yun), [KSY22] (joint with Siddarth Kannan, and Claudia He Yun) and [Ser22]. The purpose of our work is not only to develop results concerning the moduli Theory of tropical curves itself but also the connections between this setting and the original algebro-geometric setting of Hassett moduli spaces. Througout this introduction, we will retrace the history of the connections between the moduli problems of algebraic and tropical curves, starting from the foundations of the algebraic setting and Hassett birational models of the moduli stacks of curves $\mathcal{M}_{g,\mathcal{A}}$ and $\overline{\mathcal{M}}_{g,\mathcal{A}}$, and arriving to their tropical version and the interplay between the two.

Moduli of algebraic curves. In algebraic geometry, classification problems are usually called moduli problems, and consist in the research of a moduli space parametrizing the objects of interest. As it is written in [Cha00], informally speaking a moduli space is a parameter space for classes of geometric objects of interest, such that nearby points specify "nearby" geometric objects, meaning that those geometric objects are close to be the same. In its more recent formulations, a moduli problem in the algebro-geometric category consists in considering a set of algebro-geometric objects, together with an equivalence relation, and a notion of family for this objects. To state such a problem, one considers a so called moduli functor

$$F : \mathit{schemes} \to \mathit{Sets}$$

from the category of schemes to the one of sets, which is a contravariant functor sending a scheme S into the set of families of isomorphism classes of the algebro-geometric objects over S. By the word family, we mean that we do not only parametrize the object themselves, but we do it taking into account their deformations.

The problem of constructing a space to parametrize a certain class of algebro-geometric objects, along with the introduction of the term "moduli" has its origins in the work "The theory of Abelian functions" by Riemann in 1857, who originally dealt with the problem of classifying isomorphisms classes of compact Riemann surfaces. The first invariant which already Riemann considered back then was the topological genus of the surfaces, thanks to which it is possible to bound the class of objects so they can be parametrized conveniently in a finite dimensional "space." These objects are identified with smooth projective algebraic curves over $\mathbb{C}$, as there is an equivalence of categories between the category of smooth irreducible projective algebraic varieties of dimension one over $\mathbb{C}$ (which we will call just smooth curves from now on) with non-constant regular maps as morphisms, and the category of compact Riemann surfaces with non-constant holomorphic maps as morphisms. With this equivalence we can rephrase this problem as the classification problem of all the possible smooth curves up to isomorphism of algebraic varieties. In order to state conveniently the moduli problem of curves, let us recall the notion of family of smooth curves of fixed genus.

Definition. *Let $g \geq 2$, and let S be a scheme. A family of smooth curves of genus g over S is a proper and flat family $C \to S$ whose geometric fibers are smooth, connected 1-dimensional schemes of genus g.*

A fine moduli space $\mathcal{M}$ for the moduli functor F is a scheme with a universal family $\mathcal{U} \to \mathcal{M}$ such that, for every scheme S and for every family $C \to S$ over it, there is a unique $f : S \to \mathcal{M}$ making the following diagram Cartesian:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ S \arrow[r, "f"] & \mathcal{M}. \end{tikzcd}$$

Once we have $\mathcal{M}$, its points are in bijection with the objects we want to classify. In the case of smooth curves, it is impossible to find such a space in the category of schemes, so we have to weaken the request on this space. What we obtain is called a coarse moduli space, i.e., a scheme M_g together with a natural transformation of functors $\phi : F \to Hom(-, M_g)$ such that

- For any algebraically closed field k, the map

  $$F(Spec(k)) \to Hom(Spec(k), M_g)$$

  is a bijection.

- Given any scheme M' and a transformation $\phi' : F \to Hom(-, M')$ there is a unique transformation

  $$\psi : Hom(-, M_g) \to Hom(-, M')$$

  such that $\phi' = \psi \circ \phi$.

Theorem. *([MFK94]) Given an algebraically closed field k there is a coarse moduli scheme M_g of dimension $3g - 3$ for the moduli functor of smooth curves defined over $Spec(k)$, which is quasi-projective and irreducible.*

The scheme M_g was constructed using the geometric invariant theory developed by Mumford, see [Edi00] for an account on the construction. Working with moduli spaces, a desiderable property they should have is properness, which is the algebraic geometry analogue of compactness. Unfortunately, the space M_g is not proper. To solve this problem, one looks for a so called Compactification, i.e., a proper space which contains M_g as an open dense subset. With the purpose of showing that M_g was irreducible, Deligne and Mumford in [DM69] described the compactification of M_g adding to the moduli problem (families of) stable curves.

Definition. *([DM69], Definition 1.1) Let $g \geq 2$ be an integer. A family of stable curves of genus g over a scheme S is a proper flat family $\pi : C \to S$ whose geometric fibers C_s are reduced, connected, 1-dimensional schemes (which we call curves) such that:*

- *C_s has only ordinary double points as singularities;*
- *If E is a non-singular rational component of C_s, then E meets the other components of C_s in more than 2 points;*
- *The fiber C_s has genus g.*

Analogously to M_g, by using G.I.T. we can show that there is a projective coarse moduli space $\overline{M}_g$ of dimension $3g-3$ for the moduli problem of stable curves and such that $M_g \subset \overline{M}_g$ as an open dense subset, as required. Furthermore, in the same work the authors introduced the fundamental notion of Deligne-Mumford stack, which was later generalized to the notion of algebraic stack (see Definition 1.1.8). A stack is a category fibered in groupoids such that isomorphisms form a sheaf and such that every descent datum is effective, which informally speaking means that it is a category with a functor on the category of schemes (in the case of algebraic stacks) that "pulls back and glues like bundles", see [Fan01] for a short introduction to the theory. In particular, Deligne and Mumford in [DM69] defined two Deligne-Mumford stacks $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ which turned out to be fine moduli spaces for the moduli problem of smooth (respectively stable) curves of genus g. Moreover, the inclusion $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ can be seen as an open dense inclusion and $\overline{\mathcal{M}}_g$ works as a compactification, since it is proper.

Marked points on curves. A natural generalization of this problem is to consider the moduli problem of curves with a distinguished set of marked points. The idea of considering this moduli problem firstly appeared in the work of Knudsen [Knu83] in 1983, but curves with marked points were already considered before, as for example elliptic curves. In [Knu83], Knudsen generalized the notion of stable curve to curves with marked points as follows.

Definition. *([Knu83], Definition 1.1) Let S be a scheme, $g \geq 0$, $n \geq 0$ be two integers such that $2g-2+n > 0$. A n-pointed family of stable curves of genus g over S is a proper flat family $\pi : C \to S$ together with n distinct sections $s_i : S \to C$ such that:*

- *the geometric fibers C_s are reduced, connected curves with at most ordinary double points as singularities;*
- *Each C_s is smooth at $P_i := s_i(s)$, for every i from 1 to n;*
- *We have $P_i \neq P_j$ whenever $i \neq j$;*
- *The number of points where a nonsingular rational component E of C_s meets the rest of C_s plus the number of marked points P_i on E is at least 3;*
- *Each fiber C_s has arithmetic genus g.*

Following the work of Deligne and Mumford, Knudsen constructed two fine moduli stacks $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ for the moduli problems of smooth and respectively stable curves of genus g with n marked points, provided that $2g - 2 + n > 0$ (otherwise there is no stable curve), where the inclusion is again a compactification since $\overline{\mathcal{M}}_{g,n}$ is proper. Both these spaces come with their corresponding coarse moduli schemes $M_{g,n} \subset \overline{M}_{g,n}$ obtained later using analogous techniques to the ones used for the unmarked case (seen also as the case where $n = 0$). The universal family over these spaces coming from the fine moduli space Structure, called universal curve, is one of the first motivations which led to consider this enhanced version of the problem. In fact, Knudsen shown that there is an identification of the universal curve $\mathcal{C}_g$ over $\mathcal{M}_g$ with the space $\mathcal{M}_{g,1}$, and in general $\mathcal{C}_{g,n}$ with $\mathcal{M}_{g,n+1}$ (and an analogous result holds considering the compactifications, see again [Knu83]).

A further generalization of this moduli problem came in 2003, with the work of Hassett, [Has03]. The starting idea of this generalization was to consider pointed curves as log varieties, i.e., pairs (X, D) where X is a variety and $D = \sum a_i p_i$ is an effective $\mathbb{Q}$-divisor on X. There is a notion of stability for log varieties, coming from the Minimal Model Program, which allows for the construction of a moduli space provided these pairs are stable. Namely, we ask that (X, D) has relatively mild singularities and that the divisor $K_X + D$ is ample. Given $\mathcal{A} = (a_1, \dots, a_n) \in ((0, 1] \cap \mathbb{Q})^n$ and $g \geq 0$ such that $2g - 2 + a_1 + \cdots + a_n > 0$ (such a pair of g and $\mathcal{A}$ is called input datum), we can construct a moduli stack $\overline{\mathcal{M}}_{g,\mathcal{A}}$ called the moduli space of $\mathcal{A}$-stable curves of genus g, which is the paremeter space for the following families of objects

Definition. *([Has03]) Let $\pi : C \to S$ be a proper flat family together with n distinct sections $s_i : S \to C$. Let $(g, \mathcal{A})$ be an input datum, $\mathcal{A} = (a_1, \dots a_n)$. We say that $\pi : C \to S$ is stable of type $(g, \mathcal{A})$ (or $(g, \mathcal{A})$-stable) if*

- *geometric fibers are nodal connected curves of arithmetic genus g;*
- *the sections $s_1, \dots s_n$ lie in the smooth locus of π, and for any subset $\{s_{i_1}, \dots, s_{i_r}\}$ with nonempty intersection we have $a_{i_1} + \cdots + a_{i_r} \leq 1$;*
- *the twisted canonical sheaf $K_\pi + p_1 s_1 + \cdots + a_n p_n$ is relatively π-ample, where by p_i we denote the image of s_i.*

Theorem. *([Has03], Theorem 2.1) Let $(g, \mathcal{A})$ be an input datum. There exists a connected Deligne–Mumford stack $\overline{\mathcal{M}}_{g,\mathcal{A}}$ smooth and proper over $\mathbb{Z}$ representing the moduli problem of pointed stable curves of type $(g, \mathcal{A})$. The corresponding coarse moduli scheme $\overline{M}_{g,\mathcal{A}}$ is projective over $\mathbb{Z}$.*

It is easy to see that when D is reduced (i.e., when $\mathcal{A} = (1, \dots, 1)$) the resulting moduli space is the Deligne-Mumford-Knudsen moduli space of marked stable curves. It comes with a locus of smooth curves $\mathcal{M}_{g,\mathcal{A}}$ such that $\mathcal{M}_{g,\mathcal{A}} \subset \overline{\mathcal{M}}_{g,\mathcal{A}}$ is a compactification, and admits coarse moduli schemes $M_{g,\mathcal{A}}$ and $\overline{M}_{g,\mathcal{A}}$. Special cases of these spaces appeared before 2003 in the work of Kapranov, Keel, and Losev and Manin. It is possible to show that for every $\mathcal{A} = (a_1, \dots, a_n)$, the resulting Hassett moduli space $\overline{M}_{g,\mathcal{A}}$ is birational to $\overline{M}_{g,n}$, but in general it is non-isomorphic, and an analogous result holds for the stacks $\mathcal{M}_{g,\mathcal{A}}$ and $\overline{\mathcal{M}}_{g,\mathcal{A}}$.

It is also possible to construct tautological maps analogous to the ones constructed for $\overline{M}_{g,n}$ in [Knu83], see 1.1.10 and 1.1.17.

Among these spaces, a special class of them with

$$\mathcal{A} = (1^{(m)}|\varepsilon^{(n)}) := \left(\underbrace{1,\ldots,1}_{m},\underbrace{\varepsilon,\ldots,\varepsilon}_{n}\right),$$

are called the universal curve moduli spaces, and are denoted $\overline{\mathcal{M}}_{g,m|n}$ (or $\mathcal{M}_{g,m|n}$ if we consider the locus of smooth curves, and analogously $\overline{M}_{g,m|n}$ and $M_{g,m|n}$ for the coarse moduli schemes). These spaces have been studied in several algebro-geometric contexts. They can be viewed as a resolution of singularities of the n-fold product of the universal curve over $\overline{M}_{g,m}$, and as g, m, and n vary, the spaces $\overline{M}_{g,m|n}$ form the components of Losev–Manin's extended modular operad [LM04]. When $g = 0$ and $m = 2$, the space $\overline{M}_{0,2|n}$ is a toric variety, and it coincides with the Losev–Manin moduli space of stable chains of $\mathbb{P}^1$'s [LM00].

Tropical curves and their moduli. Tropical geometry appeared as a branch of mathematics in its own right in the 90's, through the reunification of many works and ideas of people coming from various fields, and became famous in the last thirty years thanks to its applications different areas. It has deep connections and allows to obtain a lot of new results in other fields of mathematics, in particular in algebraic geometry, of which tropical geometry is considered a sub-field sometimes. In our work we are interested in the study of tropical curves and its connections and applications to the moduli problem of algebraic curves introduced before.

In the text, by tropical curve we will always refer to abstract tropical curves, which are, loosely speaking, weighted graphs with a metric. In particular we will be interested in abstract tropical curve whose underlying graph is $\mathcal{A}$-stable.

Definition. *Let $g, n \in \mathbb{Z}_{\geq 0}$, and $\mathcal{A} = (a_1, \ldots, a_n) \in ((0, 1] \cap \mathbb{Q})^n$ be a weight datum such that $2g - 2 + a_1 + \cdots + a_n > 0$. An n-marked $\mathcal{A}$-stable tropical curve of genus g is a pair $\Gamma := (G, l)$ where G is a weighted n-marked graph of genus g with n legs and l is a function*

$$l : E(G) \cup L(G) \to \mathbb{R}_{>0}$$

such that, for each vertex $v \in V(G)$ verifies

$$2w(v) - 2 + val(v) + |v|_{\mathcal{A}} > 0,$$

where $w : V(G) \to \mathbb{Z}_{\geq 0}$ is the weight function, $m : \{1 \ldots n\} \to V(G)$ is the marking function on vertices and $|v|_{\mathcal{A}} = \sum_{i \in m^{-1}(v)} a_i$.

It makes sense to ask for a moduli space of $\mathcal{A}$-stable tropical curves, which is much easier to construct than the algebraic one. In this context, a moduli space can be constructed as follows. First, for each $\mathcal{A}$-stable graph G of genus g we pick a copy of $\mathbb{R}_{\geq 0}^{|E(G)|}$. A point in such a cone identifies a tropical curve, as it is the data of a graph and an edge length function on it (with a small exception as we allow some lengths to be zero).

For a $\mathcal{A}$-stable graph G of genus g let $Aut(G)$ be its automorphism group (preserving m and w). It acts on the set of edges $E(G)$, and hence on the orthant $\mathbb{R}_{\geq 0}^{|E(G)|}$ by permuting coordinates. We define $C(G)$ to be the quotient space of the orthant by this action. Next, we define an equivalence relation on the points in the union $\bigsqcup C(G)$, as G ranges over all the considered graphs. The relation identifies two points x and x' if one of them is obtained from the other by contracting all edges of length zero (for more details about the rules of weighted contractions, see Figure 1.6). Denote by $\sim$ this relation. The moduli space $M_{g,\mathcal{A}}^{trop}$ is the topological space

$$M_{g,\mathcal{A}}^{trop} := \bigsqcup C(G)/\sim .$$

The resulting topological spaces are called Hassett moduli spaces of tropical curves, and they were defined in [Uli15] following and generalizing the previous constructions of M_g^{trop} (i.e., tropical curves without markings) made in [BMV11] and the one of $M_{g,n}^{trop}$ (obtained when $a_i = 1$ for every $i = 1, \ldots, n$) from [Cap11], see 1.2.2 for more details about the construction. An important subset of these spaces considered in many works and fundamental for the connection with the algebraic setting is the link space $\Delta_{g,\mathcal{A}}$, i.e., the locus of tropical curves whose volume (i.e., the sum of its edge lengths) is 1.

Connections between algebraic and tropical curves. The deep connection between the Hassett moduli space of algebraic curves and the Hassett moduli space of tropical curves is the following, firstly shown in [ACP15] for Deligne-Mumford stability and and later generalized for Hassett stability.

Theorem. *([ACP15], Theorem 1.2.1; [Uli15], Theorem 1.2) Let $g, n \in \mathbb{Z}_{\geq 0}$, and consider $\mathcal{A} = (a_1, ..., a_n) \in ((0,1] \cap \mathbb{Q})^n$ such that $2g - 2 + a_1 + \cdots + a_n > 0$. Let $\overline{\mathfrak{S}}(\overline{\mathcal{M}}_{g,\mathcal{A}})$ be the canonical Thuillier skeleton of the Hassett moduli space (see 1.3.2 for further details). There is a natural isomorphism*

$$J_{g,\mathcal{A}} : \overline{M}_{g,\mathcal{A}}^{trop} \to \overline{\mathfrak{S}}(\overline{\mathcal{M}}_{g,\mathcal{A}})$$

of extended generalized cone complexes. Let $\mathfrak{S}(\mathcal{X})$ denote the interior of the generalized cone complex $\overline{\mathfrak{S}}(\mathcal{X})$. Such isomorphism restricts to an isomorphism of cone complexes (see 1.2.19):

$$J_{g,\mathcal{A}} : M_{g,\mathcal{A}}^{trop} \to \mathfrak{S}(\overline{\mathcal{M}}_{g,\mathcal{A}}).$$

The link spaces $\Delta_{g,\mathcal{A}}$ play an important role in the understanding of the so called Top Weight Cohomology of $\mathcal{M}_{g,\mathcal{A}}$. The compactification $\overline{\mathcal{M}}_{g,\mathcal{A}}$ is a toroidal compactification of the moduli stack $\mathcal{M}_{g,\mathcal{A}}$, since the boundary divisor $\partial\overline{\mathcal{M}}_{g,\mathcal{A}} := \overline{\mathcal{M}}_{g,\mathcal{A}} \setminus \mathcal{M}_{g,\mathcal{A}}$ has normal crossings. As a Deligne-Mumford stack, the rational cohomology of $\mathcal{M}_{g,\mathcal{A}}$ carries a mixed Hodge structure, and one can consider its graded pieces

$$\mathrm{Gr}_i^W H^j(\mathcal{M}_{g,\mathcal{A}}, \mathbb{Q}) = W_i \cap H^j(\mathcal{M}_{g,\mathcal{A}}; \mathbb{Q})/W_{i-1} \cap H^j(\mathcal{M}_{g,\mathcal{A}}; \mathbb{Q}).$$

The top graded piece of this filtration of $\mathcal{M}_{g,\mathcal{A}}$, denoted by $\mathrm{Gr}_{6g-6+2n}^W H^{6g-6+2n-j}(\mathcal{M}_{g,\mathcal{A}}; \mathbb{Q})$, is called Top Weight Cohomology, and is the main object of study in [CGP21] and [CGP22]. Through a generalization of previous works by Danilov ([Dan75]) and Payne ([Pay13]), Chan, Galatius and Payne were able to construct a combinatorial object, called dual boundary complex and denoted by $\Delta(\partial\overline{\mathcal{M}}_{g,\mathcal{A}})$, encoding the combinatorics of the boundary divisor.

The dual boundary complex has the structure of a symmetric Δ-complex (see Section 1.2.3 for generalities about these objects) and is such that

$$\widetilde{H}_{j-1}(\Delta(\partial\overline{\mathcal{M}}_{g,\mathcal{A}});\mathbb{Q}) \cong \mathrm{Gr}^W_{6g-6+2n} H^{6g-6+2n-j}(\mathcal{M}_{g,\mathcal{A}};\mathbb{Q}).$$

The beautiful idea of Chan, Galatius and Payne is that this dual boundary complex can be identified with the link space, i.e., $\Delta(\partial\overline{\mathcal{M}}_{g,\mathcal{A}}) = \Delta_{g,\mathcal{A}}$, therefore we have the following important equality:

$$\widetilde{H}_{j-1}(\Delta_{g,\mathcal{A}};\mathbb{Q}) \cong \mathrm{Gr}^W_{6g-6+2n} H^{6g-6+2n-j}(\mathcal{M}_{g,\mathcal{A}};\mathbb{Q}).$$

We refer again to [CGP21] for the unmarked case, to [CGP22] for the case of curves with markings and to [Uli15] for the case of Hassett stable curves. In [CGP21] and [CGP22], the authors developed an automorphism between $\widetilde{H}_{j-1}(\Delta_{g,n};\mathbb{Q})$ and the rational homology of a special chain complex generated by isomorphism classes of n-marked genus g stable graphs, called Graph Complex (see Section 4.3). In fact, they proved that there is a natural injection of chain complexes

$$G^{(g,n)} \to C_*(\Delta_{g,n},\mathbb{Q})$$

decreasing degrees by $2g-1$, and inducing isomorphisms on homology

$$\widetilde{H}_{k+2g-1}(\Delta_{g,n};\mathbb{Q}) \to H_k(G^{(g,n)})$$

for all k's. We work out a generalization for this isomorphism in Section 4.3, where we define analogous complexes $G^{(g,\mathcal{A})}$ and show the following:

Theorem. *4.3.2 Let $g \geq 1$ and $\mathcal{A} \in ((0,1]\cap\mathbb{Q})^n$ such that $2g-2+a_1+\cdots+a_n > 0$. There is a natural injection of chain complexes*

$$G^{(g,\mathcal{A})} \to C_*(\Delta_{g,\mathcal{A}},\mathbb{Q})$$

decreasing degrees by $2g-1$, inducing isomorphisms on homology

$$\widetilde{H}_{k+2g-1}(\Delta_{g,\mathcal{A}};\mathbb{Q}) \to H_k(G^{(g,\mathcal{A})})$$

for all k's.

Originals results of [CGP21] and [CGP22] were used to develop results on the dimension of the cohomology of moduli spaces of curves, as for example we mention the following:

Theorem. *([CGP21], Theorem 1.1) The cohomology $H^{4g-6}(\mathcal{M}_g;\mathbb{Q})$ is nonzero for $g=3$, $g=5$, and $g \leq 7$. Moreover, $dim H^{4g-6}(\mathcal{M}_g;\mathbb{Q})$ grows at least exponentially. More precisely,*

$$dim H^{4g-6}(\mathcal{M}_g;\mathbb{Q}) > \beta_g + constant,$$

for any $\beta < \beta_0$, where β_0 is the real root of $t^3-t-1=0$.

Combinatorial properties of Hassett spaces of algebraic curves and their relation with their tropical counterpart motivates the three papers which are the core of this work, each of which is described in a different chapter. In the first chapter of this thesis, we will give a more detailed account on the notions introduced so far and we also develop all the necessary background on algebraic curves, tropical curves and Hassett generalized stability conditions.

Topology of the link. In the second chapter we study the homotopy type of the spaces of tropical curves of volume one, the link spaces $\Delta_{g,\mathcal{A}}$. Previous to this work, a lot of different works with the same purpose came out, and we will give an account on them in Section 2.1. We start the chapter with a mild generalization of results on the contractibility of some subloci of the $\Delta_{g,n}$'s obtained in [CGP22], to the $\Delta_{g,\mathcal{A}}$'s:

Theorem. *2.1.1 Let $g \geq 1$, $n \geq 1$ be integers, $\mathcal{A} \in ((0,1] \cap \mathbb{Q})^n$ be a weight datum. Then the following subcomplexes are either empty or contractible:*

1. *The subset $\Delta^{w}_{g,\mathcal{A}}$ of tropical curves with at least a strictly positive weighted vertex;*
2. *The subset $\Delta^{lw}_{g,\mathcal{A}}$ of tropical curves with at least a strictly positive weighted vertex and/or loops;*
3. *The closure of the subset of tropical curves with bridges $\Delta^{br}_{g,\mathcal{A}}$.*

We prove this Theorem in Appendix A. Right after, we study the fundamental group of $\Delta_{g,\mathcal{A}}$ showing the following:

Theorem. *2.1.6 For any $g, n \geq 1$ and $\mathcal{A} \in ((0,1] \cap \mathbb{Q})^n$ such that $2g-2+a_1+\cdots+a_n > 0$, the space $\Delta_{g,\mathcal{A}}$ is simply connected.*

To show this Theorem, we generalize the argument of [ACP22]. In particular, we use the properties of symmetric Δ-complexes which ensure that there is a surjection from a particular group (the group generated from the 1-skeleton, see [ACP22], Theorem 3.1) onto the fundamental group, and we show that the former is trivial (this follows from the contractibility on a particular sublocus of $\Delta_{g,\mathcal{A}}$, see Proposition 2.1.5). In Appendix A there is also an alternative proof of this based on the combinatorics of $\Delta_{g,\mathcal{A}}$. We show an analogous result for moduli spaces of quasistable and simple tropical curves, i.e., curves where the standard stability condition is relaxed (see Section 2.1 for more details).

Our second result of the chapter concerns the Euler Characteristic of $\Delta_{g,\mathcal{A}}$ in terms of the Top Weight Euler Characteristics of the moduli spaces $\mathcal{M}_{g,r}$. Set $[n] := \{1, \ldots, n\}$, we call a partition $P_1 \sqcup \cdots \sqcup P_r \vdash [n]$ $\mathcal{A}$-admissible if $\sum_{i \in P_j} a_i \leq 1$ for all $1 \leq j \leq r$. Let $N_{r,\mathcal{A}}$ denote the number of $\mathcal{A}$-admissible partitions of $[n]$ with r parts.

Theorem. *2.2.12 Denote by $\chi^W_{6g-6+2r}$ the Euler Characteristic of $\mathrm{Gr}^W_{6g-g+2r} H^*(\mathcal{M}_{g,r};\mathbb{Q})$. Then*

$$\chi(\Delta_{g,\mathcal{A}}) = 1 - \sum_{r=1}^{n} N_{r,\mathcal{A}} \cdot \chi^W_{6g-6+2r}(\mathcal{M}_{g,r}).$$

By previous computations for the numbers $\chi^W_{6g-6+2r}(\mathcal{M}_{g,r})$ this allows for the calculation of $\chi(\Delta_{g,\mathcal{A}})$ for arbitrary g and $\mathcal{A}$, yielding the following closed forms, expressed as Corollaries of this Theorem.

Corollary. *2.2.13 Given $\mathcal{A}$, such that $N_{r,\mathcal{A}} = 0$ for $r \leq g+1$, the Euler Characteristic of $\Delta_{g,\mathcal{A}}$ is*

$$\chi(\Delta_{g,\mathcal{A}}) = 1 + \sum_{r=1}^{n} N_{r,\mathcal{A}} (-1)^r \frac{(g+r-2)!}{g!} B_g.$$

Corollary. 2.2.14 *Let $S(n,r)$ denote the number the Stirling numbers of the second kind (see the proof of Corollary 2.2.14 for their Definition). Given a universal curve weight vector $\mathcal{A}$ of m heavy and n light components, where $m \geq g+1$, $n > 0$, and $0 < \varepsilon < 1/n$, we have*

$$\chi(\Delta_{g,\mathcal{A}}) = 1 + \sum_{r=1}^{n} \sum_{\ell=0}^{g} (-1)^{m+r+\ell} \frac{(g+m+r-2)!\ell!}{g!(\ell+1)} S(n,r)S(g,\ell).$$

Using this Corollary above, we compute explicitly the Euler Characteristics of link spaces with universal curve vectors in Table 2.1. To show the Theorem, we decompose isomorphism classes of Hassett spaces in the Grothendieck Ring of varieties as the sum of isomorphims classes of Deligne-Mumford spaces, and then show a relation between the Top Weight Euler Characteristic of $\mathcal{M}_{g,\mathcal{A}}$ and the Euler Characteristic of $\Delta_{g,\mathcal{A}}$ (further details in Section 2.2.2).

Equivariant Hodge Polynomials of universal curve moduli spaces. In chapter 3 we derive the formulas which encode the combinatorial relationships between the generating functions for the equivariant Hodge polynomials of universal curve Hassett spaces. Here we denote by $\overline{\mathcal{M}}_{g,m|n}$ the universal curve space with m heavy and n light components.

Let X be a d-dimensional complex variety with an action of a group G, then its complex cohomology groups are G-representations in the category of mixed Hodge structures. When $G = S_n$ is the group of permutations of n elements, the S_n-equivariant Hodge–Deligne polynomial of X is given by the formula

$$h_X^{S_n}(u,v) := \sum_{i,p,q=0}^{2d} (-1)^i ch_n \left(\mathrm{Gr}_p^F \mathrm{Gr}_{p+q}^W H_c^i(X;\mathbb{C}) \right) u^p v^q \in \Lambda^{(2)}[u,v],$$

where Λ is the ring of symmetric functions, and $ch_n(V) \in \Lambda$ is the Frobenius characteristic of an S_n-representation V. Analogously for a variety X with action of $S_m \times S_n$, we have set $h_X^{S_m \times S_n}(u,v)$ for the $S_m \times S_n$-equivariant Hodge–Deligne polynomial of X:

$$h_X^{S_m \times S_n}(u,v) := \sum_{i,p,q=0}^{2d} (-1)^i ch_{m,n} \left(\mathrm{Gr}_p^F \mathrm{Gr}_{p+q}^W H_c^i(X;\mathbb{C}) \right) u^p v^q \in \Lambda^{(2)}[u,v],$$

where $\Lambda^{(2)} = \Lambda \otimes \Lambda$, and the function $ch_{m,n}(V) \in \Lambda^{(2)}$ is the Frobenius characteristic of an $(S_m \times S_n)$-representation V. If X is proper, and the mixed Hodge structure on each cohomology group is pure, as is the case for the coarse moduli space $\overline{M}_{g,m|n}$ associated to $\overline{\mathcal{M}}_{g,m|n}$, the Hodge–Deligne polynomial specializes to the usual Hodge polynomial

$$\sum_{p,q=0}^{2d} (-1)^{p+q} ch_{m,n} \left(H^{p,q}(X;\mathbb{C}) \right) u^p v^q.$$

In order to state the main Theorem, we need some generalities about Λ and $\Lambda^{(2)}$. First, given $f \in \Lambda$, we set $f^{(j)} \in \Lambda^{(2)}$ for the inclusion of f into the jth tensor factor, $j \in \{1,2\}$. These extend to maps $\Lambda[[u,v]] \to \Lambda^{(2)}[[u,v]]$. Let $p_i \in \Lambda$ be the ith power sum symmetric function, i.e., $p_i = \sum_{k>0} x_k^i$, then Λ is generated by these elements:

$$\Lambda = \mathbb{Q}[[p_1, p_2, \ldots]].$$

The coproduct $\Lambda \to \Lambda^{(2)}$ defined by

$$p_i \mapsto p_i^{(1)} + p_i^{(2)}$$

also extends to a map

$$\Delta : \Lambda[[u,v]] \to \Lambda^{(2)}[[u,v]].$$

There is an associative operation $\circ$, called plethysm on Λ, characterized by the following properties:

(i) for any $g \in \Lambda$, the map $f \mapsto f \circ g$ defines an algebra homomorphism $\Lambda \to \Lambda$;

(ii) for all n, the map $f \mapsto p_n \circ f$ defines an algebra homomorphism $\Lambda \to \Lambda$;

(iii) $p_n \circ p_m = p_{nm}$.

This induces two plethysm operations $\circ_1$ and $\circ_2$ on $\Lambda^{(2)}$, and these extend to $\Lambda^{(2)}[[u,v]]$ by

$$p_n^{(i)} \circ_i q = q^n,$$

$$p_n^{(i)} \circ_j q = p_n^{(i)},$$

for $\{i,j\} = \{1,2\}$ and $q \in \{u,v\}$.

The main Theorem of this chapter is the following.

Theorem. 3.2.1 *Let $h_n \in \Lambda$ denote the nth homogeneous symmetric function. For $f \in \Lambda^{(2)}[[u,v]]$ set*

$$Exp^{(2)}(f) = \sum_{n>0} h_n^{(2)} \circ_2 f.$$

Then we have

$$\mathsf{a}_g = \Delta(\mathsf{b}_g) \circ_2 Exp^{(2)}\left(p_1^{(2)}\right)$$

and

$$\bar{\mathsf{a}}_g = \Delta(\bar{\mathsf{b}}_g) \circ_2 \left(p_1^{(2)} - \frac{\partial \mathsf{b}_0^{(2)}}{\partial p_1^{(2)}}\right) \circ_2 Exp^{(2)}\left(p_1^{(2)}\right),$$

where

$$\mathsf{a}_g := \sum_{m,n} h^{S_m \times S_n}_{\mathcal{M}_{g,m|n}}(u,v), \quad \bar{\mathsf{a}}_g := \sum_{m,n} h^{S_m \times S_n}_{\overline{\mathcal{M}}_{g,m|n}}(u,v) \in \Lambda^{(2)}[[u,v]]$$

and

$$\mathsf{b}_g := \sum_{n} h^{S_n}_{\mathcal{M}_{g,n}}(u,v), \quad \bar{\mathsf{b}}_g := \sum_{n} h^{S_n}_{\overline{\mathcal{M}}_{g,n}}(u,v) \in \Lambda[[u,v]].$$

We can show a numerical analogue of the Theorem above which deals with the non-equivariant Hodge–Deligne polynomials, defined by the assignment

$$h_X(u,v) := \sum_{i,p,q=0}^{2d} (-1)^i \dim\left(\mathrm{Gr}_p^F \mathrm{Gr}_{p+q}^W H_c^i(X;\mathbb{C})\right) u^p v^q \in \mathbb{Q}[u,v].$$

Set
$$a_g := \sum_{m,n} h_{\mathcal{M}_{g,m|n}}(u,v)\frac{x^m y^n}{m!n!}, \quad \overline{a}_g := \sum_{m,n} h_{\overline{\mathcal{M}}_{g,m|n}}(u,v)\frac{x^m y^n}{m!n!} \in \mathbb{Q}[[u,v,x,y]],$$
and similarly put
$$b_g := \sum_{n} h_{\mathcal{M}_{g,n}}(u,v)\frac{x^n}{n!}, \quad \overline{b}_g := \sum_{n} h_{\overline{\mathcal{M}}_{g,n}}(u,v)\frac{x^n}{n!} \in \mathbb{Q}[[u,v,x]].$$

Corollary. *3.2.2 We have*
$$a_g = b_g|_{x \to w}$$
where $w = x + e^y - 1$, *and*
$$\overline{a}_g = \overline{b}_g|_{x \to z},$$
where
$$z = x + e^y + \frac{e^{uvy} - uv \cdot e^y + uv - 1}{uv - u^2v^2} - 1.$$

These results allow for many explicit computations aided by previous work in the area, see Tables 3.1, 3.2 and 3.3.

In genus zero, the problem of computing the equivariant Hodge polynomials of $\overline{\mathcal{M}}_{0,m|n}$ has been studied by Bergström–Minabe [BM13, BM14] and by Chaudhuri [Cha16]. Our formula gives a third approach to this problem, which applies in arbitrary genus. Also in genus zero, the Chow groups of $\overline{\mathcal{M}}_{0,m|n}$ have been computed by Ceyhan [Cey09], while the Chow ring has been computed by Petersen [Pet17] and by Kannan–Karp–Li [KKL21]. The techniques of this paper are based on prior work on the operad structure of moduli of stable curves and maps, by Getzler [Get95, Get98], Getzler–Kapranov [GK98], and Getzler–Pandharipande [GP06]. In particular, the main tool of the paper is a generalization of Getzler–Pandharipande's Grothendieck ring of $\mathbb{S}$-spaces, which encodes sequences of varieties with S_n-actions, to the setting of $\mathbb{S}^2$-spaces, which allows us to keep track of $(S_m \times S_n)$-actions as m and n vary.

Wall Crossing and filtrations of tropical moduli spaces. In chapter 4 we study how varying the vector $\mathcal{A}$ in $((0,1] \cap \mathbb{Q})^n$ i.e. varying the Hassett stability condition, affects the topology of Hassett moduli spaces of tropical curves, and also we study some relations that arise between different spaces. We consider the set of real hyperplanes called fine chamber decomposition, defined in [Has03]:
$$W_f = \left\{ \sum_{j \in S} a_j = 1 : \{S \subset \{1, ..., n\}\}, 2 \leq |S| \leq n - 2\delta_{g,0} \right\},$$
where $\delta_{i,j}$ is the Kronecker Delta. We denote by $\mathbf{K}$ the set of connected components of
$$((0,1] \cap \mathbb{Q})^n \setminus \bigsqcup_{w \in W_f} w,$$
and we call these components chambers.

Then we can see that different weight data in the same chamber give rise to the same moduli space, and the fine chamber decomposition is the coarsest one with this property (see Proposition 4.2.3 and 4.2.10). Let $g \geq 0$, $n \geq 1$ and $\mathcal{A} \in ((0,1] \cap \mathbb{Q})^n$, and let $C \subset M^{trop}_{g,\mathcal{A}}$ be a closed subset. We say it is a sub-moduli space if it is homeomorphic to $M^{trop}_{g,\mathcal{B}}$ for some $\mathcal{B}$, and points of C are in bijection with $(g,\mathcal{B})$-stable tropical curves. Given two weight data $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \leq \mathcal{B}$ if $a_i \leq b_i$ component-wise. It is easy to show that if $\mathcal{A} \leq \mathcal{B}$ there is a closed inclusion

$$M^{trop}_{g,\mathcal{A}} \subseteq M^{trop}_{g,\mathcal{B}}. \tag{1}$$

preserving the moduli space structure.

For a given weight datum $\mathcal{A}$, we denote by $Ch_{\mathcal{A}}$ its chamber. We define a partial order relation in the set $\mathbf{K}$ which extends the previous one on weight data as follows. Let $Ch_1, Ch_2 \in \mathbf{K}$, we say that $Ch_1 \leq Ch_2$ if they are equal or there are $S_1, ..., S_t \subset \{1, ..., n\}$ such that for every $\mathcal{A} \in Ch_1$, $\sum_{i \in S_j} a_i < 1$ and for every $\mathcal{B} \in Ch_2$, $\sum_{i \in S_j} a_i > 1$, for every j from 1 to t, while for any $S' \neq S_j$ for every j from 1 to t, the two chambers belong to the same half-space of $((0,1] \cap \mathbb{Q})^n$ induced by the wall $\sum_{i \in S'} a_i = 1$. It is easy to see that if we pick $\mathcal{A} \in Ch_1$ and $\mathcal{B} \in Ch_2$ with $Ch_1 \leq Ch_2$, the inclusion still holds.

Given a weight datum $\mathcal{A} = (a_1, ..., a_n)$ and a permutation $\sigma \in S_n$, let $\sigma(\mathcal{A})$ be the weight datum obtained by permuting the weights of $\mathcal{A}$ according to σ, i.e., $\sigma(\mathcal{A}) = (a_{\sigma(1)}, ..., a_{\sigma(n)})$. Then $M^{trop}_{g,\mathcal{A}}$ is homeomorphic to $M^{trop}_{g,\sigma(\mathcal{A})}$: the homeomorphism, called relabeling homeomorphism, consists of sending a tropical curve into the curve with the same underlying graph and the same length function, but with the legs marked according to the permutation.

We can consider the action of S_n induced on $\mathbf{K}$ given by $\sigma(Ch_{\mathcal{A}}) = Ch_{\sigma(\mathcal{A})}$ as well. We call the orbits of this action chambers up to symmetry, and we denote the set of chambers up to symmetry by $[\mathbf{K}]$. Denote by $[Ch]$ the orbit of $Ch \in \mathbf{K}$. We say that $[Ch_1] \leq [Ch_2]$ if there are two chambers $Ch_{\mathcal{A}_1} \in [Ch_1]$ and $Ch_{\mathcal{A}_2} \in [Ch_2]$ such that $Ch_{\mathcal{A}_1} \leq Ch_{\mathcal{A}_2}$, for some $\mathcal{A}_1$ and $\mathcal{A}_2$. In that case, each time we pick

$$\mathcal{A} \in Ch_{\mathcal{A}_1} \in [Ch_{\mathcal{A}_1}] \leq [Ch_{\mathcal{A}_2}] \ni Ch_{\mathcal{A}_2} \ni \mathcal{B},$$

there is a permutation $\sigma \in S_n$ giving a topological embedding

$$M^{trop}_{g,\mathcal{A}} \hookrightarrow M^{trop}_{g,\mathcal{B}}$$

obtained by combining the relabeling homeomorphism $M^{trop}_{g,\mathcal{A}} \cong M^{trop}_{g,\sigma(\mathcal{A})}$ with the inclusion $M^{trop}_{g,\sigma(\mathcal{A})} \subset M^{trop}_{g,\mathcal{B}}$. In particular, there is only a moduli space of tropical curves up to homeomorphism for each chamber up to symmetry. All these properties work if we replace $M^{trop}_{g,\mathcal{A}}$ with $\Delta_{g,\mathcal{A}}$.

We illustrate the situation with an example. Suppose we have $g \geq 1$ and $n = 3$, and we have the weight data $\mathcal{A}_1 = (\frac{12}{27}, \frac{14}{27}, 1 - \varepsilon)$ and $\mathcal{A}_2 = (\frac{14}{27} - \varepsilon, \frac{12}{27}, \frac{14}{27})$, for some $0 < \varepsilon < \frac{1}{27}$. Clearly they are not comparable with respect to the partial order on weight data, and we can verify that they belong to different chambers, i.e., $Ch_{\mathcal{A}_1} \neq Ch_{\mathcal{A}_2}$. It is also possible to verify that $[Ch_{\mathcal{A}_1}] \neq [Ch_{\mathcal{A}_2}]$. This implies that their moduli spaces $M^{trop}_{g,\mathcal{A}_1}$ and $M^{trop}_{g,\mathcal{A}_2}$ are different, and none of them is the subspace of the other. But if we reorder the weights of the second weight by the permutation $\sigma = (1 \quad 3 \quad 2) \in S_3$ the datum we obtain is

$\sigma(\mathcal{A}_2) = (\frac{12}{27}, \frac{14}{27}, \frac{14}{27} - \varepsilon)$, and by the relabeling homemorphism we know that

$$M^{trop}_{g,\mathcal{A}_2} \approx M^{trop}_{g,\sigma(\mathcal{A}_2)}.$$

Moreover $\sigma(\mathcal{A}_2)$ and $\mathcal{A}_1$ are comparable, in particular $\sigma(\mathcal{A}_2) \leq \mathcal{A}_1$, so we have the inclusion as a subspace

$$M^{trop}_{g,\sigma(\mathcal{A}_2)} \subset M^{trop}_{g,\mathcal{A}_1}.$$

Composing the relabeling homeomorphism with this inclusion gives an embedding as a subspace

$$M^{trop}_{g,\mathcal{A}_2} \hookrightarrow M^{trop}_{g,\mathcal{A}_1}.$$

In the case $g \geq 1$ and $n = 3$ there are five chambers up to symmetry, so choosing representative weight data up to relabeling homeomorphisms we get the filtration

$$M^{trop}_{g,(\frac{1}{3},\frac{1}{3},\frac{1}{3}-\varepsilon)} \subset M^{trop}_{g,(\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon)} \subset M^{trop}_{g,(\frac{12}{27},\frac{14}{27},\frac{14}{27}-\varepsilon)} \subset M^{trop}_{g,(\frac{12}{27},\frac{14}{27},1-\varepsilon)} \subset M^{trop}_{g,3}$$

where we take ε's are taken in order to take weight data in the interior of the chamber decomposition.

Following the previous discussion, we prove the following Theorem.

Theorem. 4.2.2 *Let $g \geq 0$, $n \geq 1$ be two integers. Fix $\mathcal{A} = (a_1, \ldots, a_n) \in ((0,1] \cap \mathbb{Q})^n$ such that $2g - 2 + a_1 + \cdots + a_n > 0$. There are filtrations of $M^{trop}_{g,\mathcal{A}}$ given by embeddings as sub-moduli spaces induced by the partial order on the set of chambers up to symmetry. Namely, given a sequence*

$$[Ch_{\mathcal{A}_1}] \leq [Ch_{\mathcal{A}_2}] \leq ... \leq [Ch_{\mathcal{A}_p}] \leq ... \leq [Ch_{\mathcal{A}_{N-1}}] \leq [Ch_{\mathcal{A}}],$$

the filtration is

$$M^{trop}_{g,\mathcal{A}_1} \hookrightarrow M^{trop}_{g,\mathcal{A}_2} \hookrightarrow ... \hookrightarrow M^{trop}_{g,\mathcal{A}_p} \hookrightarrow ... \hookrightarrow M^{trop}_{g,\mathcal{A}_{N-1}} \hookrightarrow M^{trop}_{g,\mathcal{A}}.$$

The same result holds if we replace $M^{trop}_{g,\mathcal{A}}$ with the moduli space of $(g,\mathcal{A})$-stable tropical curves of volume 1, $\Delta_{g,\mathcal{A}}$.

We also define the graph complexes $G^{(g,\mathcal{A})}$, see Section 4.3. In order to study further these graph complexes and their homology, we generalize Theorem 1.4 of [CGP22] showing that there is a natural surjection of chain complexes $C_*(\Delta_{g,\mathcal{A}}) \to G^{(g,\mathcal{A})}$ decreasing degrees by $2g-1$ inducing isomorphisms on homology

$$\widetilde{H}_{k+2g-1}(\Delta_{g,\mathcal{A}};\mathbb{Q}) \to H_k(G^{(g,\mathcal{A})})$$

for all k's. As stated before, combining this with the isomoprhism of the Top Weight Cohomology with the rational homology of the dual boundary complex we get also a natural isomorphism

$$Gr^W_{6g-6+2n}H^{4g-6+2n-k}(\mathcal{M}_{g,\mathcal{A}};\mathbb{Q}) \to H_k(G^{(g,\mathcal{A})}),$$

between the rational Top Weight Cohomology of $\mathcal{M}_{g,\mathcal{A}}$ and the rational homology of the complex $G^{(g,\mathcal{A})}$. We also deduce a Filtration Theorem analogous to the one stated for the moduli spaces.

Theorem. *4.3.11 Let $g \geq 0$, $n \geq 1$ be two integers. Fix $\mathcal{A} = (a_1, \ldots, a_n) \in ((0,1] \cap \mathbb{Q})^n$ such that $2g - 2 + a_1 + \cdots + a_n > 0$. There are filtrations of $G^{(g,\mathcal{A})}$ induced by the partial order on the set of chambers up to symmetry given by inclusions of complexes. Namely a ordered sequence*

$$[Ch_{\mathcal{A}_1}] \leq [Ch_{\mathcal{A}_2}] \leq ... \leq [Ch_{\mathcal{A}_p}] \leq ... \leq [Ch_{\mathcal{A}_{N-1}}] \leq [Ch_{\mathcal{A}}],$$

induces a filtration of chain complexes

$$G^{(g,\mathcal{A}_1)} \hookrightarrow G^{(g,\mathcal{A}_2)} \hookrightarrow ... \hookrightarrow G^{(g,\mathcal{A}_p)} \hookrightarrow ... \hookrightarrow G^{(g,\mathcal{A}_{N-1})} \hookrightarrow G^{(g,\mathcal{A})},$$

with $G^{(g,\mathcal{A}_{p-1})} \hookrightarrow G^{(g,\mathcal{A}_p)}$ being an injective map of chain complexes for every $p = 2, ..., N$.

This gives them the structure of Filtered Chain Complexes. There is a spectral sequence associated to a Filtered Chain Complex which can be used to compute the homology of the complex itself. In particular, the structure of bounded Filtered Chain Complex given to each $G^{(g,\mathcal{A})}$, combined with the shifting degree isomorphism of the Top Weight Cohomology of $\mathcal{M}_{g,\mathcal{A}}$ with the homology of the complex gives us the following Theorem.

Theorem. *4.3.13 Fix $g \geq 1$, $n \geq 2$. Assume we have a sequence of chambers up to symmetry $[Ch_{\mathcal{A}_1}] \leq ... \leq [Ch_{\mathcal{A}_p}] \leq ... \leq [Ch_{\mathcal{A}_N}]$, and let $G^{(g,\mathcal{A}_1)} \hookrightarrow ... \hookrightarrow G^{(g,\mathcal{A}_p)} \hookrightarrow ... \hookrightarrow G^{(g,\mathcal{A}_N)}$ be the induced filtration on graph complexes. Then*

$$Gr^W_{6g-6+2n} H^{4g-6+2n-k}(\mathcal{M}_{g,\mathcal{A}_N}; \mathbb{Q}) \cong \bigoplus_{p=1}^{N} E^{\infty}_{p,k-p},$$

where the terms $E^{\infty}_{p,k-p}$ are the ones to which the spectral sequence induced by the above filtration converges.

At the end of chapter 3, we discuss the relative Homology of the inclusions of the link spaces, and how it can be interpreted again as the homology of a Graph Complex.

Chapter 1

Background

This chapter is devoted to introduce all the notation and the results we will use in the rest of the document. It is divided in three sections: section 1.1 introduces the moduli problem of algebraic curves and the Hassett's generalization. Section 1.2 deals with the analogous tropical problem. Section 1.3 exposes the connections between the two moduli problems.

1.1 Moduli spaces of algebraic curves

Here we talk about the moduli problem of algebraic curves, from its foundations in the nineteenth century until its most modern generalization made by Hassett in 2003.

1.1.1 The Moduli Problem of Curves

From smooth to stable curves

The origin of moduli theory of curves is usually attributed to Riemann, who firstly studied the classification of Riemann surfaces, i.e., dimension one varieties over the complex field, and firstly introduced the genus as an invariant of the problem. Later in the twentieth century, the problem evolved along with the development of algebraic geometry with the introduction of the theory of schemes by Grothendieck. A complete solution to the problem was obtained in the late 60's by the work of Deligne and Mumford, who gave the construction of a moduli space (which we are going to introduce) for curves in the category of algebraic stacks. The original formulation of the algebro-geometric version of the problem dealt with the classification of the following class of objects, which are families of smooth curves of given genus.

Definition 1.1.1. Let $g \geq 2$, and let S be a scheme. A family of smooth curves of genus g over S is a proper, flat, family $C \to S$ whose geometric fibers are smooth, connected 1-dimensional schemes (which we call curves from now on) of genus g.

Informally, a family over S is a scheme morphism over S such that its fibers are smooth curves of fixed given genus, and we obtain a single curve whenever the base scheme S is $Spec(k)$, where k is an algebraically closed field.

In modern algebraic geometry, moduli problems are stated through moduli functors. A moduli functor is a contravariant functor which sends a scheme S into the set of isomorphism classes of families over S, and sends a morphism $f : S \to S'$ into the morphism of families $f^*(C' \to S') = (C' \times_{S'} S \to S)$. The moduli functor of curves, defined for every $g \geq 2$, is then:

$$F_g : Schemes \longrightarrow Sets$$

$$S \longmapsto \left\{ \begin{array}{r} \text{isomorphism classes} \\ \text{of families of curves} \\ \text{of genus } g \text{ over } S \end{array} \right\}$$

Given a moduli functor F, a fine moduli space $\mathcal{M}$ for F is a scheme (later we will relax this request allowing objects which are not schemes) with a universal family $\mathcal{U} \to \mathcal{M}$ such that, for every scheme S and for every family $C \to S$ over it, there is a unique $f : S \to \mathcal{M}$ such that the following diagram holds and is Cartesian:

$$\begin{array}{ccc} \mathcal{U} \times_{\mathcal{M}} S \sim C & \longrightarrow & \mathcal{U} \\ \downarrow & & \downarrow \\ S & \xrightarrow{f} & \mathcal{M}. \end{array}$$

In other words, if it exists, a fine moduli space $\mathcal{M}$ for the moduli functor F is such that F is isomorphic as a functor to $Hom(-, \mathcal{M})$. In particular, $F(S)$ is in bijection with $Hom(S, \mathcal{M})$, so in the case $S = Spec(k)$ we see that points of $\mathcal{M}$ over k are in bijection with the objects we want to classify. Moreover, it can be shown that the universal family $\mathcal{U}$ which comes with it is unique up to isomorphism.

Concerning the problem of curves, a fine moduli Space for the moduli functor F_g of smooth curves of give genus g does not exists (in the category of schemes). In fact, if we take a curve X with non-trivial automorphisms (like for instance hyperelliptic curves), we can describe non-trivial families $C \to S$ where each fiber has the same isomorphism class of X, even if the family is not isomorphic to the trivial product family $X \times S \to S$. However, if there was a moduli space $\mathcal{M}_g$ for F_g, such a family should be isomorphic to the trivial product family since the image of S under the corresponding map to the moduli space is a point. A way to solve this is to weaken the request on the moduli space. For instance, we can look for a coarse moduli scheme, i.e., a scheme M together with a natural transformation of functors $\phi : F \to Hom(-, M)$ such that

- For any algebraically closed field k, the map

$$F(Spec(k)) \to Hom(Spec(k), M)$$

  is a bijection.

- Given any scheme M' and a natural transformation $\phi' : F \to Hom(-, M')$, there is a unique natural transformation $\psi : Hom(-, M) \to Hom(-, M')$ such that $\phi' = \psi \circ \phi$.

Theorem 1.1.2. *([DM69]) Given an algebraically closed field k there is a coarse moduli scheme M_g of dimension $3g-3$ for the moduli functor of smooth curves defined over $Spec(k)$, which is quasi-projective and irreducible.*

The proof of this theorem relies on tecnhiques coming from geometric invariant theory. The irreducibility of the moduli space of smooth curves M_g was addressed by Deligne and Mumford in [DM69]. To show this property, since M_g is not proper, they constructed a proper space which contains M_g as an open dense subset: such a procedure is now called compactification of the moduli space. In order to do so, they allowed some singular degenerations of the smooth curves to be allowed by the moduli functor, i.e., they changed the classification problem from considering only smooth curves to consider what they called stable curves.

Definition 1.1.3. ([DM69], Definition 1.1) Let $g \geq 2$ be an integer. A family of stable curves of genus g over a scheme S is a proper flat family $\pi : C \to S$ whose geometric fibers C_s are curves such that:

- C_s has only ordinary double points as singularities;
- If E is a non-singular rational component of C_s, then E meets the other components of C_s in more than 2 points;
- The fiber C_s has genus g.

With the same techniques used to construct the coarse moduli scheme M_g, Deligne and Mumford constructed a coarse moduli scheme $\overline{M}_g$ of dimension $3g-3$ for the moduli problem of stable curves and such that $M_g \subset \overline{M}_g$ as an open subset, as required.

Deligne-Mumford Stacks

The introduction of the notion of stability was not the only big innovation that Deligne and Mumford introduced in their breakthrough paper [DM69]. The main tool of [DM69] was in fact the newly introduced notion of algebraic stack, which later was worked out and generalized by Artin. Stacks today are widely used in different areas of geometry and the basics of the subject grew enormously. Stacks are defined as category fibered in groupoids over a fixed base category $\mathcal{C}$, such that isomorphisms form a sheaf and every descent datum is effective (see [Fan01] for a very short and friendly introduction). Their application to different base categories other than the one of schemes lead for instance to the notions of differential stack, topological stack or cone stacks, and nowadays algebraic stacks are only one example of them. Anyways, we are going to describe Stacks following the original idea of [DM69], i.e., thinking of them as a generalization of schemes.

Let S be a scheme, and let Sch_S be the category of schemes over S. We say that a category $\mathcal{T}$ with a functor $p_{\mathcal{T}} : \mathcal{T} \to Sch_S$ is a category over S. If $B \in Obj(Sch_S)$ we say $X \in Obj(\mathcal{T})$ lies over B if $p_{\mathcal{T}}(X) = B$.

Definition 1.1.4. ([Edi00], Definition 2.2) If $(\mathcal{T}, p_\mathcal{T})$ is a category over S, then it is a groupoid over S if the following conditions hold:

1) If $f : B' \to B$ is a morphism in Sch_S, and X is an object of $\mathcal{T}$ lying over B, then there is an object X' over B' and a morphism $\phi : X' \to X$ such that $p_\mathcal{T}(\phi) = f$.

2) Let X, X', X'' be objects of $\mathcal{T}$ lying over B, B', B'', respectively. If $\phi : X' \to X$ and $\psi : X'' \to X$ are morphisms in $\mathcal{T}$, and $h : B' \to B''$ is a morphism such that $p_\mathcal{T}(\psi) \circ h = p_\mathcal{T}(\phi)$ then there is a unique morphism $\chi : X' \to X''$ such that the composition $\psi \circ \chi = \phi$ and $p_\mathcal{T}(\chi) = h$.

Define $\mathcal{T}(B)$ to be the subcategory consisting of all objects X such that $p_\mathcal{T}(X) = B$ and morphisms f such $p_\mathcal{T}(f) = id_B$. Then $\mathcal{T}(B)$ is a groupoid, i.e., a category where all morphisms are isomorphisms. This is the reason we say $\mathcal{T}$ is a groupoid over S or a category fibered in groupoids over Sch_S. Note that condition 2 of Definition 1.1.4 implies that the scheme X' over B' is unique up to canonical isomorphism, and will be called the pull-back of X via f and denoted f^*X.

Let $(\mathcal{T}, p_\mathcal{T})$ be a groupoid over S. Let B be an S-scheme and let X and Y be any objects in $\mathcal{T}(B)$. We define a functor $Iso_B(X, Y) : Sch_B \to Set$ by associating to any morphism $f : B' \to B$, the set of isomorphisms in $\mathcal{T}(B')$ between f^*X and f^*Y. If $X = Y$ then $Iso_B(X, X)$ is the functor whose sections over B' mapping to B are the automorphisms of the pull-back of X to B'.

Definition 1.1.5. ([Edi00], Definition 2.3) A groupoid $(\mathcal{T}, p_\mathcal{T})$ over S is a stack if

1. $Iso_B(X, Y)$ is a sheaf in the étale topology for all B, X and Y.

2. If $\{B_i \to B\}$ is a covering of B in the étale topology, and X_i is a collection of objects in $\mathcal{T}(B_i)$ with isomorphisms

$$\phi_{i,j} : X_j|_{B_i \times_B B_j} \to X_i|_{B_i \times_B B_j}$$

in $\mathcal{T}(B_i \times_B B_j)$ satisfying the cocycle condition. Then there is an object $X \in \mathcal{T}(B)$ with isomorphisms

$$X|B_i \to X_i$$

inducing the isomorphisms $\phi_{i,j}$ above.

Stacks over S form a category. Let $\mathcal{T}$ and $\mathcal{U}$ be stacks over S. A morphism of stacks is just a functor of groupoids which commutes with the projection functor to S. If $f : \mathcal{T} \to \mathcal{U}$ and $h : \mathcal{W} \to \mathcal{U}$ are morphisms of stacks, then we can define the fiber product $\mathcal{T} \times_\mathcal{U} \mathcal{W}$ as the groupoid whose sections over a base B are pairs $(x, y) \in \mathcal{T}(B) \times \mathcal{W}(B)$ such that $f(x)$ is isomorphic to $h(y)$. It can be easily checked that this groupoid is a stack. As mentioned, stacks were obtained as a generalization of schemes. In fact, given an S-scheme T, we can see it as a stack through its functor of points $p : Sch_T \to Sch_S$ which sends a T-scheme $U \to T$ into an S-scheme $U \to S$ by composition with the structure morphism $T \to S$. This is easily seen as a category over Sch_S, and one can check that this structure verifies Definition 1.1.5 (see also [DM69], section 4). From now we also view them as stacks, via their functor of points.

The stacks introduced by Deligne and Mumford in [DM69] where the moduli stacks of smooth and respectively stable algebraic curves of genus g, and are denoted $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$, respectively. They are constructed as follows: fix $Sch := Sch_{Spec(\mathbb{Z})}$ as base category, and let $\mathcal{M}_g$ be the category were objects are families $C \to S$ of smooth curves of genus g, and morphisms are given by commutative diagrams

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

inducing isomorphisms $C' \sim C \times_S S'$. The projection functor from $\mathcal{M}_g$ to Sch sends a family $C \to S$ into its base S, and a morphism into the lower part of the diagram. Analogously, $\overline{\mathcal{M}}_g$ is defined in the same way by picking families of stable curves instead of families of smooth curves. The first proof that these are stacks appeared in [DM69], and follows as a corollary of a more general fact. Indeed, it is possible to show that for a scheme X with an action of a group scheme G, one can define a category fibered in groupoids denoted $[X/G]$ by picking G-torsors $T \to B$ as objects, and projecting them to B (see also [Vis89] for further details). The Stack $[X/G]$ is also called Quotient Stack. It can be shown that such a category is always a Stack ([Vis89], Proposition 7.17). We can get $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ as quotient stacks by considering the subscheme $H_{g,l}$ (respectively $\overline{H}_{g,l}$) of the Hilbert scheme corresponding to l-canonically embedded smooth (respectively stable) curves with the action of $PGL((2l-1)(g-1)-1)$, for some $l \geq 3$ (see also [Edi00], Theorem 3.2).

Even though $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ were introduced by Deligne and Mumford as tools for the proof of the irreducibility of M_g, later on people started to consider stacks fundamental objects not only in the theory of curves, but in general for any moduli problem in algebraic geometry. In particular, a special class of stacks started to be considered as they share many properties with schemes: these stacks were later called Deligne-Mumford stacks (or DM stacks).

Definition 1.1.6. ([Edi00] Definition 2.4) A morphism $f : \mathcal{T} \to \mathcal{U}$ of stacks is said to be representable if for any map of a scheme $B \to \mathcal{U}$ the fiber product $\mathcal{T} \times_{\mathcal{U}} B$ is represented by a scheme.

Definition 1.1.7. ([Edi00] Definition 2.5) A representable morphism of stacks $f : \mathcal{T} \to \mathcal{U}$ has property **P**, if for all maps of scheme $B \to \mathcal{U}$ the corresponding morphism of schemes $\mathcal{T} \times_{\mathcal{U}} B \to B$ has property **P**.

Definition 1.1.8. A stack $\mathcal{X}$ is said to be Deligne–Mumford if it satisfies the following two properties:

(1) The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable, quasi-compact and separated;

(2) There exists a scheme U and a morphism $U \to \mathcal{X}$ which is étale and surjective.

All the schemes are Deligne-Mumford stacks. Two examples of Deligne-Mumford stacks which are not schemes are $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$. A reason of this can be found in Corollary 2.2 of [Edi00]: the Corollary says that whenever we have a Noetherian scheme of finite type X and a smooth group scheme G acting on X with finite, reduced stabilizers, then $[X/G]$ is a Deligne-Mumford stack, which can be shown to be our case. One important difference between the moduli stacks and the coarse moduli spaces we described before is that, even if these are not schemes, they are fine moduli Spaces for the moduli functors of curves. As for the coarse moduli spaces, we can identify the locus of smooth curves sub-stack parametrizing points representing smooth curves $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$. Both $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ have dimension $3g-3$. The fact that they are fine moduli spaces implies that they both come with a universal family, usually called universal curve, denoted $\mathcal{C}_g$ and $\overline{\mathcal{C}}_g$, respectively.

Marked points on curves

A natural enrichment of the previous problem is to consider curves of given genus with pairwise distinct marked points, along with a concept of stability generalizing the one for unmarked curves: this moduli problem was considered by Knudsen in [Knu83], and was posed for the following families of curves.

Definition 1.1.9. ([Knu83], Definition 1.1) Let S be a scheme, $g \geq 0$, $n \geq 0$ be two integers such that $2g-2+n > 0$. An n-pointed family of stable curves of genus g over S is a proper flat family $\pi : C \to S$ together with n distinct sections $s_i : S \to C$ such that:

- the geometric fibers C_s are reduced, connected curves with at most ordinary double points as singularities;
- Each C_s is smooth at $P_i := s_i(s)$, for every i from 1 to n;
- We have $P_i \neq P_j$ whenever $i \neq j$;
- The number of points where a nonsingular rational component E of C_s meets the rest of C_s plus the number of marked points P_i on E is at least 3;
- Each fiber C_s has arithmetic genus g.

Analogously to the case of unmarked curves, we can construct a Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}$ of dimension $3g-3+n$ parametrizing n-marked stable curves of genus g, with a sub-locus $\mathcal{M}_{g,n}$ parametrizing smooth curves. The condition $2g-2+n > 0$ is necessary for the existence of at least one stable curve with the given parameters. When $n = 0$, we have $\mathcal{M}_{g,0} = \mathcal{M}_g$ and $\overline{\mathcal{M}}_{g,0} = \overline{\mathcal{M}}_g$. In either case, the embedding $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is again a compactification of the moduli space of smooth curves, since the bigger space is proper.

Both these spaces come with a universal curve $\overline{\mathcal{C}}_{g,n}$ over $\overline{\mathcal{M}}_{g,n}$, and $\mathcal{C}_{g,n}$ over $\mathcal{M}_{g,n}$, respectively, as they are fine moduli spaces for the respective moduli problem. The universal curve $\overline{\mathcal{C}}_{g,n}$ is isomorphic to the category fibered in groupoids whose fibers over a scheme S consist of isomorphism classes of tuples $(\pi : C' \to S; s'_1, \dots s'_n, q' : S \to C')$ such that

- $(\pi : C' \to S; s'_1, \dots s'_n : S \to C')$ is an n-pointed genus g family;
- q' is any section of π.

There is a natural stabilization functor $s : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ which is actually an isomorphism of algebraic stacks. The natural transformation $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ forgetting q' is then equivalent to the map

$$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

which forgets the last section of the curve and contracts the components which become unstable, as shown in [Knu83]. We will call π the forgetting morphism.

All the stacks $\overline{\mathcal{M}}_{g,n}$ come with an associated projective coarse moduli scheme $\overline{M}_{g,n}$ parametrizing stable curves of genus g with n marked points, containing the moduli scheme $M_{g,n}$ parametrizing smooth curves of genus g with n marked points. These are not fine moduli spaces, as they do not represent the associated moduli functor, but still their points over algebraically closed fields are in bijection with isomorphism classes of curves. For more details about their construction we refer to the books [ACGH13] and [HM06], or to the notes [Edi00].

Moduli spaces of curves come usually with two particular morphisms called clutching and gluing, which we define here below. Let C, C_1 and C_2 be three curves with genus respectively g, g_1 and g_2 with $n+2$, n_1+1 and n_2+1 marked points (for simplicity, to describe these morphisms we can consider curves over an algebraically closed field k instead of families). On C, we can take the two marked points p_{n+1} and p_{n+2} and identify them. The resulting curve C' has n marked points, a new nodal point, genus $g+1$ and n marked points, as seen in Figure 1.1. This defines the gluing morphism

$$\Phi : \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}.$$

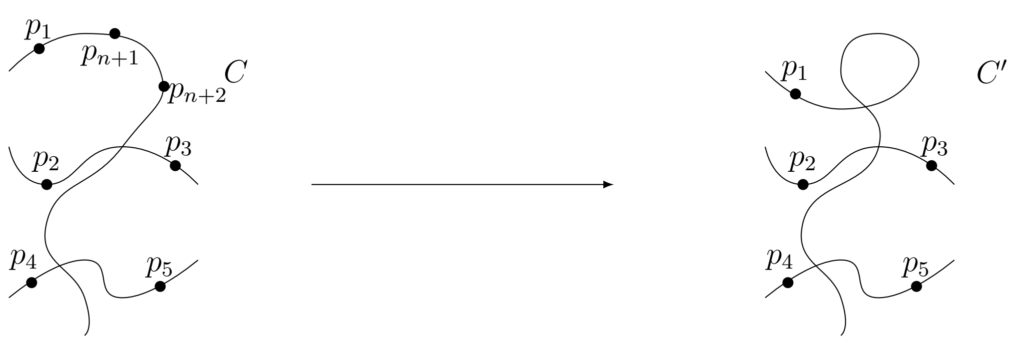

Figure 1.1: Example of gluing. The new node on C' is the result of the gluing between the last two marked points on the left hand side.

Analogously, by taking the two curves C_1 and C_2 and identifying the two marked points of index n_i+1 into a non-marked node we get a curve C'' of genus g_1+g_2 and n_1+n_2 marked points, as shown in Figure 1.2. This gives the clutching morphism

$$\Psi : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}.$$

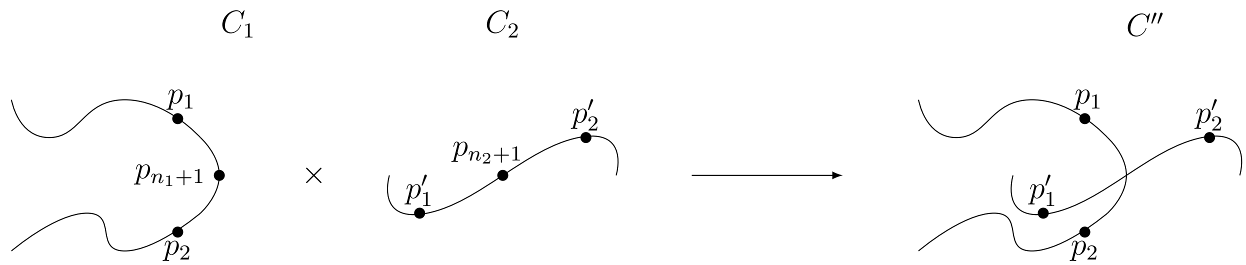

Figure 1.2: Example of clutching. The new node on C'' is the result of the identification between the last marked point on each of the two curves on the left.

Remark 1.1.10. The gluing, clutching and forgetting morphisms are also called tautological maps.

1.1.2 Hassett's Moduli Spaces

As we saw in the previous section, in the context of algebraic curves, the notion of stable curves was introduced by Deligne and Mumford in order to compactify the moduli space of smooth curves. A natural question one may ask then is if besides the Deligne-Mumford moduli spaces of stable curves, there are other modular compactifications of the moduli space of smooth marked curves, i.e., if there are other interesting proper moduli stacks containing $\mathcal{M}_{g,n}$ as a sub-stack. In fact, the answer is positive, and Hassett spaces, which we now introduce, are as well remarkable alternatives as modular compactifications of $\mathcal{M}_{g,n}$.

In [Has03], Hassett considered curves as log varieties, i.e., pairs (C, D) where C is a curve and D is an effective $\mathbb{Q}$-divisor on C. In the context of the Minimal Model Program, for such pairs, there is a notion of stability for families which can be used to construct a moduli space of such pairs provided they respect some conditions. It turns out that when varieties are curves, these Hassett stability conditions are a mild generalization of the Deligne-Mumford-Knudsen ones. In order to define Hassett stable curves and to see why they generalize Deligne-Mumford-Knudsen stability, we need first to introduce some notation.

Definition 1.1.11. ([Has03]) An input datum $(g, \mathcal{A})$ consists of a non-negative integer g together with a collection $\mathcal{A} = (a_1, ..., a_n)$ of weights $a_i \in (0, 1] \cap \mathbb{Q}$ such that

$$2g - 2 + a_1 + \cdots + a_n > 0.$$

The collection $\mathcal{A}$ is also called weight datum. We say that n is the lenght of the weight datum.

We denote the domain of all admissible weight data for genus g and length n with

$$\mathcal{D}_{g,n} := \{(a_1, ..., a_n) \in ((0, 1] \cap \mathbb{Q})^n \text{ such that } a_1 + ... + a_n > 2 - 2g\}.$$

Note that for a fixed n this space is $\mathcal{D}_{g,n} = ((0, 1] \cap \mathbb{Q})^n$ for every $g \geq 1$ since the condition $a_1 + ... + a_n > 2 - 2g$ becomes trivial. When $g = 0$, weight data have to respect also the condition $a_1 + ... + a_n > 2$. We call $\mathcal{D}_{g,n}$ the space of Hassett stability conditions.

Definition 1.1.12. Let $\pi : C \to S$ be a proper flat family together with n distinct sections $s_i : S \to C$. Let $(g, \mathcal{A})$ be an input datum, $\mathcal{A} = (a_1, \dots, a_n)$. We say that $\pi : C \to S$ is stable of type $(g, \mathcal{A})$ (or it is $(g, \mathcal{A})$-stable) if

- geometric fibers are nodal connected curves of arithmetic genus g;
- the sections $s_1, \dots, s_n$ lie in the smooth locus of π, and for any subset $\{s_{i_1}, \dots, s_{i_r}\}$ with non-empty intersection we have $a_{i_1} + \dots + a_{i_r} \leq 1$;
- the twisted canonical sheaf $K_\pi + p_1 s_1 + \dots + a_n p_n$ is relatively π-ample, where by p_i we denote the image of s_i.

Theorem 1.1.13. *([Has03], Theorem 2.1) Let $(g, \mathcal{A})$ be an input datum. There exists a connected Deligne–Mumford stack $\overline{\mathcal{M}}_{g,\mathcal{A}}$ smooth and proper over $\mathbb{Z}$ representing the moduli problem of pointed stable curves of type $(g, \mathcal{A})$. The corresponding coarse moduli scheme $\overline{M}_{g,\mathcal{A}}$ is projective over $\mathbb{Z}$.*

We will refer to these as Hassett spaces.

Remark 1.1.14. When

$$\mathcal{A} = 1^{(n)} := \left(\underbrace{1, \dots, 1}_{n} \right)$$

is a sequence of n ones, Hassett stability coincides with the Deligne-Mumford-Knudsen stability (which from now on we will refer to as the standard stability condition), and $\overline{\mathcal{M}}_{g,\mathcal{A}} = \overline{\mathcal{M}}_{g,n}$. One has to to check that the condition on the twisted sheaf $K_C + p_1 + ... + p_n$ to be ample for each fiber is equivalent to ask that the number of points where a non-singular rational component meets the rest of the curve plus its marked points is at least three (see [Liu02]).

In general, we will denote a weight datum with all entries being equal to a number a as $a^{(n)}$.

Remark 1.1.15. Hassett spaces are modular compactifications of the moduli space $\mathcal{M}_{g,n}$ of smooth curves with n marked points, i.e., $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,\mathcal{A}}$ for every $\mathcal{A} \in \mathcal{D}_{g,n}$. However, differently from the standard case, there may be some extra smooth curves were sections are allowed to coincide depending on their weights. We set $\mathcal{M}_{g,\mathcal{A}} \subset \overline{\mathcal{M}}_{g,\mathcal{A}}$ for the locus of smooth curves (and we always have $\mathcal{M}_{g,n} \subset \mathcal{M}_{g,\mathcal{A}}$.) The same happens for the coarse moduli scheme $\overline{M}_{g,\mathcal{A}}$, which comes with a sub-scheme of smooth curves $M_{g,\mathcal{A}}$. The locus $M_{g,\mathcal{A}}$ parameterizes smooth curves of genus g with n markings, such that whenever $\sum_{i \in S} a_i \leq 1$ for some $S \subseteq [n]$, the markings indexed by S are allowed to coincide. Analogously, $\overline{M}_{g,\mathcal{A}}$ parameterizes $\mathcal{A}$-stable curves of genus g, i.e., curves such that whenever $\sum_{i \in S} a_i \leq 1$ for some $S \subseteq [n]$, the markings indexed by S are allowed to coincide.

These spaces come equipped with two kinds of morphisms called respectively reduction and forgetting morphism.

Theorem 1.1.16. *([Has03], Theorem 4.1) Fix g and n and let $\mathcal{A} = (a_1, \ldots, a_n) \in \mathcal{D}_{g,n}$ and $\mathcal{B} = (b_1, \ldots, b_n) \in \mathcal{D}_{g,n}$ be weight data so that $b_j \leq a_j$ for each $j = 1, \ldots n$. Then there exists a natural birational reduction morphism $\rho_{\mathcal{B},\mathcal{A}} : \overline{\mathcal{M}}_{g,\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{B}}$. Given an element $(C, s_1, \ldots, s_n) \in \overline{\mathcal{M}}_{g,\mathcal{A}}$, $\rho_{\mathcal{B},\mathcal{A}}(C, s_1, \ldots, s_n)$ is obtained by successively collapsing components of C along which $K_C + b_1 s_1 + \cdots + b_n s_n$ fails to be ample (see Figure 1.3).*

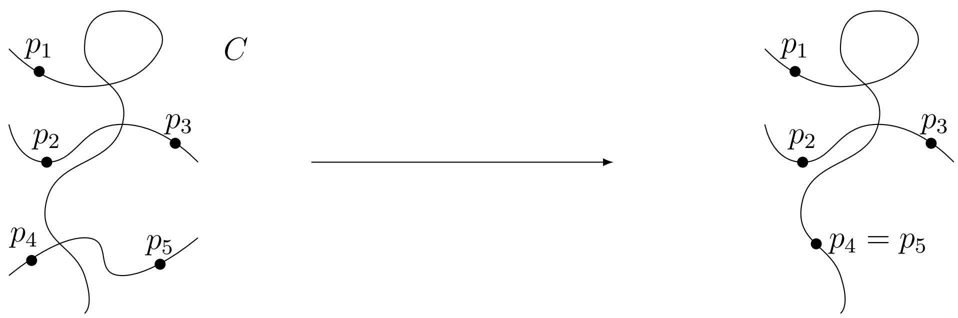

Figure 1.3: Suppose we have the weight data $\mathcal{A} = (1, 1, 1, 1, 1)$ and $\mathcal{B} = (1, 1, 1, \varepsilon, \varepsilon) \leq (1, 1, 1, 1, 1)$ in $\mathcal{D}_{g,5}$. This picture shows an example of reduction morphism on a curve over $Spec(k)$.

Being birational for this morphism implies in particular that for every fixed g and n, the moduli space $\overline{\mathcal{M}}_{g,n}$ is birational to each $\overline{\mathcal{M}}_{g,\mathcal{A}}$, for any $\mathcal{A} \in \mathcal{D}_{g,n}$.

Theorem 1.1.17. *([Has03], Theorem 4.2) Fix g and n and let $\mathcal{A} = (a_1, \ldots, a_n)$ be a weight datum, and $\mathcal{A}' := (a_{i_1}, \ldots a_{i_r})$ such that $\{a_{i_1}, \ldots a_{i_r}\} \subset \{a_1, ..., a_n\}$ a subset so that*

$$2g - 2 + a_{i_1} + \cdots + a_{i_r} > 0.$$

Then there exists a natural forgetting morphism $\phi_{\mathcal{A},\mathcal{A}'} : \overline{\mathcal{M}}_{g,\mathcal{A}} \to \overline{\mathcal{M}}_{g,\mathcal{A}'}$. Given an element $(C, s_1, \ldots, s_n) \in \overline{\mathcal{M}}_{g,\mathcal{A}}$, $\phi_{\mathcal{A},\mathcal{A}'}(C, s_1, \ldots, s_n)$ is obtained by successively collapsing components of C along which $K_C + a_{i_1} s_{i_1} + \cdots + a_{i_r} s_{i_r}$ fails to be ample (see Figure 1.4).

The forgetting morphism plays a similar role to the morphism π in the previous section, and reduces to it when $\mathcal{A} = 1^{(n)}$ and $\mathcal{A}' = 1^{(n-1)}$. From now on, when we talk about tautological maps we consider also the reduction morphism and this generalized version of the forgetting morphism.

Hassett spaces appear often in the literature when one studies the birational geometry of $\overline{\mathcal{M}}_{g,n}$, but they have been studied also indipendently. In [LM00] and [LM04], Losev and Manin study certain classes of Hassett Spaces of rational curves, called heavy/light spaces in the recent literature, in the context of cyclic operads. In general, we call an Hassett space heavy/light if

$$\mathcal{A} = (1^{(m)} | \varepsilon^{(n)}) := \left(\underbrace{1, \ldots, 1}_{m}, \underbrace{\varepsilon, \ldots, \varepsilon}_{n} \right),$$

where the components of weight 1 are called heavy and the ones of weight $\varepsilon \leq \frac{1}{n}$ are called light.

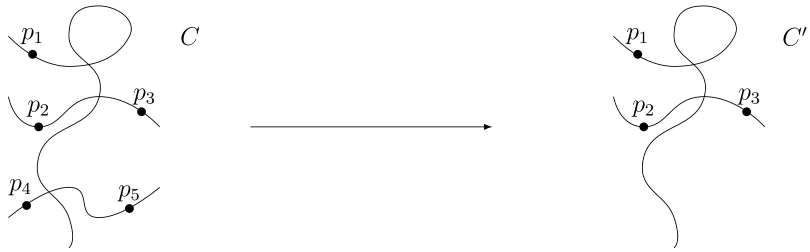

Figure 1.4: Suppose we have the weight data $\mathcal{A} = (1, 1, 1, 1, 1)$ and $\mathcal{A}' = (1, 1, 1)$, so that we are forgetting the last two marked points. This picture shows an example of forgetting morphism on a curve over $Spec(k)$. If the component containing p_4 and p_5 is rational, then $K_C + p_1 + p_2 + p_3$ is not ample there anymore, so it gets contracted.

Hassett spaces already made their appearance in the work of Keel and Tevelev [Kee92],[KT04] and Kapranov [Kap92]. In [CT20], the authors construct a collection Hassett spaces of weighted stable rational curves identified with symmetric GIT quotients of $(\mathbb{P}^1)^n$ by the diagonal action of $\mathbb{G}_m$. Their automorphism group was studied in [MM17].
In [Fed11] it is proven that $\overline{\mathcal{M}}_{g,\mathcal{A}}$ is a log-canonical model of $\overline{\mathcal{M}}_{g,n}$ whenever $\mathcal{A} \in \mathcal{D}_{g,n}$, and in [Moo11] the author prove a formula of log canonical models for moduli space of pointed stable curves which describes all Hassett's moduli spaces of weighted pointed stable curves in a single equation. In [Cey09] the author studies the Chow group of rational Hassett spaces, and for heavy/light Hassett spaces with no restricition on the genus this was done in [KKL21]. In [Swi08], the author constructs coarse moduli spaces $M_{g,\mathcal{A}}$ and $\overline{M}_{g,\mathcal{A}}$ using GIT techniques, differently from what was done by Hassett who used Kollár semi-ampleness criterion (see [Kol90], Theorem 2.2).

1.2 Moduli spaces of tropical curves

This section is devoted to develop the theory of tropical curves and their moduli spaces necessary in the next chapters.

1.2.1 Graphs and tropical curves

We start this section with the following definition:

Definition 1.2.1. A weighted graph with n legs G is the data of:

1) A finite non-empty set $V(G)$ called the set of vertices;

2) A finite set of half-edges $H(G)$;

3) An involution $\iota : H(G) \to H(G)$ with n fixed elements, called legs, whose set is denoted by $L(G)$;

4) An endpoint map $\epsilon : H(G) \to V(G)$;

5) A weight function on the vertices $w : V(G) \to \mathbb{Z}_{\geq 0}$.

We also introduce some notation related to this Definition:

i) A non-ordered pair $e = \{h, h'\}$ of distinct elements in $H(G)$ interchanged by the involution is called an edge of the weighted graph, and the set of edges is denoted by $E(G)$. An edge whose endpoints coincide is called a loop-edge. Two or more different edges with the same endpoints are called parallel or multiple edges.

ii) If $\epsilon(h) = v$ we say that h is adjacent to v, and that v is the endpoint of h. The same definition works for edges.

iii) The valence of a vertex v is the number of half-edges adjacent to v not fixed by the involution, and it is denoted with $val(v)$ i.e., it is the number of edges adjacent to it, counting loops twice.

iv) Two legs are called disjoint if their endpoints are distinct. For a given vertex v, the set of legs incident to it is denoted by $L(v) \subset L(G)$.

Definition 1.2.2. The genus of a weighted graph is

$$g(G) = b_1(G) + \sum_{v \in V(G)} w(v)$$

with

$$b_1(G) := |E(G)| - |V(G)| + c,$$

where c is the number of connected components of the weighted graph.

Remark 1.2.3. Here we need to clarify a conflict of terminology concerning our definition of "weight". In this context, the weight function assigns a weight to each vertex of the graph. Later on, we will assign weights to legs according to a weight datum as defined in 1.1.11. We will always distinguish between the two when it is not clear by the context, but we prefer to mantain these name to be consistent with the literature.

Example 1.2.4. Consider the weighted graph obtained by the following data. We take $V(G) = \{v_1, v_2\}$ and $H(G) = \{h_1, h_2, k_1, k_2, l_1, l_2, l_3\}$ with $\iota(h_i) = h_j$, $\iota(k_i) = k_j$, $i \neq j$ and $\iota(l_i) = l_i$ for $i = 1, 2, 3$. The endpoint map is given by $\epsilon(k_1) = \epsilon(k_2) = \epsilon(l_1) = \epsilon(h_1) = v_1$, while $\epsilon(h_2) = \epsilon(l_2) = \epsilon(l_3) = v_2$. The weight function is given by $w(v_1) = 0$, $w(v_2) = 1$. The genus of the weighted graph is

$$|E(G)| - |V(G)| + 1 + \sum_{v \in V(G)} w(v) = 2 - 2 + 1 + 1 = 2.$$

We depict its geometric realization in Figure 1.5.

We adopt the following conventions to depict the geometric realization of a weighted graph: if the weight is zero we will depict the vertex as a dot, while if it is not zero we will depict the vertex as a circle with its weight written inside. We depict non-loop edges as segments between the vertices, loops as circles closing on their vertex and legs as half-segments. From now on, we will work with all the graphs from their geometric realization. We will also omit labels of the half-edges and vertices unless we need them. We will also consider only graphs which are connected, i.e., $c = 1$.

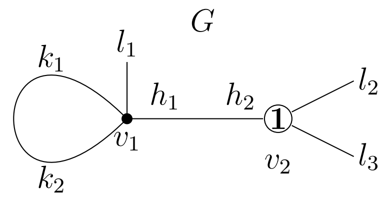

Figure 1.5: A weighted graph of genus 2 with 3 legs.

We now define morphisms of weighted graphs.

Definition 1.2.5. A morphism between weighted graphs is a map

$$\alpha : V(G) \cup H(G) \to V(G') \cup H(G')$$

such that $\alpha(L(G)) \subset L(G')$ and the following diagrams commute:

$$\begin{array}{ccc} V(G) \cup H(G) & \xrightarrow{\alpha} & V(G') \cup H(G') \\ \big\downarrow{\scriptstyle (id_{V'}, \epsilon)} & & \big\downarrow{\scriptstyle (id_{V''}, \epsilon')} \\ V(G) \cup H(G) & \xrightarrow{\alpha} & V(G') \cup H(G') \end{array} \qquad \begin{array}{ccc} V(G) \cup H(G) & \xrightarrow{\alpha} & V(G') \cup H(G') \\ \big\downarrow{\scriptstyle (id_{V'}, \iota)} & & \big\downarrow{\scriptstyle (id_{V''}, \iota')} \\ V(G) \cup H(G) & \xrightarrow{\alpha} & V(G') \cup H(G'). \end{array}$$

In particular, we have that $\alpha(V(G)) \subseteq V(G')$.

Definition 1.2.6. A morphism $\alpha : G \to G'$ is said to be an isomorphism if it induces by restriction three bijections: $\alpha_V : V(G) \to V(G')$, $\alpha_E : E(G) \to E(G')$ and $\alpha_L : L(G) \to L(G')$. An automorphism of a weighted graph G is an isomorphism of G with itself: in this case α_L must preserve the adjacence of the legs, i.e., endpoints of the legs are preserved.

Remark 1.2.7. Loops and parallel edges play a role in the automorphisms of the weighted graph G. For every loop $l = \{h, k\}$, an automophism can fix the loop or can send h in k and vice versa. Similarly if we have two parallel edges $e_1 = \{h_1, k_1\}$ and $e_2 = \{h_2, k_2\}$, where the h_i's and the k_i's have respectively the same endpoints, then automorphisms can switch h_1 with h_2 and k_1 with k_2, i.e., can send the edge e_1 in the edge e_2.

If the image of an edge e is $v' \in V(G')$, it follows from the definition that also its endpoints are mapped into v'; in this case we say that e is contracted by α, and that α is a contraction. Let G be a weighted graph, and let $T \subseteq E(G)$: notation G/T denotes the weighted graph obtained by contracting the edges $e \in T$. Vertices of a contracted edge are identified, while the other vertices remain unaltered. So the contraction morphism $\alpha : G \to G/T$ is a surjection (i.e. edges and vertices of G/T are the images of some edge or vertex of G). Moreover, we have a natural identification between $E(G) \setminus T$ and $E(G/T)$, a surjection of the vertices and a bijection between the sets of legs. One can prove that any morphism of weighted graphs is given by compositions of contractions and isomorphisms.

Given $T \subset E(G)$, a weighted contraction is $(G/T, w/T)$ where G/T is a contraction and w/T is the weight function defined by setting, for every $v \in V(G/T)$

$$w/T(v) = b_1(\sigma^{-1}(v)) + \sum_{u \in \sigma^{-1}(v)} w(u) \tag{1.1}$$

with σ being the contraction morphism.

It is clear from Equation 1.1 that the genus of a weighted graph remains constant after contraction. Indeed, it is easy to see that if e is a non-loop edge, then the vertex obtained by its contraction has weight equal to the sum of the weights of the endpoints of e. If it is a loop, the weight of its endpoint vertex grows by one.

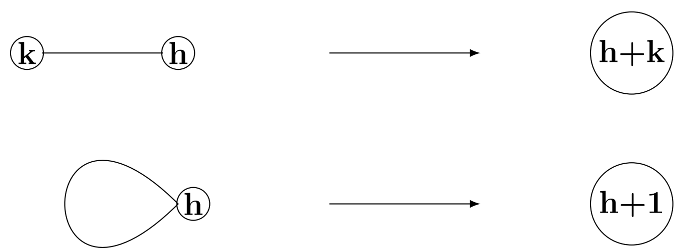

Figure 1.6: Rules for the weight of a vertex after a contraction morphism

Let G be a weighted graph, and let $L(G)$ be the set of its legs. The assignment of a number from 1 to n to each leg is called marking, and a graph with a marking is called marked graph. We write also $L(G) = \{x_1, ..., x_n\}$ and we usually identify this set with $[n] := \{1, \dots, n\}$. A morphism of marked graphs $\phi : G \to G'$ is a morphism of graphs which preserves the marking, i.e., $\phi(x_i) = x'_i$ for every i from 1 to n.

Remark 1.2.8. In some literature, instead of giving the definition of marked graph through legs, one defines a graph as we did above but asking that the involution has no fixed points, and then defines a marking function

$$m : [n] \to V(G).$$

The two definitions are equivalent, so we will use both notations interchangeably.

From now on, we will refer to marked weighted graphs just calling them graphs when no confusion arises, so for us a graph G will be $(V(G), H(G), \iota, \epsilon, w, m)$, where by m we indicate either the marking function or the label of the legs.

Example 1.2.9. We consider the same graph of Figure 1.5, but we add a marking on it. In particular, we label the leg incident to v_1 by 3, while the other two are labeled 1 and 2 (since they are adjacent to the same vertex, the order is not important). On the picture, we put the label near the leg. In this case, the marking function would have been $m : [3] \to V(G)$ so that $m(3) = v_1$ and $m(1) = m(2) = v_2$.

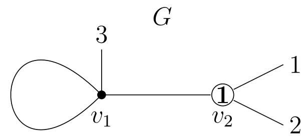

Figure 1.7: Example of the marking of legs on a graph.

We can now define tropical curves:

Definition 1.2.10. ([Cap11], Definition 2.28) An n-marked tropical curve of genus g is a pair $\Gamma := (G, l)$ where G is a weighted marked graph of genus g with n legs and l is a function

$$l : E(G) \cup L(G) \to \mathbb{R}_{>0} \cup \{\infty\}$$

such that $l(x) = \infty$ if and only if x is a leg. or an edge adjacent to a vertex of valence 1 and weight 0.

In the literature, these are also known as Abstract tropical curves, as people usually use the name tropical curves for tropical varieties of dimension one. We will call them just tropical curves avoiding to recall the number of legs and the genus when no confusion arises. Note that later in this work edges of valence 1 and weight zero will disappear from our graphs and tropical curves, due to the stability conditions we ask for them. Let $\Gamma = (G, l)$ be a tropical curve. We call G its underlying graph, and we denote it with $G(\Gamma)$ when it is necessary. Legs are called marked points of the tropical curve. We also write $V(\Gamma)$, $E(\Gamma)$ and so on to indicate vertices, edges and other characteristics of the tropical curve, meaning the ones of the underlying graph. Let $\Gamma = (G, l)$ be a tropical curve. We have the following notations:

i) Let w be the weight function of G; if $w(v) = 0$ for every $v \in V(G)$, we write $w = \underline{0}$ and we say that the tropical curve is pure.

ii) A tropical curve is called regular if it is pure and if G is a 3-regular graph, i.e., all of its vertices have valence 3.

iii) The volume of a tropical curve is defined as the sum of its edge lenghts.

Definition 1.2.11. Let $g, n \geq 0$ be such that $2g - 2 + n > 0$. We say that a graph G of genus g is stable if for every vertex $v \in V(G)$

$$2w(v) - 2 + |\epsilon^{-1}(v)| > 0.$$

Notice that the number $|\epsilon^{-1}(v)|$ for a vertex v equals the sum of its valence plus the number of legs incident to it, i.e.,

$$|\epsilon^{-1}(v)| = val(v) + |L(v)|. \tag{1.2}$$

Definition 1.2.12. A tropical curve is stable if its underlying graph is stable.

We say that two tropical curves Γ and Γ' with n marked points $L(\Gamma) = \{x_1, ..., x_n\}$ and $L(\Gamma') = \{x'_1, ..., x'_n\}$ are isomorphic if there exists an isomorphism of weighted marked graphs α from $G(\Gamma)$ to $G(\Gamma')$ such that $l(e) = l'(\alpha(e))$. We denote by $Aut(\Gamma)$ the group of automorphisms of a tropical curve Γ. Note that $Aut(\Gamma) \subset Aut(G(\Gamma))$. We give also the following definition of extended tropical curve:

Definition 1.2.13. ([Cap11], Definition 3.25) An n-marked extended tropical curve of genus g is a pair $\Gamma := (G, l)$ where G is a weighted marked graph of genus g and l is a function

$$l : E(G) \cup L(G) \to \mathbb{R}_{>0} \cup \{\infty\}$$

such that $l(x) = \infty$ for every leg.

The only difference with Definition 1.2.10 is that we possibly have infinite length edges. An edge with infinity length is seen as a copy of $(\mathbb{R}_{\geq 0} \sqcup \{\infty\}) \sqcup (\{-\infty\} \sqcup \mathbb{R}_{\leq 0})$ where we identify the two points ∞ and $-\infty$. All the other notions we have defined are valid also for extended tropical curves.

As in the case of algebraic curves, the notion of stability for tropical curves (and for graphs) can be extended through Hassett stability conditions in the following way. Let G be a weighted marked graph, and let $L(G) = \{x_1, ..., x_n\}$ be the set of its legs (with a label). For every $v \in V(G)$, we set

$$|v|_{\mathcal{A}} = \sum_{x_i \in L(v)} a_i.$$

Definition 1.2.14. ([Uli15], Definition 2.1) Let $(g, \mathcal{A} = (a_1, ..., a_n))$ be an input datum and G a weighted marked graph with n legs. We say that G is stable of type $(g, \mathcal{A})$ (or that it is $(g, \mathcal{A})$-stable) if it is of genus g and if for every vertex $v \in V(G)$

$$2w(v) - 2 + val(v) + |v|_{\mathcal{A}} > 0.$$

A tropical curve is $(g, \mathcal{A})$-stable if its underlying graph is $(g, \mathcal{A})$-stable.

When the genus is clear from the context, we will also write only $\mathcal{A}$-stable to mean $(g, \mathcal{A})$-stable. Note that whenever $\mathcal{A} = 1^{(n)}$, we recover Definition 1.2.11, as in the case of algebraic curves. Indeed, observe that $|v|_{1^{(n)}} = |L(v)|$, and so it suffices to observe Equation 1.2.

Lemma 1.2.15. *A weighted contraction of a stable graph of type $(g, \mathcal{A})$ is still a $(g, \mathcal{A})$-stable graph.*

Proof. Consider first a contraction $\pi : G \to G'$ of a single edge e. If e is not a loop, in G it has two endpoints v_1 and v_2 identified in the vertex $v' \in V(G')$. Denote by w' the weight function on G'. We have

$$2w'(v) - 2 + val(v') + |v'|_{\mathcal{A}} = 2w(v_1) + 2w(v_2) - 2 + val(v_1) + val(v_2) - 2 + |v_1|_{\mathcal{A}} + |v_2|_{\mathcal{A}} > 0$$

because G is $(g, \mathcal{A})$-stable, so $2w(v_i) + val(v_i) + |v_i|_{\mathcal{A}} > 2$.

If e is a loop, it has a single endpoint v in G and

$$2w'(v') - 2 + val(v') + |v'|_{\mathcal{A}} = 2(w(v)+1) - 2 + val(v) - 2 + |v|_{\mathcal{A}} =$$

$$= 2w(v) + 2 - 2 + val(v) - 2 + |v|_{\mathcal{A}} = 2w(v) - 2 + val(v) + |v|_{\mathcal{A}} > 0$$

again by hypothesis.

For a generic contraction, we can always write it as a sequence of one-edge contraction, so the result follows. □

Remark 1.2.16. If we fix g, n and $\mathcal{A} \in \mathcal{D}_{g,n}$, the set of $(g, \mathcal{A})$-stable graphs form a category $\mathcal{G}_{g,\mathcal{A}}$ generated by isomorphisms and contractions. The number of objects in this category is finite since the maximal number of edges for a $(g, \mathcal{A})$-stable graph is $3g - 3 + n$, and this category has a natural structure of poset induced by edge contraction. We set $G' \leq G$ if G' is obtained from G after a contraction. When $\mathcal{A} = 1^{(n)}$, $\mathcal{G}_{g,\mathcal{A}} := \mathcal{G}_{g,n}$.

Example 1.2.17. Let $g = 1, n = 2$. In $\mathcal{G}_{1,1^{(2)}} := \mathcal{G}_{1,2}$ we the have five graphs in 1.8. If instead we consider the weight datum $\varepsilon^{(2)}$, for $\varepsilon < \frac{1}{2}$, its poset is made by the graphs in blue, again in Figure 1.8.. In general, is it possible to see that if a graph is $\mathcal{A}$-stable for a certain $\mathcal{A}$, then it is also stable in the standard sense. So $\mathcal{G}_{1,\varepsilon^{(2)}} \subset \mathcal{G}_{1,2}$. We will come back to this particular property in Section 4.1.

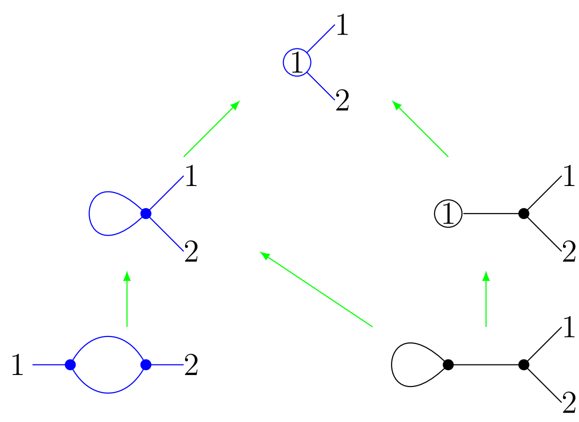

Figure 1.8: The poset for $\mathcal{G}_{1,2}$. In blue, the poset of $\mathcal{G}_{1,\varepsilon^{(2)}}$. Green arrows indicate the direction of the order of the poset as we defined above.

1.2.2 Construction of the Tropical Moduli Spaces

As in the case of algebraic curves, for fixed g, n and $\mathcal{A} \in \mathcal{D}_{g,n}$ one can construct a space whose points are in bijection with isomorphism classes of $(g, \mathcal{A})$-stable tropical curves. Here in this section we recall some tools and techniques and we briefly see how to construct such Moduli Spaces of tropical curves. We start by introducing Rational Polyhedral Cones and Rational Polyhedral Cone Complexes.

Definition 1.2.18. A strictly convex rational polyhedral cone is a pair (σ, N) consisting of a finitely generated free abelian group N and a $\sigma \subset N_{\mathbb{R}} := N \otimes \mathbb{R}$ that is a finite intersection of half spaces $H_i = \{u \in N_{\mathbb{R}} | \langle u, v_i \rangle \geq 0\}$, where $\langle -, - \rangle$ is the duality pairing, $v_i \in Hom(N, \mathbb{Z})$, such that σ does not contain any nontrivial linear subspace.

A morphism $f : (\sigma, N) \to (\sigma', N')$ of rational polyhedral cones is given by an element $f \in Hom(N, N')$ such that $f(\sigma) \subset \sigma'$. We restrict our attention to sharp rational polyhedral cones, that is, those cones (σ, N) whose linear span in $N_{\mathbb{R}}$ is equal to $N_{\mathbb{R}}$. We also drop the adjectives "rational polyhedral" when it is clear by the context. We denote the category of sharp cones by **RPC** and we refer to its objects, i.e., to cones, only with σ without explicit reference to N. A face morphism $\tau \to \sigma$ is a morphism of cones that induces an isomorphism onto a not necessarily proper face of σ. If it is isomorphic to a proper face of σ, we say it is as a proper face morphism. Denote by $Int(T)$ the interior of a topological space.

Definition 1.2.19. A rational polyhedral cone complex Σ is a topological space $|\Sigma|$ with a collection of rational polyhedral cones $\{\sigma_\alpha\}_{\alpha \in \Omega}$ and continuous maps $\phi_\alpha : \sigma_\alpha \to |\Sigma|$ such that the following properties hold:

(i) The maps ϕ_α are injective and induce a bijection

$$\bigsqcup_\alpha Int(\sigma_\alpha) \to |\Sigma|.$$

(ii) Given a proper face τ of a cone σ_α, then τ is also in the collection $\{\sigma_\alpha\}_{\alpha \in \Omega}$.

(iii) A subset A of $|\Sigma|$ is closed if and only if its preimages $\phi_\alpha^{-1}(A)$ are closed in σ_α for all α.

A morphism $f : \Sigma \to \Sigma'$ of rational polyhedral cone complexes is given by a continuous map $|f| : |\Sigma| \to |\Sigma'|$, a choice, for each σ_α in the collection of cones of $|\Sigma|$, of a $\sigma'_{\beta(\alpha)}$ in the collection of cones of $|\Sigma|'$, and a family of morphisms of cones $\sigma_\alpha \to \sigma'_{\beta(\alpha)}$, such that the diagrams

$$\begin{tikzcd}
\sigma_\alpha \arrow[r, "\phi_\alpha"] \arrow[d] & {|\Sigma|} \arrow[d, "|f|"] \\
\sigma'_{\beta(\alpha)} \arrow[r, "\phi'_{\beta(\alpha)}"] & {|\Sigma'|}
\end{tikzcd}$$

commute for all α, and is called $\mathbb{Z}$-linear morphism. Denote the category of (Rational Polyhedral) cone complexes with piece-wise and $\mathbb{Z}$-linear morphisms by **RPCC**.

We consider also extended cones and extended cone complexes. An extended cone complex is a topological space together with a finite collection of closed subspaces and an identification of each of these closed subspaces with an extended cone, which is $\overline{\sigma} \cong \overline{\mathbb{R}}^r_{\geq 0}$, where $\overline{\mathbb{R}}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{\infty\}$, such that the intersection of any two of these extended cones is a union of extended faces of each. A morphism of extended cone complexes is a defined as a morphism of cone complexes.

For every $\mathcal{A} \in \mathcal{D}_{g,n}$, there are Moduli Spaces of tropical curves $M_{g,\mathcal{A}}^{trop}$ and extended tropical curves $\overline{M}_{g,\mathcal{A}}^{trop}$ which carry respectively the structure of a cone complex and extended cone complex. Their construction is due to Ulirsch, [Uli15], and generalizes the previous constructions made firstly in the case on unmarked tropical curves for M_g^{trop} and $\overline{M}_g^{trop}$ in [BMV11] and then for stable marked tropical curves for $M_{g,n}^{trop}$ and $\overline{M}_{g,n}^{trop}$ in [Cap11]. Moreover, it is possible to define the locus $\Delta_{g,\mathcal{A}}$ of $(g, \mathcal{A})$-stable tropical curves of genus g and volume 1 inside of them in analogy with the ones considered in [CGP21] and [CGP22], respectively Δ_g for the unmarked version and $\Delta_{g,n}$ for the marked one. Consider the category $\mathcal{G}_{g,\mathcal{A}}$ of isomorphism classes of $(g, \mathcal{A})$-stable marked weighted graphs with morphisms generated by isomorphisms and contractions. We will refer to $M_{g,\mathcal{A}}^{trop}$ and $\overline{M}_{g,\mathcal{A}}^{trop}$ and $\Delta_{g,\mathcal{A}}$ as Tropical Hassett Spaces, while when necessary we will call the unweighted versions of these Standard Tropical Moduli Spaces.

We define a natural contravariant functor:

$$\Sigma : \mathcal{G}_{g,\mathcal{A}} \to \mathbf{RPCC}$$

on the category **RPCC** of rational polyhedral cone complexes as follows: to each isomorphism class of $(g, \mathcal{A})$-stable marked weighted graph G we associate the rational polyhedral cone $\sigma_G = \mathbb{R}_{\geq 0}^{|E(G)|}$. A weighted edge contraction $\pi : G \to G'$ induces the natural embedding $i_\pi : \sigma_{G'} \to \sigma_G$ of a face of σ_G. An automorphism of G induces an automorphism of σ_G. Similarly there is also a natural functor $\overline{\Sigma}$ from $\mathcal{G}_{g,\mathcal{A}}$ into the category of extended rational polyhedral cone complexes that is given by sending G into $\overline{\sigma}_G = \overline{\mathbb{R}}_{\geq 0}^{|E(G)|}$. The Moduli Space $M_{g,\mathcal{A}}^{trop}$ of $(g, \mathcal{A})$-stable tropical curves is defined to be the colimit

$$M_{g,\mathcal{A}}^{trop} := \lim_{\to} \sigma_G$$

taken over $(\mathcal{G}_{g,\mathcal{A}})^{op}$. The Moduli Space $\overline{M}_{g,\mathcal{A}}^{trop}$ of $(g, \mathcal{A})$-stable extended tropical curves is defined analogously using $\overline{\sigma}_G$'s. The Moduli Space $\Delta_{g,\mathcal{A}}$ of volume 1 $(g, \mathcal{A})$-stable tropical curves marked points is obviously a subspace of $M_{g,\mathcal{A}}^{trop}$. It can be defined in the same way we did for the other spaces. Let

$$\pi_G := \left\{ \ell : E(\mathbf{G}) \to \mathbb{R}_{\geq 0} \mid \sum_{e \in E(\mathbf{G})} \ell(e) = 1 \right\} \subset \mathbb{R}_{\geq 0}^{E(G)}.$$

Then we have

$$\Delta_{g,\mathcal{A}} := \lim_{\to} \pi_G,$$

where the limit is taken again over $(\mathcal{G}_{g,\mathcal{A}})^{op}$.

Remark 1.2.20. The Moduli Spaces of tropical curves can be embedded as subspaces one into each other in the following way: fix g, n and $\mathcal{A} \in \mathcal{D}_{g,n}$, then $\Delta_{g,\mathcal{A}} \subset M_{g,\mathcal{A}}^{trop} \subset \overline{M}_{g,\mathcal{A}}^{trop}$. The space $\Delta_{g,\mathcal{A}}$ can be identified with the link of $M_{g,\mathcal{A}}^{trop}$, i.e. its intersection with a sphere of radius one with center in its cone point, which is exactly the point representing the only graph with a vertex, no edges and n legs attached to it.

In the past years these Moduli Spaces where deeply studied. In [BMV11] authors construct a tropical Torelli map $t_g : M_g^{trop} \to A_g^{trop}$ into the Moduli Space of tropical principally polarized abelian varieties of dimension g constructed in the same work, sending a tropical curve into its Jacobian as it was defined in [MZ08]. This map was further studied in [Cha12] and [Viv13]. The space M_g^{trop} can be also seen as a quotient of the tropical Teichmuller space by the action of the Outer Automorphism group $Out(F_g)$ as shown in [CMV13].

Here we collect some known facts about the topology of these Moduli Spaces. In [BMV11], authors show that M_g^{trop} is of pure (real) dimension $3g-3$. This was later generalized by Caporaso in [Cap11] for $M_{g,n}^{trop}$, which has pure dimension equal to $3g-3+n$. For a generic $M_{g,\mathcal{A}}^{trop}$, for $\mathcal{A} \in \mathcal{D}_{g,n}$, dimension is still $3g-3+n$, but it may happen that it is not pure.

Example 1.2.21. Let $g = 1$, $n = 3$ and $\mathcal{A} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3} - \varepsilon)$, for $\varepsilon < \frac{1}{3}$. The graph G in Figure 1.9 is $(g, \mathcal{A})$-stable, so there is a locus of dimension $|E(G)| = 1$ inside $M_{g,\mathcal{A}}^{trop}$ parametrizing tropical curves with G as underlying graph not embedded in the boundary of any locus of greater dimension. But dimension of $M_{g,\mathcal{A}}^{trop}$ is $3g-3+n = 3$, as for example we have the locus of curves with underlying graph being the one of Figure 1.10

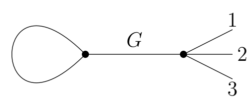

Figure 1.9: The graph G is clearly $(g, \mathcal{A})$-stable, but none of the possible uncontractions of G is $(g, \mathcal{A})$-stable.

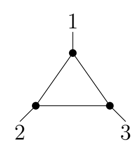

Figure 1.10: This graph is stable for every $\mathcal{A} \in \mathcal{D}_{1,3}$. This implies that for every $\mathcal{A} \in \mathcal{D}_{1,3}$ the space $M_{1,\mathcal{A}}^{trop}$ has dimension 3.

For any g and n, one can also show that $M_{g,n}^{trop}$ is connected through codimension one (see [BMV11] for $n = 0$, [Cap11] for the general standard case). Also we can show that $M_{g,\mathcal{A}}^{trop}$ is normal (hence Hausdorff), locally compact, paracompact, locally contractible, metrizable and second countable (see [CMV13] and [Viv13] for $n = 0$, [Cap11] for the general standard case and [Uli15] for the general case). In [Cap11] there are also some considerations on particular subsets of $M_{g,n}^{trop}$:

1. Let $M_{g,n}^{reg}$ be the subset of $M_{g,n}^{trop}$ parametrizing 3-regular curves, i.e., curves such that each vertex has valence 3. Then $M_{g,n}^{reg}$ is open and dense.

2. The subset $M_{g,n}^{pure}$ parametrizing pure tropical curves is open and dense.

Fact 1.2.22. *We record some results about the spaces $\Delta_{g,\mathcal{A}}$:*

1) *([ACP22] Theorem 1.1.) If $g \geq 2$, $n = 0$, the space Δ_g is simply connected.*

2) *([ACP22], Theorem 1.4.) The space $\Delta_{g,n}$ is simply connected for every (g,n) different from $(0,4),(0,5)$ such that $2g-2+n>0$.*

3) *([CGP22] Theorem 1.1.) Assume $g>0$ and $2g-2+n>0$. Each of the following subcomplexes of $\Delta_{g,n}$ is either empty or contractible.*

   1. *The subcomplex $\Delta^{w}_{g,n}$ parametrizing tropical curves with at least one vertex of positive weight.*
   2. *The subcomplex $\Delta^{lw}_{g,n}$ parametrizing tropical curves with loops or vertices of positive weight.*
   3. *The subcomplex $\Delta^{rep}_{g,n}$ parametrizing tropical curves in which at least two marked points coincide.*
   4. *The closure $\Delta^{br}_{g,n}$ of the locus of tropical curves with bridges.*

4) *([CGP22]) For $n \geq 3$, the space $\Delta_{1,n}$ is homotopy equivalent to a wedge sum of $\frac{(n-1)!}{2}$ spheres of dimension $n-1$.*

5) *([Kan21] Theorem 1.1.) Fix integers $g,n \geq 0$ such that $3g-3+n>0$. If $2g-2+n \geq 3$, then*
$$Aut(M^{trop}_{g,n}) \cong Aut(\overline{M}^{trop}_{g,n}) \cong Aut(\Delta_{g,n}) \cong S_n.$$
If $2g-2+n<3$, then $(g,n) \in \{(0,4),(1,1),(1,2)\}$, and in these cases we have $Aut(\Delta_{0,4}) \cong S_3$ while $Aut(\Delta_{1,1})$ and $Aut(\Delta_{1,2})$ are both trivial.

We will generalize Fact 3 of 1.2.22 in Chapter 2, giving the proof in Appendix A.1.

Abstract simplicial complexes

A collection K of non-empty finite subsets of a set X is called a set-family. A set-family K is called an abstract simplicial complex if, for every set S in K, and every non-empty subset $T \subset S$, the set T also belongs to K. The finite sets that belong to K are called faces of the complex, and a face T is said to belong to another face S if $T \subset S$, so the definition of an abstract simplicial complex can be restated as saying that every face of a face of a complex K is itself a face of K. The vertex set of K is defined as the union of all faces. For every vertex v the set $\{v\}$ is a face of the complex, and every face of the complex is a finite subset of the vertex set. Given a weight datum $\mathcal{A}$, we can form an abstract simplicial complex $K_{\mathcal{A}}$ with vertex set $[n]$ by declaring that a subset $S \subset [n]$ belongs to $K_{\mathcal{A}}$ if and only if $\sum_{i\in S} a_i \leq 1$. In [FHK21], the authors determine $Aut(\Delta_{g,\mathcal{A}})$ in terms of $K_{\mathcal{A}}$ for $g \geq 1$.

Theorem 1.2.23. *([FHK21], Theorem 1.1.) Let $g \geq 1$ and suppose $\mathcal{A} \in \mathcal{D}_{g,n}$ for some n such that $2g-2+n \geq 3$. Then $Aut(\Delta_{g,\mathcal{A}}) \cong Aut(K_{\mathcal{A}})$, where $Aut(K_{\mathcal{A}})$ acts by permuting the markings.*

Abstract simplicial complexes are examples of (geometric realization of) symmetric Δ-complexes, a very important tool we are going to introduce in the next section.

1.2.3 Symmetric Δ-complexes

We use this section to introduce one of the main tools we are going to use through the whole work, which is the notion of symmetric Δ-complex. We describe symmetric Δ-complexes following [CGP21] and [CGP22]. First let $p \in \mathbb{Z}$ be greater or equal than -1. Denote by σ^p the standard p-simplex, i.e., the convex hull of the standard basis vectors $e_0, \dots e_p$ in $\mathbb{R}^{p+1}$. Points of σ^p are $t = (t_0, \dots t_p)$, where $t_i \geq 0$ and $\sum_{i=0}^{p} t_i = 1$.

Given $\theta : \{0, \dots q\} \to \{0, \dots, p\}$ an injection, there is an induced inclusion θ_* of σ^q as a face of σ^p given by

$$\theta_*(t_0, \dots, t_q) = (t'_0, ..., t'_p) \tag{1.3}$$

where

$$t'_i := \begin{cases} t_j & \text{if } \theta(j) = i, \\ 0 & \text{if } \theta^{-1}(i) = \varnothing. \end{cases}$$

Let I be the category having one object for each finite set

$$\overline{[p]} := \begin{cases} \{0, \dots, p\} & \text{for } p \geq 0, \\ \varnothing & p = -1. \end{cases}$$

and morphisms consisting of all injections.

Definition 1.2.24. A symmetric Δ-complex X is a functor $X : I^{op} \to \mathsf{Sets}$, and a morphism of symmetric Δ-complexes is a natural transformation of functors.

For a given symmetric Δ-complex X, we will write X_p to denote $X(\overline{[p]})$ to lighten the notation. Whenever we have a morphism $\theta : \overline{[q]} \to \overline{[p]}$ in I (implying $q \leq p$), we denote by $\theta^* := X(\theta) : X_p \to X_q$ the map obtained through the functor. Notice that there is an induced action of S_{p+1} on X_p, for each p, given as follows: given a permutation $\phi \in S_{p+1}$, we can see it as a bijection $\phi : \overline{[p]} \to \overline{[p]}$, so in particular is an injection inducing $\phi* : X_p \to X_p$. The action is then defined as $\phi \cdot x = \phi^* x$ for every $x \in X_p$.

Each symmetric Δ-complex X comes with a geometric realization functor associating to it a topological space denoted by $|X|$ that we will define now. For each $p \geq 0$, let $X_p \times \sigma^p$ be the coproduct

$$\coprod_{x \in X_p} (\sigma^p)_x,$$

i.e., we associate to each element of X_p a p-simplex, labeled according to it, and we denote its elements by (x, t), for $x \in X_p$, $t \in \sigma^p$. The geometric realization of X is then defined as

$$|X| := \left(\coprod_{p \geq 0} X_p \times \sigma^p \right) \Big/ \sim$$

where the equivalence relation $\sim$ is generated by relations of the form $(\theta^* x, t) \sim (x, \theta_*(t))$, where $\theta^* x = X(\theta)$ as before and θ_* is the map of simplices $\theta_* : (\sigma^q)_{\theta^* x} \to (\sigma^p)_x$ associated to the injection $\theta : \overline{[q]} \to \overline{[p]}$ of Equation 1.2.3. We refer to the p-th piece of the union $X(p) \times \sigma^p / \sim$ as the p-skeleton.

Remark 1.2.25. When no confusion arises, we denote with X both the functor and its geometric realization. When this happens, we will refer to the elements of X_p as the p-simplices of the symmetric Δ-complex, as the elements of this set index the copies of σ^p (before the quotient by $\sim$) in the geometric realization (also, we will think to the elements of X_p as copies of σ^p).

Let X be (the geometric realization of) a symmetric Δ-complex, $\theta : \overline{[q]} \to \overline{[p]}$ an injection, $\theta^* : X_p \to X_q$ the induced map. Let $\sigma \in X_p$ be a simplex (according to Remark 1.2.25), we say that $\tau = \theta^*\sigma \in X_q$ is a face of σ, and we write $\tau \precsim \sigma$. This induces a partial order on the set $\bigsqcup_{p\geq 0} X_p$. The order $\precsim$ descends to an order $\preceq$ on $\bigsqcup_{p\geq 0} X_p/S_{p+1}$, since the actions of S_{p+1} on the X_p's are compatible with injections.

Remark 1.2.26. Again, the term "face" is used to indicate the fact that when we identify a symmetric Δ-complex with its geometric realization, we are thinking elements of X_p as the simplices they index. In the same way, in what follows we will call elements of X_0 vertices of a symmetric Δ-complex.

Definition 1.2.27. ([CGP22], Definition 4.1) Let X be a symmetric Δ-complex. A Property on X is a subset of the vertices $P \subset X_0$.

Remark 1.2.28. By the word "Property" with a capital P, from now on, we refer to this definition.

Let P be a Property, $\sigma \in X_p$. For $i = 0, ..., p$, the i-th vertex of σ is $v_i = i^*(\sigma)$ where $i : \overline{[0]} \to \overline{[p]}$ sends 0 to i. We write

$$P(\sigma) = \{i \in [p] : v_i \in P\}$$

for the set of P-vertices of σ. Similarly,we write $P^c(\sigma)$ for its complementary set, and we call these the non-P-vertices of σ. Write $Simp(X)$ for the set of simplices of X, i.e., it is the set $Simp(X) = \bigsqcup_{p\geq 0}\{(\sigma^p)_x \in X_p\}$. We define also

$$P(X) = \{\sigma \in Simp(X) | P(\sigma) \neq \varnothing\}.$$

Elements of $P(X)$ are called P-simplices of X. If $P^c(\sigma) = \varnothing$,then we say σ is a strictly P-simplex. We write also

$$P^*(X) := \{\tau \in Simp(X) : \tau \precsim \sigma \in P(X)\}.$$

Let $\sigma \in X_p$, $\theta : [q] \to [p]$, we say that θ is a co-P face map if $[p] \setminus im(\theta) \subset P(\sigma)$. In this case we say that $\tau = \theta^*(\sigma)$ is a co-P face of σ, and then we write $\tau \precsim_P \sigma$. Then it is a reflexive, transitive relation, and it induces a partial order $\preceq_P$ on $\bigsqcup_{p\geq 0} X_p/S_{p+1}$. A face $\tau \precsim \sigma$ is canonical if, for any two injections θ_1 and θ_2 from $[q]$ to $[p]$ such that $\theta_i^*(\sigma) = \tau$, there exists $\psi \in Aut(\sigma)$ such that $\theta_1 = \psi\theta_2$. We will say that $[\tau] \preceq [\sigma]$ is canonical if $\tau \precsim \sigma$ is canonical.

Definition 1.2.29. ([CGP22], Definition 4.8) Let $Z \subset \bigsqcup_{p\geq 0} X_p/S_{p+1}$ and let P be any Property. We say that Z admits canonical co-P maximal faces if, for every $[\tau] \in Z$, the poset of those $[\sigma] \in Z$ such that $[\tau] \preceq_P [\sigma]$ has a unique maximal element $[\hat{\sigma}]$, and moreover $[\tau] \preceq [\hat{\sigma}]$ is canonical.

Definition 1.2.30. ([CGP22], Definition 4.1) Let $Y \subset Simp(X)$ be any subset and let P be any Property on X. We call Y co-P-saturated if $\tau \in Y$ and $\tau \precsim_P \sigma$ implies $\sigma \in Y$.

We cite also two important results of [CGP22] we need later. Denote by X_P the subcomplex of X generated by $P^*(X)$, i.e., the subcomplex obtained by taking all the simplices n $P^*(X)$ and then quotient by $\sim$, and by $X_{P,i}$ the image of the natural map

$$\bigsqcup_{p \geq 0} (X_P \cap V_{P,i}) \times \Delta^p \to |X|$$

where $V_{P,i}$ is the set of P-simplices with at most i non P-vertices.

Proposition 1.2.31. *([CGP22], Proposition 4.9) Let X be a symmetric Δ-complex. Suppose P,$Q \subset X_0$ are properties satisfying the following conditions.*

1. *The set of simplices $P^*(X)$ is co-Q-saturated.*

2. *The set of symmetric orbits of $X \setminus P^*(X)$ admits canonical co-Q maximal faces.*

Then there are strong deformation retracts

$$(X_P \cup X_{Q,i}) \searrow (X_P \cup X_{Q,i-1})$$

for each $i > 0$.

If, in addition, every strictly Q-simplex is in $P^*(X)$, then there is a strong deformation retract $(X_P \cup X_Q) \searrow X_P$.

Corollary 1.2.32. *([CGP22], Corollary 4.16) Let X be a symmetric Δ-complex, and let $P_1, ..., P_N$ be a sequence of properties.*

1. *Suppose that for $i = 2, ..., N$, the two properties $P = P_1 \cup ... \cup P_{i-1}$ and $Q = P_i$ satisfy that*

   - *$P^*(X)$ is co-Q-saturated,*
   - *the symmetric orbits of $X \setminus P^*(X)$ admit canonical co-Q maximal faces,*
   - *every strictly Q-simplex is in $P^*(X)$.*

   Then there exists a strong deformation retract

   $$X_{P_1 \cup ... \cup P_N} \searrow X_{P_1}.$$

2. *If in addition the symmetric orbits of X admit canonical co-P_1 maximal faces,then there exists a strong deformation retract*

   $$X_{P_1 \cup ... \cup P_N} \searrow X_{P_{1,0}}.$$

To a symmetric Δ-complex X, we can associate its group of cellular p-chains

$$C_p(X) = (\mathbb{Q}^{sign} \otimes \mathbb{Q}X_p)_{S_{p+1}}$$

where $\mathbb{Q}X_p$ is the vector space with basis X_p on which S_{p+1} acts by permuting the basis vectors. By Proposition 3.8 of [CGP21], since we are over $\mathbb{Q}$, the homology of $C_*(X)$ is identified with $\widetilde{H}_*(|X|;\mathbb{Q})$ for all the Δ-complexes we will consider.

Definition 1.2.33. ([CGP22],Definition 3.5) A subcomplex $X \subset Y$ of a symmetric Δ-complex is a subfunctor X of $Y : I^{op} \to Sets$, in which for each p, X_p is a subset of Y_p, with the subfunctor being given by the canonical inclusions $X_p \hookrightarrow Y_p$. The inclusion $\iota : X \to Y$ then induces an injection $|\iota| : |X| \to |Y|$ which we shall use to identify $|X|$ with its image $|X| \subset |Y|$.

Whenever $X \subset Y$ is a subcomplex, for every $p \geq -1$ one can consider the exact sequence

$$0 \to C_p(X) \to C_p(Y) \to C_p(Y, X) \to 0.$$

Example 1.2.34. We can regard our $\Delta_{g,\mathcal{A}}$ as symmetric Δ-complexes in the sense of [CGP21] and [CGP22] generalizing the construction of Example 3.2,[CGP22], as follows. Let $X = X_{g,\mathcal{A}} : I^{op} \to Sets$ be a functor, with

$$X_p = \{\text{equivalence classes of pairs } (G, \tau)\},$$

with $G \in Ob(\mathcal{G}_{g,\mathcal{A}})$ and $\tau : E(G) \to [p]$ a bijection, where we consider $\tau = \tau'$ if they are in the same orbit under the evident action of $Aut(G)$. For every $\theta : [p'] \to [p]$ we have $\theta^* : X_p \to X_{p'}$ as follows: given an element of X_p represented by $(G, \tau : E(G) \to [p])$ we contract the edges of G whose labels are not in $\theta([p']) \subset [p]$, and then we relabel the remaining edges with labels $[p']$ as prescribed by θ. The result is a $[p']$ edge labeled graph G', and we set it to be $\theta^*(G)$. So by what we saw above there is a chain complex $C_*(\Delta_{g,\mathcal{A}})$ whose homology can be identified with $\widetilde{H}_*(\Delta_{g,\mathcal{A}};\mathbb{Q})$.

Note that whenever $\mathcal{A}, \mathcal{A}' \in (\mathbb{Q} \cap (0,1])^n$ satisfy $a_i \leq a_i'$ for all i, we can identify $\Delta_{g,\mathcal{A}}$ as a subcomplex (subfunctor) of $\Delta_{g,\mathcal{A}'}$, since there is an equality

$$\Delta_{g,\mathcal{A}}(p) = \{[\mathbf{G}, \tau] \in \Delta_{g,\mathcal{A}'}(p) \mid \mathbf{G} \text{ is } \mathcal{A}\text{-stable}\}.$$

We will return later on this property more deeply in Section 4.2.

Example 1.2.35. Let $g = 1$, $n = 3$ and $\mathcal{A} = \varepsilon^{(3)}$ for $0 < \varepsilon < 1/3$. We describe (the geometric realization of) $\Delta_{1,\mathcal{A}}$ by its skeleta. The 0-skeleton of $\Delta_{1,\mathcal{A}}$, i.e., $\Delta_{1,\mathcal{A}}(0) = \{G \in \Delta_{1,\mathcal{A}} : |E(G)| = 1\}$ has only one 0-simplex parametrizing the graph shown in Figure 1.11.

The elements in $\Delta_{1,\mathcal{A}}(1) = \{G \in \Delta_{1,\mathcal{A}} : |E(G)| = 2\}$ have three combinatorial types shown in Figure 1.12. Since contracting any edge in any graph in Figure 1.12 gives G_0, the endpoints of all 1-simplices indexed by combinatorial types of $\Delta_{1,\mathcal{A}}(1)$ are identified with the point corresponding to G_0. Moreover, each combinatorial type of an element in $\Delta_{1,\mathcal{A}}(1)$ admits a $\mathbb{Z}/2\mathbb{Z}$ automorphism induced by permuting the top and bottom edges. The 1-skeleton of $\Delta_{1,\mathcal{A}}$ is thus three half-edges glued at the point G_0; see Figure 1.13. Lastly, there is only one combinatorial type in $\Delta_{1,\mathcal{A}}(2)$, yielding one 2-simplex $\mathbf{T}$; see Figure 1.14.

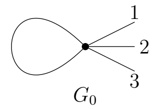

Figure 1.11: The combinatorial type of the only curve in $\Delta_{1,\mathcal{A}}(0)$ for $\mathcal{A} = \varepsilon^{(3)}$.

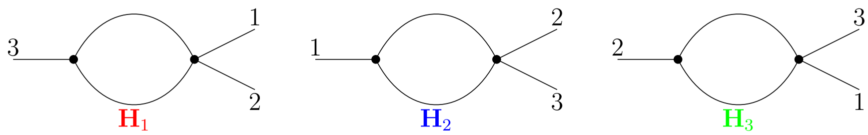

Figure 1.12: Combinatorial types in $\Delta_{1,\mathcal{A}}(1)$, for $\mathcal{A} = \varepsilon^{(3)}$.

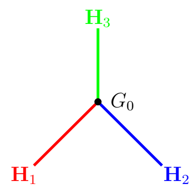

Figure 1.13: The geometric realization of the 1-skeleton of the symmetric Δ-complex $\Delta_{1,\mathcal{A}}$ when $\mathcal{A} = \varepsilon^{(3)}$. We use colors to distinguish the simplices, and we label them with the name of the corresponding graph.

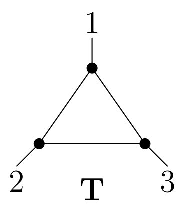

Figure 1.14: The only $\varepsilon^{(3)}$-stable graph with 3 edges.

The 2-simplex has no self-identification in its interior since $\mathrm{Aut}(\mathbf{T})$ is trivial. The only gluings happen on the boundary of the 2-simplex: these are exactly the self-gluings of the 1-simplices seen before. It is glued to the 1-skeleton as shown in Figure 1.15. The resulting space $\Delta_{1,\mathcal{A}}$ is homeomorphic to a 2-sphere and indeed simply connected.

Example 1.2.36. Let $g = 1$, $\mathcal{A} = \varepsilon^{(3)}$, and $\mathcal{A}' = (1, \varepsilon, \varepsilon)$. Since $a_i \leq a'_i$ fore every i, $\Delta_{1,\mathcal{A}'}$ contains $\Delta_{1,\mathcal{A}}$. The space $\Delta_{1,\mathcal{A}'}$ contains new simplices in dimensions 0, 1 and 2, corresponding to graphs shown in Figure 1.16, 1.17 and 1.18 respectively.

The 1-simplices have as endpoints the 0-simplices according to these graph contractions: $\mathbf{E}_1$ contracts to $\mathbf{G}_1$ and $\mathbf{G}_2$, $\mathbf{E}_2$ contracts to $\mathbf{G}_3$ and $\mathbf{G}_2$, $\mathbf{F}_1$ contracts to $\mathbf{G}_1$ and G_0, $\mathbf{F}_2$ contracts to $\mathbf{G}_2$ and G_0, $\mathbf{F}_3$ contracts to G_0. Modding out by appropriate automorphisms and folding some 1-simplices into half-edges, the 1-skeleton of $\Delta_{1,\mathcal{A}'}$ is shown in Figure 1.19.

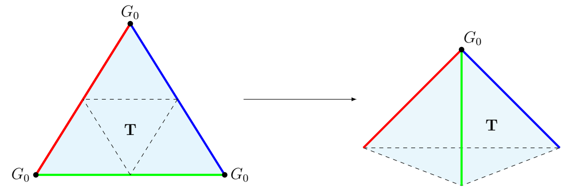

Figure 1.15: On the left, we have the 2-simplex before the gluings given by automorphisms of the combinatorial types in $\Delta_{1,\mathcal{A}}(1)$. The three vertices of the simplex are glued together since they correspond to the same combinatorial types in $\Delta_{1,\mathcal{A}}(0)$. The geometric realization of $\Delta_{1,\mathcal{A}}$ is the (hollow) tetrahedron on the right.

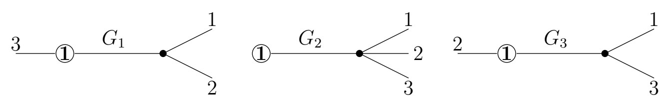

Figure 1.16: Combinatorial types in $\Delta_{1,\mathcal{A}'}(1) \setminus \Delta_{1,\mathcal{A}}(1)$.

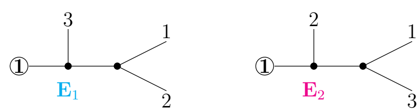

Figure 1.17: Combinatorial types in $\Delta_{1,\mathcal{A}'}(1) \setminus \Delta_{1,\mathcal{A}}(1)$ made by non-pure graphs.

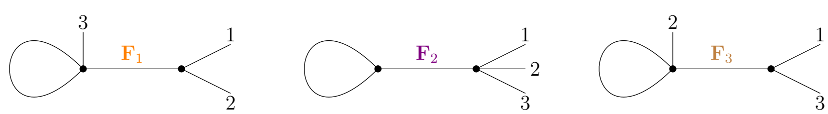

Figure 1.18: Combinatorial types in $\Delta_{1,\mathcal{A}'}(1) \setminus \Delta_{1,\mathcal{A}}(1)$ made by pure graphs.

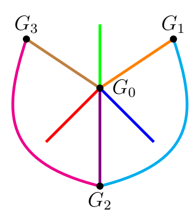

Figure 1.19: The geometric realization of the 1-skeleton of the symmetric Δ-complex $\Delta_{1,\mathcal{A}'}$ when $\mathcal{A}' = (1, \varepsilon, \varepsilon)$. The inclusion $\Delta_{1,\mathcal{A}}(1) \subset \Delta_{1,\mathcal{A}'}(1)$ is clearly visible in the picture.

The 2-simplices corresponding to the graphs in $\Delta_{1,\mathcal{A}'}(2)$ have boundaries formed by 1-simplices corresponding to 1-edge graphs: $\mathbf{R}_1$ contracts to $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{E}_1$; $\mathbf{R}_2$ contracts to $\mathbf{F}_3$, $\mathbf{F}_2$ and $\mathbf{E}_2$; $\mathbf{P}_1$ contracts to $\mathbf{H}_1$ and to $\mathbf{F}_1$ on its parallel edges, and $\mathbf{P}_2$ contracts to $\mathbf{H}_2$ and to $\mathbf{F}_2$ on its parallel edges. Their identifications are described in Figures 1.21 and 1.22.

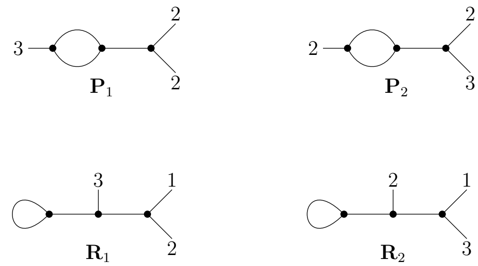

Figure 1.20: Combinatorial types in $\Delta_{g,\mathcal{A}'}(2) \setminus \Delta_{g,\mathcal{A}}(2)$.

All the quotients of the two simplices are then glued together according to edge contractions. The resulting space $\Delta_{1,\mathcal{A}'}$ is evidently simply connected and contains $\Delta_{1,\mathcal{A}}$; see Figure 1.23.

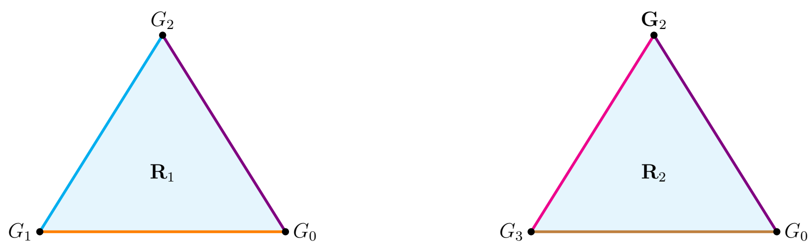

Figure 1.21: Each of $\mathbf{R}_1$ and $\mathbf{R}_2$ has a nontrivial automorphism which "flips" the loop but does not induce self-gluings. Moreover, there are no gluings of edges or vertices, so the 2-simplices in $\Delta_{1,\mathcal{A}'}(2)$ remain solid triangles with edges being 1-simplices and with 3 distinct vertices.

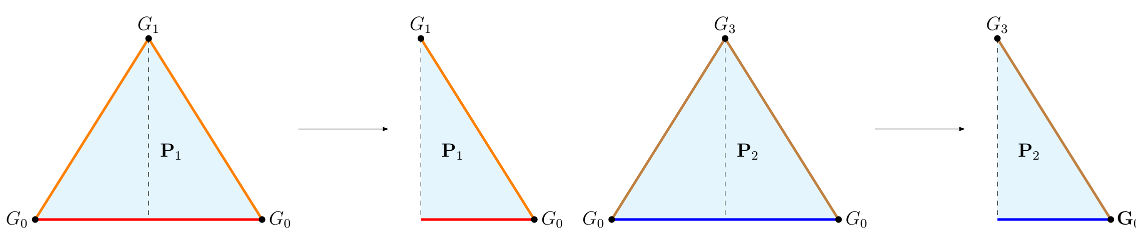

Figure 1.22: Each of $\mathbf{P}_1$ and $\mathbf{P}_2$ has a nontrivial automorphism which induces a self-gluing on the corresponding simplex. The resulting 2-simplex has an edge corresponding to a simplex with two distinct vertices, an edge corresponding to the self-gluing of a 1-simplex, and an edge which is not a simplex.

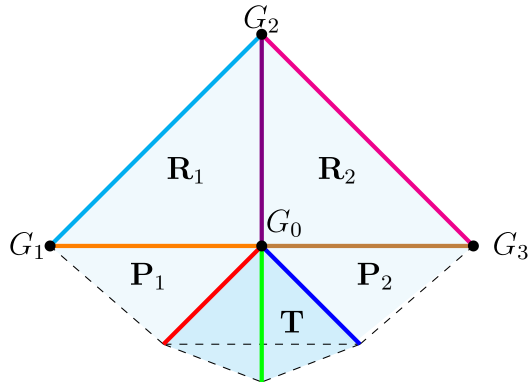

Figure 1.23: The geometric realization of the space $\Delta_{1,\mathcal{A}'}$, for $\mathcal{A}' = (1, \varepsilon, \varepsilon)$. Again we can see $\Delta_{1,\mathcal{A}} \subset \Delta_{1,\mathcal{A}'}$ and that $\Delta_{1,\mathcal{A}'}$ is simply connected.

1.3 Connections between Algebraic and Tropical Setting

1.3.1 Normal Crossings Divisors and Dual Boundary Complexes

The theory of dual complexes for simple normal crossings divisors was introduced by Danilov in [Dan75], and some developments of the theory appear in [Pay13]. The theory was later enhanced in [CGP21] to make it work for Stacks rather than varieties, and to normal crossings divisors which are not simple normal crossings. We will use the generalization of this latter work to describe how to construct the dual boundary complex of a normal crossings in a Deligne-Mumford Stack. These boundary divisors arise naturally as (geometric realizations of) symmetric Δ-complexes.

First of all, the dual complex $\Delta(D)$ of a simple normal crossings divisor D in a d-dimensional smooth variety X is naturally defined as a symmetric Δ-complex: for each $p \geq -1$, $\Delta(D)(p)$ is the set of equivalence classes of pairs (x, σ), where x is a point in a stratum of codimension p in D and σ is an ordering of the $p+1$ analytic branches of D that meet at x. The equivalence relation is generated by paths within strata: if there is a path from x to x' within the codimension p stratum and a continuous assignment of orderings of branches along the path, starting at (x, σ) and ending at (x', σ'), then we set $(x, \sigma) \sim (x', \sigma')$.

We now generalize this construction to normal crossings divisors D in a smooth Deligne-Mumford Stack $\mathcal{X}$. We refer to [CGP21] for a deeper study of the issues we have to solve in this situation, and we limit our discussion on the construction of $\Delta(D)$. A divisor $D \subset \mathcal{X}$ has normal crossings if and only if there is an étale cover by a smooth variety $X_0 \to \mathcal{X}$ in which the preimage of D is a divisor with simple normal crossings (this étale local characterization of normal crossings divisors works both for varieties and Deligne-Mumford Stacks). In this situation the dual complex may be defined directly as a functor $I^{op} \to Sets$ as follows. Let $\tilde{D} \to \mathcal{X}$ denote the normalization of $D \subset \mathcal{X}$, and for $[\overline{p}] \in I$ write

$$\tilde{D}_p = (\tilde{D} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \tilde{D}) \setminus \{(z_0, \ldots, z_p) | z_i = z_j \text{ for some } i \neq j\}.$$

We have $\tilde{D}_0 = \tilde{D}$ and $\tilde{D}_{-1} = \mathcal{X}$.

Remark 1.3.1. In this situation, $\tilde{D}_p \to \mathcal{X}$ is a local complete intersection morphism whose conormal sheaf is a vector bundle of rank $(p+1)$ (see [Sta22, Tag 0CBR]). In particular $\tilde{D}_p$ is smooth over $\mathbb{C}$ of dimension $d-p$ if $\mathcal{X}$ is smooth over $\mathbb{C}$ of dimension $d+1$.

Definition 1.3.2. ([CGP21], Definition 5.2) Let $\mathcal{X}$ be a Deligne-Mumford Stack, let $D \subset \mathcal{X}$ be a normal crossings divisor. Consider, for all $p \geq -1$, the construction $\tilde{D}_p \to \mathcal{X}$. We define the symmetric Δ-complex $\Delta(D)$ by letting $\Delta(D)[\overline{p}]$ be the set of irreducible components (which coincide with the connected components) of $\tilde{D}_p$.

1.3.2 The skeleton of the analitification

A very nice relation between Moduli Spaces of tropical curves and Moduli Spaces of algebraic curves was shown in [ACP15] for the standard case, and then was generalized for Hassett Spaces in [Uli15]. In order to explain it, let us introduce some notation.

Let X be a separated algebraic space, and consider its analytification X^{an} (as defined in [CT07]). As explained in [ACP15], given a proper toroidal Deligne–Mumford Stack $\mathcal{X}$ with Coarse space X, one can construct its canonical skeleton $\overline{\mathfrak{S}}(\mathcal{X})$, which is an extended generalized cone complex (see [Thu07, ACP15]). The space $\overline{\mathfrak{S}}(\mathcal{X})$ is both a topological closed subspace of X^{an} and also the image of a canonical retraction $\mathbf{p}_{\mathcal{X}} : X^{an} \to \overline{\mathfrak{S}}(\mathcal{X})$. Let now $\overline{M}_{g,\mathcal{A}}$ be the Coarse Moduli Space of a given Hassett space, for parameters g, n and $\mathcal{A} \in \mathcal{D}_{g,n}$. There is a tropicalization map $trop_{g,\mathcal{A}} : \overline{M}^{an}_{g,\mathcal{A}} \to \overline{M}^{trop}_{g,\mathcal{A}}$ defined as follows: a point $x \in \overline{M}^{an}_{g,\mathcal{A}}$ can be viewed as a morphism $Spec(K) \to \overline{\mathcal{M}}_{g,\mathcal{A}}$ for a non-Archimedean field extension K of the base algebraically closed field k. By the valuative criterion for properness, since $\overline{\mathcal{M}}_{g,\mathcal{A}}$ is proper, this morphism extends uniquely to a morphism $Spec(R) \to \overline{\mathcal{M}}_{g,\mathcal{A}}$ where R denotes the valuation ring of K. This datum is equivalent to a curve $C \to Spec(R)$ that is stable of type $(g, \mathcal{A})$. We define its image by $trop_{g,\mathcal{A}}$ to be the tropical curve whose underlying graph is the dual graph of the special fiber, endowed with the edge length given by $l(e) = v(f_e)$, where v denotes the valuation on R and f_e comes from the equation $xy = f_e$ defining the node corresponding to the edge e. We are ready to state the main result. Originally this was proven for $\mathcal{A} = 1^{(n)}$ in [ACP15], Theorem 1.2.1, and then it was generalized for all the Hassett Spaces, so we write down it in the more general version.

Theorem 1.3.3. *([Uli15], Theorem 1.2) There is a natural isomorphism*

$$J_{g,\mathcal{A}} : \overline{M}^{trop}_{g,\mathcal{A}} \to \overline{\mathfrak{S}}(\overline{\mathcal{M}}_{g,\mathcal{A}})$$

of extended generalized cone complexes such that the diagram

$$\begin{tikzcd} & \overline{M}^{an}_{g,\mathcal{A}} \arrow[dl, "trop_{g,\mathcal{A}}"'] \arrow[dr, "\mathbf{p}_{\overline{\mathcal{M}}_{g,\mathcal{A}}}"] & \\ \overline{M}^{trop}_{g,\mathcal{A}} \arrow[rr, "J_{g,\mathcal{A}}"] & & \overline{\mathfrak{S}}(\overline{\mathcal{M}}_{g,\mathcal{A}}) \end{tikzcd}$$

commutes.

Let $\mathfrak{S}(\mathcal{X})$ denote the interior of the generalized cone complex $\overline{\mathfrak{S}}(\mathcal{X})$. A corollary of the latter Theorem is the following.

Corollary 1.3.4. *([Uli15], Corollary 1.4) The isomorphism $J_{g,\mathcal{A}}$ restrict to an isomorphism of generalized cone complexes*

$$J_{g,\mathcal{A}} : M_{g,\mathcal{A}}^{trop} \to \mathfrak{S}(\overline{\mathcal{M}}_{g,\mathcal{A}})$$

such that the diagram

$$\begin{tikzcd} & M_{g,\mathcal{A}}^{an} \arrow[dl, "trop_{g,\mathcal{A}}"'] \arrow[dr, "\mathbf{p}_{\mathcal{M}_{g,\mathcal{A}}}"] & \\ M_{g,\mathcal{A}}^{trop} \arrow[rr, "J_{g,\mathcal{A}}"] & & \mathfrak{S}(\overline{\mathcal{M}}_{g,\mathcal{A}}) \end{tikzcd}$$

commutes.

In words, this Theorem says that Tropical Hassett Moduli Spaces have a nice interpretation as Skeletons of the analytification of the corresponding Algebraic Hassett Spaces. We do not develop anything new about this specific topic, this material is included for the reader to appreciate the appearance of tropical Hassett spaces as skeletons of analytifications.

1.3.3 Stratification of the boundary

One of the most important properties of the Deligne-Mumford compactification of the Moduli Space of smooth curves is the following:

Theorem 1.3.5. *([Knu83], Theorem 2.7) The boundary divisor $\partial\overline{\mathcal{M}}_{g,n} := \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ has (Stack-theoretically) normal crossings.*

Hassett Spaces are alternative compactifications of the Stack of smooth curves $\mathcal{M}_{g,n}$, so one could ask if the same property holds. Unfortunately, the embedding $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,\mathcal{A}}$ does not verify the property of Theorem 1.3.5. Anyway, one can consider the locus $\mathcal{M}_{g,\mathcal{A}}$ of smooth $\mathcal{A}$-stable curves and show the following, which generalizes Theorem 1.3.5:

Theorem 1.3.6. *([Uli15], Theorem 1.1) The boundary divisor $\partial\overline{\mathcal{M}}_{g,\mathcal{A}} := \overline{\mathcal{M}}_{g,\mathcal{A}} \setminus \mathcal{M}_{g,\mathcal{A}}$ has (Stack-theoretically) normal crossings.*

In the situation above, i.e., whenever we have a normal crossing divisor in a variety or a Deligne-Mumford Stack, we aim to find a structure in the divisor called stratification. The notion of stratification for Stacks comes from its counterpart in topology, which is the following: for a topological space X, a stratification is a decomposition of X into pairwise disjoint locally closed subsets such that if X_j meets the closure of X_i, then X_j is contained in the closure of X_i. We define it for Stacks only in our context:

Definition 1.3.7. Let $\partial\overline{\mathcal{X}} := \overline{\mathcal{X}} \setminus \mathcal{X}$ be a normal crossings divisor of a Deligne-Mumford Stack $\overline{\mathcal{X}}$ of dimension d. The strata of $\partial\overline{\mathcal{X}}$ may be defined inductively as follows. First, the $(d-1)$-dimensional strata of $\partial\overline{\mathcal{X}}$ are the irreducible components of $\partial\overline{\mathcal{X}}$. For each $i < d-1$, the i-dimensional strata are the irreducible components of the regular locus of the complement in $\partial\overline{\mathcal{X}}$ of the union of all strata of D of dimension greater than i.

The divisor $\partial\overline{\mathcal{M}}_{g,\mathcal{A}}$ has a natural stratification indexed by the so called dual graph or combinatorial type of the curve, i.e., each stratum is exactly the locus of curves with a given dual graph, in a sense that will be clear in a moment. We recall the construction of the dual graph for a given nodal curve with n-marked points, m irreducible components and r nodes $(C, p_1, \ldots, p_n)$. Given such a curve, we define the graph G_C as follows:

- For every irreducible component E_i of C, we associate a vertex v_i, for every $i = 1, \ldots, m$;
- Given a node of C, we have two possibilities:
    1) If the node lies on two different components, we add an edge e_j connecting the vertices that correspond to the two components where the node lies;
    2) If the node lies on a single component, we add a loop on the correspondent vertex;
- For every marked point p_i lying on a component we add a leg x_i to the correspondent vertex;
- The weight of every vertex is the geometric genus of the corresponding component in the curve.

Example 1.3.8. In Figure 1.24 we construct the dual graph of a curve over $\mathbb{C}$, which is a Riemann Surface. This makes it easier to understand the genus of the various components, as it coincides with their topological genus as surfaces. The black dots are the nodes of the curve: notice that there is a self-node on the component on the right, resulting in a loop on the dual graph, while there are multiple nodes between the two components in the center of the picture, resulting in multiple edges on the dual graph. In red we depict the marked points: there are three of them, which give raise to three legs on the graph labeled according to the label of the corresponding point.

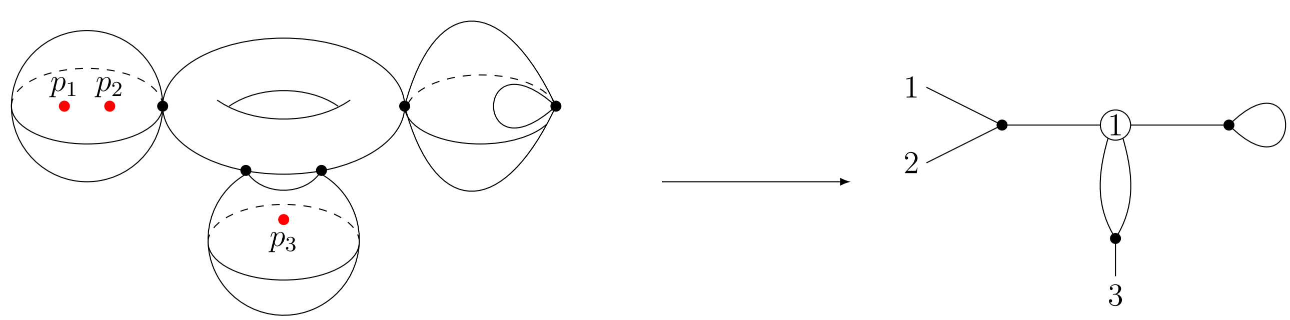

Figure 1.24: Construction of the dual graph of a curve

It is easy to show that a curve C is stable if and only if its dual graph G_C is stable (see [Cap11] Remark 4.2.), and that the genus of the curve corresponds to the genus of its dual graph. More generally, we have the following proposition.

Proposition 1.3.9. *([Uli15], Proposition 3.3) Let $(g, \mathcal{A})$ be an input datum. For curve $(C, p_1, \ldots, p_n)$ of genus g with n marked points the following properties are equivalent:*

i) The twisted canonical divisor $K_C + a_1p_1 + \cdots + a_np_n$ on C is ample;

ii) The dual graph G_C of C is stable of type $\mathcal{A}$.

The stratification by dual graphs was originally introduced in [Arb74] for $\partial\overline{\mathcal{M}}_g$, its generalization for $\partial\overline{\mathcal{M}}_{g,n}$ appears in [ACGH13], and lastly was done for $\partial\overline{\mathcal{M}}_{g,\mathcal{A}}$ in [Uli15].

Let G be a marked weighted graph. We can define $\mathcal{M}_G^{\mathcal{A}} \subset \overline{\mathcal{M}}_{g,\mathcal{A}}$ to be the locus of curves with G as dual graph. When $\mathcal{A} = 1^{(n)}$, usually the superscript is avoided, i.e., $\mathcal{M}_G^{1^{(n)}} =: \mathcal{M}_G$. We have the following facts:

1) Let $\bullet_{g,n}$ be the graph made by a single vertex of genus g with n legs attached to it and no edges. Then $\mathcal{M}_{g,\mathcal{A}} = \mathcal{M}^{\mathcal{A}}_{\bullet_{g,n}}$;

2) The locus $\mathcal{M}_G^{\mathcal{A}}$ has codimension equal to the number of edges of G;

3) The closure $\overline{\mathcal{M}}_G^{\mathcal{A}}$ of $\mathcal{M}_G^{\mathcal{A}}$ in $\overline{\mathcal{M}}_{g,\mathcal{A}}$ is given by

$$\overline{\mathcal{M}}_G^{\mathcal{A}} = \bigsqcup_{G \leq G'} \mathcal{M}_{G'}^{\mathcal{A}},$$

where the graphs are taken in the poset $\mathcal{G}_{g,\mathcal{A}}$.

This indeed defines a stratification of $\partial\overline{\mathcal{M}}_{g,\mathcal{A}}$, where the strata are exactly the $\overline{\mathcal{M}}_G^{\mathcal{A}}$'s. This stratification can be used to do some cohomological computations, as we will mention in Section 1.3.4.

1.3.4 Cohomology of Moduli Spaces of Curves

In the last few years, there has been a lot of effort in understanding the cohomology of Moduli Spaces of Curves. First results about the cohomology of standard Moduli Spaces of curves can be found in [AC98], where they show that $H^1(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, $H^3(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ and $H^5(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ are zero for all the meaningful values of g and n, while $H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ is generated by tautological classes, recovering some results from [Har83] and [BP00]. We introduce briefly these results.

First, denote by δ_G the orbifold fundamental class of $\mathcal{M}_G$, that is, the fundamental class of $\mathcal{M}_G$ divided by the order of the automorphism group of a general element of this locus.

Degree two classes correspond to graphs with one edge, and there are two kinds: one is the graph G_{irr}, with one vertex of weight $g-1$, a loop and all the legs attached to it, and the other kind are the graphs $G_{a,A}$, which have two vertices, one of genus a, with attached the legs indexed by $A \subset [n]$, and one of genus $g-a$ with attached the legs indexed by $[n] \setminus A$ (see Figure 1.25). We write δ_{irr} and $\delta_{a,A}$ for the corresponding classes. It is clear that $\delta_{a,A} = \delta_{g-a,A^c}$, and also that $\delta_{a,A}$ is defined only if $2a-2+|A| \geq 0$ and $2(g-a)-2+|A^c| \geq 0$. The class δ_{irr} is set to zero when the genus is zero. Those classes are also called boundary classes.

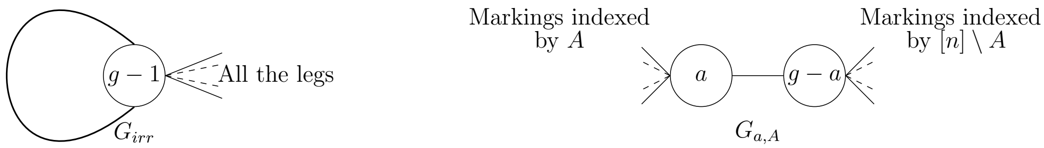

Figure 1.25: The two kind of graphs which induce Tautological Classes.

The other two kinds of tautological classes are the ψ-classes and the κ-classes. Let ω_π be the relative dualizing sheaf of the Universal Curve π, then for each i from 1 to n we have the section σ_i which attaches to any curve $(C, p_1, \ldots, p_n)$ the $n+1$-pointed curve obtained by attaching to C a copy of $\mathbb{P}^1$ by identifying p_i and $0 \in \mathbb{P}^1$, and labelling the points 1 and ∞ by i and $n+1$ Let also D_i be the image of σ_i. Then $\psi_i := \sigma_i^*(c_1(\omega_\pi))$ is a class of degree 2 and $\kappa_i := \pi_*(c_1(\omega_\pi(\sum D_i))^{i+1})$ has degree $2i$.

Theorem 1.3.10. *([AC98], Theorem 2.2)*

i) Let $g \geq 3$, then the tautological classes form a basis for $H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

ii) Let $g = 2$, the tautological classes satisfy the following relation:

$$5\kappa_1 = 5\sum_{i=1}^{n} \psi_i + \delta_{irr} - 5\sum_{A} \delta_{0,A} + 7\sum_{A} \delta_{1,A}.$$

iii) Let $g = 1$. There are relations

$$\kappa_1 = 5\sum_{i=1}^{n} \psi_i - \sum_{|A|\geq 2} \delta_{0,A}$$

and

$$12\psi_i = \delta_{irr} + 12\sum_{A \ni i} \delta_{0,A}.$$

iv) Let $g = 0$. The relations are generated by

$$\kappa_1 = \sum_{A \ni x,y} (|A|-1)\delta_{0,A}, \quad \psi_i = \sum_{A \ni x,y} \delta_{0,A}, \quad \delta_{irr} = 0,$$

where i,x and y run over elements of $[n]$.

Rational Hassett spaces, i.e., spaces where $g = 0$, have been used in [BM13] to develop a recursive algorithm for computing the character of the cohomology of the Moduli Space $\overline{\mathcal{M}}_{0,n}$. The same authors determined the cohomology of Heavy/Light spaces $\overline{\mathcal{M}}_{0,(1^{(2)},\varepsilon^{(n)})}$ in [BM14]. Tautological classes on Hassett spaces have been defined and studied in analogous ways, for example, in [Jan13]. A formula for the intersection of ψ-classes on $\overline{\mathcal{M}}_{0,\frac{1}{2}^{(n)}}$ is given in [Sha18]. Lastly, [BC20] solves the combinatorics relating the intersection theory of ψ-classes of Hassett spaces to that of $\overline{\mathcal{M}}_{g,n}$.

Top Weight Cohomology and Dual Boundary Complex

Let $\mathcal{X}$ be a Deligne-Mumford Stack (or a complex algebraic variety) of pure dimension d. As such, by the work of Deligne [Del74] we know that the rational cohomology of $\mathcal{X}$ carries a mixed Hodge structure, and in particular there is a weight filtration

$$W_1 \subset \cdots \subset W_{2d} = H^*(\mathcal{X};\mathbb{Q})$$

such that, for each j, the quotient

$$\mathrm{Gr}_i^W H^j(\mathcal{X},\mathbb{Q}) := W_i \cap H^j(\mathcal{X};\mathbb{Q})/W_{i-1} \cap H^j(\mathcal{X};\mathbb{Q})$$

carries a pure Hodge structure of weight i. Assume now there is a Deligne-Mumford Stack $\overline{\mathcal{X}}$ such that $\mathcal{X} \subset \overline{\mathcal{X}}$ and $\partial\overline{\mathcal{X}} := \overline{\mathcal{X}} \setminus \mathcal{X}$ is normal crossings. We already saw that associated to this structure there is a stratification. By [CGP21], we can associate to such a stratification a (geometric realization of a) symmetric Δ-complex $\Delta(\partial\overline{\mathcal{X}})$ (see Section 1.2.3 for the Definition) called Dual Boundary Complex, with the property that its reduced homology can be identified with the top graded piece of the weight filtration ($i = 2d$), also called Top Weight Cohomology, up to a degree shift, i.e.,

$$\widetilde{H}_{j-1}(\Delta(\partial\overline{\mathcal{X}});\mathbb{Q}) \cong \mathrm{Gr}_{2d}^W H^{2d-j}(\mathcal{X};\mathbb{Q}).$$

We already said that the above situation holds for the embedding $\mathcal{M}_{g,\mathcal{A}} \subset \overline{\mathcal{M}}_{g,\mathcal{A}}$. The fascinating property of the Dual Boundary Complex in this case, shown in [CGP21] and [CGP22] when $\mathcal{A} = 1^{(n)}$, and in [Uli15, CHMR14] for generic weight data is that $\Delta(\partial\overline{\mathcal{M}}_{g,\mathcal{A}})$ is identified with the Tropical Moduli Space $\Delta_{g,\mathcal{A}}$, giving isomorphisms

$$\widetilde{H}_{j-1}(\Delta_{g,\mathcal{A}};\mathbb{Q}) \cong \mathrm{Gr}_{6g-6+2n}^W H^{6g-6+2n-j}(\mathcal{M}_{g,\mathcal{A}};\mathbb{Q}).$$

Later in Section 4.3, we will come back on this topic by identifying the reduced rational homology of $\Delta_{g,\mathcal{A}}$ with the homology of some Graph Complexes, generalizing Theorem 1.2 of [CGP21] and Theorem 1.4 of [CGP22], and allowing for some computations of the Top Weight Cohomology of Hassett Spaces.

Chapter 2

Topology of the link

In this Chapter we describe results about the topology of the spaces $\Delta_{g,\mathcal{A}}$. Most of the material of this Chapter is published in [KLSY22], which is a joint work with Siddarth Kannan, Shiyue Li and Claudia He Yun.

2.1 Simply Connectedness of the link

The main goal of this section is to show the simply connectedness of $\Delta_{g,\mathcal{A}}$, for every $g \geq 1$, $n \geq 2$ and $\mathcal{A} \in \mathcal{D}_{g,n}$. All of our statements start from genus 1 on, as the situation for $g = 0$ is different and almost completely known. In fact, when $g = 0$, the complex $\Delta_{0,\mathcal{A}}$ may be identified with various objects whose homotopy types are known. We will start by giving a brief overview of these results.

When $\mathcal{A} = 1^{(n)}$, Vogtmann, and independently Robinson and Whitehouse shown that $\Delta_{0,n}$ is homotopic to a wedge of $(n-2)!$ spheres of dimension $n-4$, see [CV86, Vog90]. When $\mathcal{A} = (1^{(m)}|\varepsilon^{(n)})$ is Heavy/Light, Cavalieri, Hampe, Markwig, and Ranganathan in [CHMR14] first, and Cerbu, Marcus, Peilen, Ranganathan, and Salmon in [CMP+20] shown that $\Delta_{0,(1^{(m)}|\varepsilon^{(n)})}$ is homotopic to a wedge of $(m-2)!(m-1)^n$ spheres of dimension $n+m-4$. When $\mathcal{A}$ has at least two weight-1 entries, Cerbu et al. in [CMP+20] shown also that $\Delta_{0,\mathcal{A}}$ is homotopic to a wedge of spheres of possibly varying dimensions. The authors also provided infinite families of $\mathcal{A}$ where $\Delta_{0,\mathcal{A}}$ is disconnected, and examples where $\pi_1(\Delta_{0,\mathcal{A}}) = \mathbb{Z}/2\mathbb{Z}$.

For higher values of g, the following results are known. When $\mathcal{A} = 1^{(n)}$, Chan, Galatius, and Payne shown in [CGP22] that $\Delta_{1,n}$ is homotopic to $\frac{1}{2}(n-1)!$ spheres of dimension $n-1$, and that $\Delta_{g,n}$ is at least $(n-3)$-connected. In [Cha21], Chan independently shown that the reduced integral homology $\widetilde{H}_*(\Delta_{2,n};\mathbb{Z})$ is supported in the top two degrees and that a subcomplex of $\Delta_{2,n}$ has torsion in high degrees. Chan also computed the reduced rational homology $\widetilde{H}_*(\Delta_{2,n};\mathbb{Q})$ for $n \leq 8$. When $\mathcal{A}$ has at least two weight-1 entries, in [CMP+20] the authors proved also that $\Delta_{1,\mathcal{A}}$ is homotopic to a wedge of spheres. Moreover, when $\mathcal{A} = (1^{(m)}|\varepsilon^{(n)})$ is Heavy/Light, the same authors shown that $\Delta_{1,\mathcal{A}}$ is homotopic to $\frac{1}{2}(m-1)!m^n$ spheres of dimension $n+m-1$.

We will now present technical results generalizing [CGP22, Theorem 1.1] of Chan, Galatius and Payne about contractible subsets of $\Delta_{g,\mathcal{A}}$.

Theorem 2.1.1. *Let $g \geq 1$, $n \geq 1$ be integers, and let $\mathcal{A} \in \mathcal{D}_{g,n}$ be a weight datum. Then the following subcomplexes are either empty or contractible:*

1. *The subset $\Delta^{w}_{g,\mathcal{A}}$ of tropical curves with at least a strictly positive weighted vertex;*

2. *The subset $\Delta^{lw}_{g,\mathcal{A}}$ of tropical curves with at least a strictly positive weighted vertex and/or loops;*

3. *The closure of the subset of tropical curves with bridges $\Delta^{br}_{g,\mathcal{A}}$.*

The proof of this result is quite technical and is a mild adaptation of the original proof contained in [CGP22] in the case $\mathcal{A} = 1^{(n)}$, we put it in Appedix A.

In order to prove the simply connectedness of the $\Delta_{g,\mathcal{A}}$'s, we need the contractibility of $\Delta^{lw}_{g,n}$, i.e. the second point of the Theorem 2.1.1, but we will include a proof of that in Proposition 2.1.5. The original Theorem from [CGP22] addressed also the question of contractibility for another sub-locus, namely the locus of curves with repeated markings, i.e., the locus of tropical curves admitting at least a vertex v with $|m^{-1}(v)| \geq 2$. The analogous locus for Tropical Hassett Spaces to this one is called locus of curves with heavy markings, and it is defined as follows.

Definition 2.1.2. Let $g \geq 0$, $n \geq 1$ and $\mathcal{A} \in \mathcal{D}_{g,n}$. The heavy marking locus of $\Delta_{g,\mathcal{A}}$, denoted $\Delta^{r}_{g,\mathcal{A}}$, is the subspace of $\mathcal{A}$-stable tropical curves for which there is a vertex v such that $|v|_{\mathcal{A}} > 1$.

Remark 2.1.3. Here we have to clarify a conflict of terminology. The term "heavy" here is not referred to heavy weights in the sense of the Notation introduced by Equation 1.1.2, even if both the words refer to the marking weights. Anyways, the notion used here will not return later, so since no confusion arises we do not change it to be consistent with the literature.

The contractibility of this locus was addressed in [CMP+20], and we refer to the original work for the proof.

Lemma 2.1.4. *([CMP+20], Lemma 8.2.) Let $g \geq 0$, $n \geq 1$ and $\mathcal{A} \in \mathcal{D}_{g,n}$ such that $\sum_{i=1}^{n} a_i > 1$, then $\Delta^{r}_{g,\mathcal{A}}$ is a contractible subset of $\Delta_{g,\mathcal{A}}$, otherwise it is empty.*

The following proposition shows that there is one more contractible sub-complex in $\Delta_{g,\mathcal{A}}$, and it is the main tool to show its simply connectedness. The proof is parallel to the one of Theorem 6.1 of [ACP22].

Proposition 2.1.5. *Let $g, n \geq 1$ and $\mathcal{A} \in \mathcal{D}_{g,n}$. The subcomplex $\Delta^{mlw}_{g,\mathcal{A}}$ of $\Delta_{g,\mathcal{A}}$ parametrizing tropical curves with loops, vertices of positive weight, or multiple edges is contractible.*

Proof. First we consider the locus $\Delta^{lw}_{g,\mathcal{A}}$ of tropical curves with at least a strictly positive weighted vertex and/or loops. This is a subcomplex, since it is closed under contractions of edges. Observe that if $g = 1$ and $\sum_{i=1}^{n} a_i \leq 1$ the locus $\Delta^{lw}_{g,\mathcal{A}}$ is a point representing the loop graph of length 1, which is trivially contractible, so we can assume $g \geq 2$ or $g = 1$ and $\sum_{i=1}^{n} a_i > 1$. In such cases, let $B(1, \varnothing)$ be the graph made by a single edge with a vertex of weight 1 and no legs, and the other vertex of weight $g-1$ and all the legs attached to it, and notice that there is a vertex in $\Delta^{lw}_{g,\mathcal{A}}$ with underlying graph $B(1,\varnothing)$. We show that $\Delta^{lw}_{g,\mathcal{A}}$ is contractible using the fact that we can see the whole space $\Delta_{g,\mathcal{A}}$ as a symmetric Δ-complex, see Example 1.2.34. Let $P=\varnothing$ and $Q=\{B(1,\varnothing)\}$ be two Properties in sense of Definition 1.2.27, i.e. they are sets of vertices of $\Delta_{g,\mathcal{A}}$. Recall that given a Property P, and a symmetric Δ-complex X, we write $P(X)$ for the set of P-simplices, i.e., simplices with vertices in P, and $P^*(X)$ for all the simplices which are faces of a P-simplex. Recall also that we denote by X_P the subcomplex of X generated by $P^*(X)$, and by $X_{P,i}$ the image of the natural map

$$\bigsqcup_{p\geq 0}(X_P\cap V_{P,i})\times\Delta^p\to|X|$$

where $V_{P,i}$ is the set of P-simplices with at most i non P-vertices. We want to apply Proposition 1.2.31 to these two Properties, and to do so we need to show that the set $P^*(\Delta_{g,\mathcal{A}})$ is co-Q-saturated, and the set of symmetric orbits of $\Delta_{g,\mathcal{A}}\setminus P^*(\Delta_{g,\mathcal{A}})$ admits canonical co-Q maximal faces. Since P is empty, $P^*(\Delta_{g,\mathcal{A}})$ is also empty so the first condition is automatically satisfied, and the second amounts to show that every graph has a canonical maximal uncontraction by 1-ends, i.e., by edges with an endpoint of valence 1, weight 1 and no legs, or an endpoint of valence three with a loop, weight zero and no legs. We call them 1-end of type one and 1-end of type two, respectively. Let $G\in Ob(\mathcal{G}_{g,\mathcal{A}})$, the uncontraction is defined as follows: if G has no loops or weights, the expansion is trivial. Otherwise, let v be a vertex with $val(v)+2w(v)>3$, for any loop based at v, we remove it and we add a 1-end of type two, and if its weight $w(v)$ is strictly greater than zero, we add $w(v)$ 1-ends of type one, and then we set $w(v)$ to 0. Then by Proposition 1.2.31 we have a strong deformation retract $(\Delta_{g,\mathcal{A}})_P\sqcup(\Delta_{g,\mathcal{A}})_{Q,i}\searrow(\Delta_{g,\mathcal{A}})_P\sqcup(\Delta_{g,\mathcal{A}})_{Q,i-1}$ for every $i>0$. Now since $P=\varnothing$, the locus $(\Delta_{g,\mathcal{A}})_P$ is empty. When i becomes greater than the number of 1-edged $(g,\mathcal{A})-$stable graphs, $(\Delta_{g,\mathcal{A}})_{Q,i}$ locus stabilizes to the locus generated by $Q-$simplices $(\Delta_{g,\mathcal{A}})_Q$. So iterating for every i we get a strong deformation retract $(\Delta_{g,\mathcal{A}})_Q\searrow(\Delta_{g,\mathcal{A}})_{Q,0}$, where $(\Delta_{g,\mathcal{A}})_{Q,0}$ is the simplex associated to the graph with one vertex of weight zero and all the legs attached with g 1-ends of type one, quotient by the automorphism group of this graph, which is contractible. To show the contractibility of $\Delta^{lw}_{g,\mathcal{A}}$ we are left to show that it coincides with $(\Delta_{g,\mathcal{A}})_Q$. The inclusion $(\Delta_{g,\mathcal{A}})_Q\subset\Delta^{lw}_{g,\mathcal{A}}$ follows by observing that for a graph G having $B(1,\varnothing)$ as contraction implies that there is at least a loop or a vertex of weight 1. For the opposite inclusion, if G has a loop or a vertex of weight 1, it admits a (eventually trivial) uncontraction by 1-ends, so curves with underlying graph G belongs to $(\Delta_{g,\mathcal{A}})_Q$. Now consider the locus $\Delta^{mlw}_{g,\mathcal{A}}$. Note that it is a subcomplex, since it is closed under contraction of edges. The subcomplex $\Delta^{mlw}_{g,\mathcal{A}}$ is obtained from $\Delta^{lw}_{g,\mathcal{A}}$ as an iterated mapping cone, so $\Delta^{mlw}_{g,\mathcal{A}}$ is homotopy equivalent to $\Delta^{lw}_{g,\mathcal{A}}$, which is contractible by what we shown above. □

The main Theorem of the section easily follows from the previous Proposition.

Theorem 2.1.6. *Fix $g\geq 1$,$n\geq 1$ and $\mathcal{A}\in\mathcal{D}_{g,n}$. Then $\Delta_{g,\mathcal{A}}$ is simply connected.*

Proof. Let $g,n\geq 1$ and $\mathcal{A}\in(\mathbb{Q}\cap(0,1])^n$. Let the 1-skeleton of a symmetric Δ-complex be its locus of one dimensional simplices. Now observe that the 1-skeleton of $\Delta_{g,\mathcal{A}}$ is contained in the contractible subcomplex $\Delta^{mlw}_{g,\mathcal{A}}$, and since $\pi_1(\Delta_{g,\mathcal{A}})$ is generated by cycles in the 1-skeleton (as shown in Theorem 3.1 [ACP22]), we conclude that $\pi_1(\Delta_{g,\mathcal{A}})$ is trivial. □

In Appendix A, we give an alternative proof of this Theorem based on the original result of [ACP22], Theorem 1.4, which is entirely combinatorial.

Quasistable graphs

In some context, it is useful to relax the stability condition on the graphs: for example, when one treats the tropical analogue of the Jacobian of Curves as in [AP20, AAPT22, MMUV21].

Definition 2.1.7. Let $g, n \in \mathbb{Z}_{\geq 0}$ such that $2g-2+n > 0$. Let G be an n-marked weighted genus g graph. We say G is a quasistable graph if for every vertex $v \in V(G)$, we have $2w(v) - 2 + val(v) + |m^{-1}(v)| \geq 0$, and any two vertices u and v such that this condition is an equality, i.e., $val(u) = 2$, $w(u) = 0$ and $m^{-1}(u) = \varnothing$, and analogously $val(v) = 2$, $w(v) = 0$ and $m^{-1}(v) = \varnothing$, then u and v are not adjacent.

We call those vertices such that $2w(v) - 2 + val(v) + |m^{-1}(v)| = 0$ exceptional vertices, and we denote their set with $V_{exc}(G)$. Notice that all the stable graphs are also quasistable: in particular, they do not have any exceptional vertex. A special class of quasistable graphs come often in play in various contexts:

Definition 2.1.8. A quasistable graph is said to be simple if the graph $G \setminus V_{exc}(G)$ obtained by removing all the exceptional vertices and their adjacent edges is connected.

Again, notice that all the stable graphs are also simple. We can define, for fixed g and n such that $2g-2+n > 0$, graph categories by considering isomorphism classes of quasistable (and respectively simple) graphs as objects, with morphisms being contractions and automorphisms, denoting them $\mathcal{G}^{qs}_{g,n}$ (and respectively, $\mathcal{G}^{spl}_{g,n}$). The considerations made so far then lead to the following inclusions of categories:

$$\mathcal{G}_{g,n} \subset \mathcal{G}^{spl}_{g,n} \subset \mathcal{G}^{qs}_{g,n}.$$

Analogously to the category of stable graphs $\mathcal{G}_{g,n}$, these categories can be endowed with the structure of poset considering the order induced by edge contractions.

We can consider also quasistable and simple tropical curves by picking quasistable and simple underlying graphs. So analogously to what was done in Section 1.2.2, we can construct moduli spaces of quasistable and simple tropical curves $(M^{trop}_{g,n})^{qs}$ and $(M^{trop}_{g,n})^{spl}$, moduli spaces of extended quasistable and simple tropical curves$(\overline{M}^{trop}_{g,n})^{qs}$ and $(\overline{M}^{trop}_{g,n})^{spl}$, and moduli spaces of quasistable and simple tropical curves of volume one $\Delta^{qs}_{g,n}$ and $\Delta^{spl}_{g,n}$. With the same reasoning adopted for the inclusions as subcategories, we have the following inclusions as subspaces:

$$M^{trop}_{g,n} \subset (M^{trop}_{g,n})^{spl} \subset (M^{trop}_{g,n})^{qs},$$

$$\overline{M}^{trop}_{g,n} \subset (\overline{M}^{trop}_{g,n})^{spl} \subset (\overline{M}^{trop}_{g,n})^{qs},$$

$$\Delta_{g,n} \subset \Delta^{spl}_{g,n} \subset \Delta^{qs}_{g,n}.$$

These spaces share many properties with the ones of stable tropical curves. In particular, we can show the following.

Theorem 2.1.9. *Let $g, n \geq 1$ be integers such that $2g-2+n > 0$. The spaces $\Delta^{spl}_{g,n}$ and $\Delta g, n^{qs}$ are simply connected.*

Proof. The proof is analogous to the one of Theorem 2.1.6, just by observing that in both cases $\Delta^{mlw}_{g,n}$ is still a contractible subcomplex contsining the 1-skeleton which is the same as the one of $\Delta_{g,n}$. □

2.2 Euler characteristic of $\Delta_{g,\mathcal{A}}$ and Top Weight Euler Characteristic of $\mathcal{M}_{g,\mathcal{A}}$

In this section we exhibit a formula for the Euler characteristic of $\Delta_{g,\mathcal{A}}$ in terms of the one of the Top Weight Euler characteristic of spaces $\mathcal{M}_{g,r}$'s, for $r \leq n$, n being the length of $\mathcal{A}$, by its connection with the Top Weight Euler characteristic of $\mathcal{M}_{g,\mathcal{A}}$. We can attain this formula using the fact that the Top Weight Euler characteristic of $\mathcal{M}_{g,\mathcal{A}}$ is the same as the one of its Coarse Moduli Space $M_{g,\mathcal{A}}$ for every $\mathcal{A}$, and then showing a decomposition for $M_{g,\mathcal{A}}$ in the Grothendieck group of varieties.

Definition 2.2.1. Let k be a field. We denote by $K_0(\mathrm{Var}/k)$ the Grothendieck group of varieties over k, i.e., the quotient of the free abelian group on k-varieties by relations of the form

$$[X] = [X \setminus Y] + [Y],$$

whenever Y is a closed subvariety of X

Such relation are called the cut-and-paste relations, and the additive identity is the empty variety $[\varnothing]$.

Definition 2.2.2. An Euler-Poincaré characteristic of $K_0(\mathrm{Var}/k)$ is a group homomorphism $\chi : K_0(\mathrm{Var}/k) \to A$ to an abelian group A.

Notice that whenever we have an Euler-Poincaré characteristic, for any closed subvariety Y of X,

$$\chi([X]) = \chi([Y]) + \chi([X \setminus Y]),$$

see [Cra04, Loe09] for more details. For what follows, we need to introduce the notion of cohomology with compact support (or compactly supported cohomology). This Definition can be given in various ways and in different contexts. In [Ive86], Section 3, it is defined as a left exact functor, and many properties about this notion are shown. Here we adopt the following Definition.

Definition 2.2.3. Let X be a topological space, and let R be a ring. We define the cohomology with compact support (or compactly supported cohomology) as

$$H^*_c(X; R) := \varinjlim_{\mathrm{K} \subset \mathrm{X} \text{ compact}} H^*(X, X \setminus K; R).$$

Example 2.2.4. Set $k = \mathbb{C}$. Let

$$\chi_c^m(X) := \sum_{j=0}^{2d} (-1)^j \dim \mathrm{Gr}_m^W H_c^j(X;\mathbb{Q}).$$

The virtual Poincaré polynomial is the group homomorphism $K_0(\mathrm{Var}/\mathbb{C}) \to \mathbb{Z}[t]$ defined by the formula $P_X(t) = \sum_{m=0}^{2d}(-1)^m \chi_c^m(X) t^m$, where $d = \dim X$ and is an Euler-Poincaré characteristic.

2.2.1 The stratification of $M_{g,\mathcal{A}}$ through coincident points

Let $g \geq 0$, $n \geq 1$ and $\mathcal{A} \in \mathcal{D}_{g,n}$, and consider the Coarse Moduli Space $M_{g,\mathcal{A}}$ of $\mathcal{M}_{g,\mathcal{A}}$. We describe a stratification of $M_{g,\mathcal{A}}$ in terms of the loci were some marked points coincide, and we will use it inside the group $K_0(Var/k)$. Denote by $P \vdash S$ a partition P of a set S. We say that a set partition of $[n]$

$$\mathcal{P} = P_1 \sqcup \cdots \sqcup P_r \vdash [n]$$

with r parts is $\mathcal{A}$-admissible if $\sum_{i \in P_j} a_i \leq 1$ for all $1 \leq j \leq r$. Given such a partition, we write $\mathcal{P} \vdash_{\mathcal{A}} [n]$. We set $N_{r,\mathcal{A}}$ to be the number of $\mathcal{A}$-admissible partitions of $[n]$ with r parts.

Proposition 2.2.5. *In $K_0(\mathrm{Var}/k)$,*

$$[M_{g,\mathcal{A}}] = \sum_{r=1}^{n} N_{r,\mathcal{A}}[M_{g,r}].$$

Proof. The locus $M_{g,\mathcal{A}}$ parameterizes smooth curves of genus g with n markings, such that whenever $\sum_{i \in S} a_i \leq 1$ for some $S \subseteq [n]$, the markings indexed by S are allowed to coincide. Given a $\mathcal{A}$-admissible partition $\mathcal{P} = P_1 \sqcup \cdots \sqcup P_r \vdash_{\mathcal{A}} [n]$, we define

$$Z_{\mathcal{P}} := \{(C, p_1, \ldots, p_n) \in M_{g,\mathcal{A}} \mid p_i = p_j \text{ if and only if } i, j \in P_s \text{ for some } s \in [r]\}.$$

Then we see that $Z_{\mathcal{P}} \cong M_{g,r}$, since it is exactly the space parametrizing curves with r distinct marked points. As $\mathcal{P}$ ranges over all $\mathcal{A}$-admissible partitions of $[n]$, the loci $Z_{\mathcal{P}}$'s are locally closed subvarieties of $M_{g,\mathcal{A}}$, so by Proposition 7.1 of [Mus], we have

$$[M_{g,\mathcal{A}}] = \sum_{\mathcal{P} \vdash_{\mathcal{A}} [n]} [Z_{\mathcal{P}}] = \sum_{r=1}^{n} \sum_{\substack{\mathcal{P} \vdash_{\mathcal{A}} [n] \\ |\mathcal{P}| = r}} [Z_{\mathcal{P}}] = \sum_{r=1}^{n} N_{r,\mathcal{A}}[M_{g,r}],$$

as claimed. □

Remark 2.2.6. The decomposition of $M_{g,\mathcal{A}}$ into the $Z_{\mathcal{P}}$'s is indeed a stratification. In fact, recall that for a topological space X, a stratification is a decomposition of X into pairwise disjoint locally closed subsets such that if X_j meets the closure of X_i, then X_j is contained in the closure of X_i. Now consider $Z_{\mathcal{P}}$: it is the locus of curves where some marked points coincide according to $\mathcal{P}$. In the the closure of $Z_{\mathcal{P}}$ we can find curves where the marked points which already coincide still agree, but eventually also some P_i's of $\mathcal{P}$ come together. So in we have $Z_{\mathcal{P}'}$ intersecting the closure of $Z_{\mathcal{P}}$, it must be that $\mathcal{P}'$ is a partition which can be refined to be $\mathcal{P}$, hence $Z_{\mathcal{P}'} \subset \overline{Z_{\mathcal{P}}}$.

The numbers $N_{r,\mathcal{A}}$ in general are not known, but there are some favorable cases were they can be computed in a combinatorial way. Let $S(n,r)$ be the Stirling number of the second kind, i.e., the number of r-partitions of a n-set.

Corollary 2.2.7. *Let $g \geq 1$, and $\mathcal{A} = (1^{(m)}|\varepsilon^{(n)})$ be an admissible Heavy/Light weight datum, then $[M_{g,\mathcal{A}}] = \sum_{r=1}^{n} S(n,r)[M_{g,m+r}]$.*

Proof. From Proposition 2.2.5, we know that $[M_{g,\mathcal{A}}] = \sum_{r=1}^{m+n} N_{r,\mathcal{A}}[M_{g,r}]$. Since $\mathcal{A} = (1^{(m)}|\varepsilon^{(n)})$, the number $N_{r,\mathcal{A}}$ is 0 for $r \leq m$, because each part of the partition must contain a weight 1, so the sum can start from $m+1$. For $r \geq m+1$, the number $N_{r,\mathcal{A}}$ is equal to $S(n,r)$. Indeed, $N_{r,\mathcal{A}}$ is the number of partitions of $[n+m]$ with r parts $P_1, \dots P_r$ such that $\sum_{i\in P_j} a_i \leq 1$. Since such partitions must have m parts with a single element corresponding to the heavy weights, and the other parts are obtainable by taking all the possible r-partitions of the set of light elements, which is an n-set, the result follows. $\square$

Let $\left\{ n \atop r \right\}_{\leq m}$ be the m-restricted Stirling number of the second kind for $n, r \geq 1$, i.e., the number of partitions of an n-set into r non-empty subsets, each of which has at most m elements.

Corollary 2.2.8. *Let $\mathcal{A} = \frac{1}{m}^{(n)}$ for $n > m > 1$ such that $2g-2+n/m > 0$. Then $[M_{g,\mathcal{A}}] = \sum_{r=\lceil \frac{n}{m} \rceil}^{n} \left\{ n \atop r \right\}_{\leq m} [M_{g,r}]$.*

Proof. For $r < \lceil \frac{n}{m} \rceil$, the number $N_{r,\mathcal{A}}$ is 0, since there are no admissible partitions with less than $\lceil \frac{n}{m} \rceil$ sets. For $r \geq \lceil \frac{n}{m} \rceil$, $N_{r,\mathcal{A}}$ equals to $\left\{ n \atop r \right\}_{\leq m}$, since we have n components in $\mathcal{A}$ but we can not put more than m together due to the condition $\sum_{i\in Pj} a_i \leq 1$. $\square$

Corollary 2.2.9. *Let $\mathcal{A} = \frac{1}{2}^{(n)}$ for $n \geq 1$ such that $2g-2+n/2 > 0$. Then*

$$[M_{g,\mathcal{A}}] = \sum_{r=\lceil \frac{n}{2} \rceil}^{n} \frac{\prod_{i=0}^{n-r-1} \binom{n-2i}{2}}{(n-r)!} [M_{g,r}].$$

Proof. A $\frac{1}{2}^{(n)}$-admissible partition having r parts must consist of exactly $(n-r)$ subsets of size two and singletons otherwise. Therefore, we have

$$\left\{ n \atop r \right\}_{\leq 2} = \frac{\prod_{i=0}^{n-r-1} \binom{n-2i}{2}}{(n-r)!}.$$

It is sufficent then to put this into Corollary 2.2.8, and the result follows. $\square$

2.2.2 The Top Weight Euler Characteristic of $\mathcal{M}_{g,\mathcal{A}}$

We can now exploit the additivity of Euler-Poincaré characteristics and the connection between $\mathcal{M}_{g,\mathcal{A}}$ and $\Delta_{g,\mathcal{A}}$. For a complex algebraic variety (or stack) X of dimension d, let χ^{tw} be the top weight Euler characteristic, defined as

$$\chi^{tw}(X) := \sum_{i=0}^{2d}(-1)^i \dim \mathrm{Gr}^W_{2d} H^i(X;\mathbb{Q}),$$

and for any space Y, let $\widetilde{\chi}(Y)$ be the reduced Euler characteristic.

Lemma 2.2.10. *Let $\mathcal{X}$ be a smooth, separated DM stack over $\mathbb{C}$ of dimension d. Let $\overline{\mathcal{X}}$ be a smooth normal crossings compactification of $\mathcal{X}$ and $\Delta(\mathcal{X} \subset \overline{\mathcal{X}})$ be the dual boundary complex. Then $\chi^{tw}(\mathcal{X}) = -\widetilde{\chi}(\Delta(\mathcal{X} \subset \overline{\mathcal{X}}))$.*

Proof. We have

$$\begin{aligned}
\chi^{tw}(\mathcal{X}) &= \sum_{i=0}^{2d}(-1)^i \dim \mathrm{Gr}^W_{2d} H^i(\mathcal{X};\mathbb{Q}) \\
&= \sum_{i=0}^{2d}(-1)^i \dim \widetilde{H}_{2d-i-1}(\Delta(\mathcal{X} \subset \overline{\mathcal{X}});\mathbb{Q}) \\
&= \sum_{i=0}^{2d}(-1)^{i+1} \dim \widetilde{H}_{i}(\Delta(\mathcal{X} \subset \overline{\mathcal{X}});\mathbb{Q}) \\
&= -\widetilde{\chi}(\Delta(\mathcal{X} \subset \overline{\mathcal{X}})).
\end{aligned}$$

□

Recall the isomorphism $\mathrm{Gr}^W_{6g-6+2n} H^{6g-6+2n-k}(\mathcal{M}_{g,n};\mathbb{Q}) \cong \widetilde{H}_{k-1}(\Delta_{g,n};\mathbb{Q})$ is a special case of the isomorphism $\mathrm{Gr}^W_{2d} H^{2d-k}(\mathcal{X};\mathbb{Q}) \cong \widetilde{H}_{k-1}(\Delta(\mathcal{X} \subset \overline{\mathcal{X}});\mathbb{Q})$, as we saw in Section 1.3.4.

Let k be an algebraically closed field, and let Z be an isomorphism invariant of quasi-projective k-varieties. We call Z motivic if whenever $Y \subset X$ is a closed subvariety we have

$$\mathrm{Z}([X]) = \mathrm{Z}([X \setminus Y]) + \mathrm{Z}([Y]),$$

and whenever X, Y are varieties we have $\mathrm{Z}([X \times Y]) = \mathrm{Z}([X])\mathrm{Z}([Y])$. Denote by χ_c^0 the weight 0 compactly supported Euler characteristic. We observe that χ_c^0 is a motivic invariant. The following lemma is proven upon realizing that for a complex algebraic variety X, we have that $\chi_c^0(X) = P_X(0)$, where P_X is the virtual Poincaré polynomial described in Example 2.2.4.

Lemma 2.2.11. *The weight 0 compactly supported Euler characteristic*

$$\chi_c^0 : K_0(Var/\mathbb{C}) \to \mathbb{Z}$$

is an Euler-Poincaré characteristic.

Proof. It is enough to observe that P_X is already an Euler-Poincaré characteristic, and the evaluation of it in 0 is a group homomorphism. □

Denote by $\chi(Y)$ the Euler Characteristic of a topological space Y. Our main goal of this section is the following Theorem.

Theorem 2.2.12. *Let $W = W_1 \subset \cdots \subset W_{6g-6+2r} \subseteq H^*(\mathcal{M}_{g,r}, \mathbb{Q})$ be the weight filtration of the rational singular cohomology of the moduli stack $\mathcal{M}_{g,r}$ and denote by $\chi^W_{6g-6+2r}$ the Euler characteristic of the top graded piece*

$$\mathrm{Gr}^W_{6g-g+2r} H^*(\mathcal{M}_{g,r}; \mathbb{Q}) = W_{6g-6+2r}/W_{6g-7+2r}$$

of the weight filtration. Then

$$\chi(\Delta_{g,\mathcal{A}}) = 1 - \sum_{r=1}^{n} N_{r,\mathcal{A}} \cdot \chi^W_{6g-6+2r}(\mathcal{M}_{g,r}).$$

Proof. By Proposition 2.2.5 and Lemma 2.2.11, we have $\chi^0_c(M_{g,\mathcal{A}}) = \sum_{r=1}^n N_{r,\mathcal{A}} \chi^0_c(M_{g,r})$. By Proposition 36 of [Beh04] and Theorem 4.40 of [Edi10], we can show that there is an isomorphism of rational cohomologies $H^*(\mathcal{X}; \mathbb{Q}) \cong H^*(X; \mathbb{Q})$ between the Coarse Moduli Scheme X of a Deligne Mumford stack $\mathcal{X}$ and the Stack itself, which is also an isomorphism of mixed Hodge structures. Therefore, $\chi^0_c(\mathcal{X}) = \chi^0_c(X)$ and $\chi^{tw}(\mathcal{X}) = \chi^{tw}(X)$. It follows from Theorem 6.23 of [PS08] that for a smooth Deligne-Mumford stack $\mathcal{X}$ of dimension d, the Poincaré duality pairing

$$H^j_c(\mathcal{X}; \mathbb{Q}) \times H^{2d-j}(\mathcal{X}; \mathbb{Q}) \to \mathbb{Q}$$

induces a perfect pairing of graded pieces

$$\mathrm{Gr}^W_m H^j_c(\mathcal{X}; \mathbb{Q}) \times \mathrm{Gr}^W_{2d-m} H^{2d-j}(\mathcal{X}; \mathbb{Q}) \to \mathbb{Q}$$

for $0 \leq m \leq 2j$. So we can write

$$\begin{aligned}\chi^0_c(\mathcal{X}) &= \sum_{j=0}^{2d} (-1)^j \dim \mathrm{Gr}^W_{2d} H^{2d-j}(\mathcal{X}; \mathbb{Q}) \\ &= \sum_{j=0}^{2d} (-1)^{2d-j} \dim \mathrm{Gr}^W_{2d} H^{2d-j}(\mathcal{X}; \mathbb{Q}).\end{aligned}$$

In particular, it follows that

$$\chi^0_c(\mathcal{X}) = \chi^{tw}(\mathcal{X}).$$

Since $\mathcal{M}_{g,r}$ and $\mathcal{M}_{g,\mathcal{A}}$ are smooth Deligne-Mumford stacks and $\Delta_{g,\mathcal{A}} = \Delta(\mathcal{M}_{g,\mathcal{A}} \subset \overline{\mathcal{M}}_{g,\mathcal{A}})$, the result now follows from Lemma 2.2.10 and the fact that $\widetilde{\chi}(\Delta_{g,\mathcal{A}}) = \chi(\Delta_{g,\mathcal{A}}) - 1$. □

Computations for $\chi(\Delta_{g,\mathcal{A}})$

In [CFGP19], authors give a generating function for the numbers $\chi^W_{6g-6+2r}(\mathcal{M}_{g,r})$. Together with their result, Theorem 2.2.12 allows for the calculation of $\chi(\Delta_{g,\mathcal{A}})$ for arbitrary g and $\mathcal{A}$.

Corollary 2.2.13. *Given a weight vector $\mathcal{A}$, such that $N_{r,\mathcal{A}} = 0$ for $r \leq g+1$, the Euler characteristic of $\Delta_{g,\mathcal{A}}$ is*

$$\chi(\Delta_{g,\mathcal{A}}) = 1 + \sum_{r=1}^{n} N_{r,\mathcal{A}}(-1)^r \frac{(g+r-2)!}{g!} B_g$$

Proof. In Corollary 8.1 of [CFGP19], the authors show that for $r > g+1$,

$$\chi^W_{6g-6+2r}(\mathcal{M}_{g,r}) = (-1)^{r+1} \frac{(g+r-2)!}{g!} B_g,$$

where B_g is the g-th Bernoulli number, characterized by

$$\frac{t}{e^t - 1} = \sum_{\ell=0}^{\infty} B_\ell \frac{t^\ell}{\ell!}.$$

Substituting into Theorem 2.2.12 yields the formula. □

We can obtain a closed form for the Euler characteristic of $\Delta_{g,\mathcal{A}}$ for Heavy/Light weights. Using the following corollary, we compute explicitly the Euler characteristics of $\Delta_{g,(1^{(m)}|\varepsilon^{(n)})}$ in Table 2.1.

Corollary 2.2.14. *Given a Heavy/Light weight vector $\mathcal{A} = (1^{(m)}|\varepsilon^{(n)})$ where $n \geq g+1$, $n > 0$, and $0 < \varepsilon < 1/n$,*

$$\chi(\Delta_{g,\mathcal{A}}) = 1 + \sum_{r=1}^{m} \sum_{\ell=0}^{g} (-1)^{n+r+\ell} \frac{(g+n+r-2)!\ell!}{g!(\ell+1)} S(m,r) S(g,\ell).$$

Proof. It suffices to expand the Bernoulli number B_g as in [Apo98] in terms of Stirling numbers

$$B_g = \sum_{\ell=0}^{g} (-1)^\ell \frac{\ell!}{\ell+1} S(g,\ell),$$

□

$g=0$	$n=1$	$n=2$	$n=3$	$n=4$
$m=2$	-	2	0	2
$m=3$	3	-3	9	-15
$m=4$	-5	19	-53	163
$m=5$	25	-95	385	-1535

$g=1$	$n=1$	$n=2$	$n=3$	$n=4$
$m=2$	2	-1	5	-7
$m=3$	-2	10	-26	82
$m=4$	13	-47	193	-767
$m=5$	-59	301	-1499	7501

$g=2$	$n=1$	$n=2$	$n=3$	$n=4$
$m=3$	3	-7	33	-127
$m=4$	-9	51	-249	1251
$m=5$	61	-359	2161	-12959
$m=6$	-419	2941	-20579	144061

$g=3$	$n=1$	$n=2$	$n=3$	$n=4$
$m=4$	1	1	1	1
$m=5$	1	1	1	1
$m=6$	1	1	1	1
$m=7$	1	1	1	1

Table 2.1: Euler characteristics of $\Delta_{g,(1^{(m)}|\varepsilon^{(n)})}$ for $g = 0, 1, 2, 3$ and some (m, n) where $m \geq g+1$ and $n > 0$. When $g = 0$, we start with $m = 2$ since the space $\Delta_{0,(1|\varepsilon^{(n)})}$ is empty. When $g = 0, m = 2, n = 1$, $\Delta_{0,(1,1,\varepsilon)}$ is also empty.

Chapter 3

Equivariant Hodge Polynomials of Heavy/Light Moduli Spaces

This section describes the material published in the joint work with S. Kannan and C. Yun [KSY22]. We study the $(S_m \times S_n)$-equivariant Hodge–Deligne polynomials of $\mathcal{M}_{g,(1^{(m)}|\varepsilon^{(n)})}$ and $\overline{\mathcal{M}}_{g,(1^{(m)}|\varepsilon^{(n)})}$. Throughout this Chapter we will work with the Coarse Moduli Spaces of these Stacks, as the Mixed Hodge Structure on the rational cohomology of a Deligne-Mumford Stack coincides with that of its Coarse Moduli Space. We start by introducing some background about Symmetric Functions and Frobenius Characteristics.

3.1 Symmetric functions and the Frobenius characteristic

3.1.1 Symmetric functions

Definition 3.1.1. The ring Λ of symmetric functions over $\mathbb{Q}$ is defined as

$$\Lambda := \varprojlim_{\text{n}} \mathbb{Q}[[x_1, \dots, x_n]]^{S_n}.$$

Elements of Λ are power series which are invariant under any permutation of the variables. A well known property about Λ is that it is generated by elements $p_i := \sum_{k>0} x_k^i$,

$$\Lambda = \mathbb{Q}[[p_1, p_2, \dots]]$$

where p_i is called the ith power sum symmetric function. Observe also that the ring Λ is graded by degree, and p_i has degree i. We define also the complete homogeneous symmetric functions h_k for any non-negative integer k as the sum of all monomials of degree k:

$$h_k = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

There is an associative operation $\circ$, called plethysm on Λ, characterized by the following properties:

(i) for any $g \in \Lambda$, the map $f \mapsto f \circ g$ defines an algebra homomorphism $\Lambda \to \Lambda$;

(ii) for all n, the map $f \mapsto p_n \circ f$ defines an algebra homomorphism $\Lambda \to \Lambda$;

(iii) $p_n \circ p_m = p_{nm}$.

The proof of the fact that these properties define a unique operation, as well as more properties about the ring of symmetric functions can be found in [Mac95], [Sta99] and in Section 7 of [GK98]. Together with power sums and homogeneous symmetric functions, there is another important class of functions called Schur functions. Here we define them as follows.

First, if $\lambda = (\lambda_1, \dots, \lambda_l)$ is an array of natural numbers, such that $\lambda_1 + \cdots + \lambda_l = s$ and $\lambda_1 \geq \cdots \geq \lambda_l$, we call λ a partition of $[s]$ of lenght l, and denote it $\lambda \vdash s$, for $s \in \mathbb{N}$. Let $\lambda \vdash s$, we define the Schur Function associated to λ as the determinant of the following matrix:

$$s_\lambda := \det(h_{\lambda_i + j - i})_{i,j=1}^{l}.$$

Example 3.1.2. Suppose we have $s = 3$, and $\lambda = (2, 1)$. The resulting matrix is

$$\begin{pmatrix} h_2 & h_3 \\ h_0 & h_1 \end{pmatrix},$$

So $s_{(2,1)} = h_1 h_2 - h_3$ (by noticing that $h_0 = 1$.)

Whenever $\lambda_i + j - i < 0$, we set $h_{\lambda_i + j - i} = 0$ by convention. Although this is not the original Definition, this is enough for our purposes. We refer to [Mac95] for further details.

Remark 3.1.3. Note that $h_n = s_n$, were by s_n we denote the Schur Function corresponding to the trivial partition of $[n]$ made by n sets of length 1.

Remark 3.1.4. The Schur functions s_λ, where λ is a partition of $[n]$, form a basis for the homogeneous degree n part of Λ.

3.1.2 S_n-representations

A representation of a group G on a vector space V over a field k is a group homomorphism from G to $GL(V)$, the general linear group on V. That is, a representation is a map

$$\rho \colon G \to \mathrm{GL}(V)$$

such that $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$, for all $g_1, g_2 \in G$. For a representation $\rho \colon G \to \mathrm{GL}(V)$, denote by $Tr_V(g)$ the trace of an element g seen as a matrix in $GL(V)$. We will refer to V itself as the representation from now on. Given an S_n-representation V, the Frobenius characteristic $ch_n(V)$ is defined by

$$ch_n(V) := \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{Tr}_V(\sigma) p_{\lambda(\sigma)},$$

where $\lambda(\sigma)$ is the cycle type of the permutation σ, and for a partition $\lambda = (\lambda_1, \dots, \lambda_s) \vdash n$ we set $p_\lambda := \prod_i p_{\lambda_i}$. The Frobenius characteristic $ch_n(V)$ determines the S_n-representation V, and by Remark 3.1.4 we can describe the Frobenius Characteristic using Schur functions. In order to do so we need to introduce Specht Modules.

Specht Modules and Young Tableaux

We will follow [Cha08] for this section. Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash s$. Then a Ferrer's diagram of shape λ is an array of s dots, with l left-justified rows, and the ith row has λ_i dots.

Example 3.1.5. The Ferrer's diagram of shape $\lambda = (4, 2, 1)$ looks like:

$$\begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & & & \end{array}$$

A Young tableau of shape λ is obtained from the corresponding Ferrer's diagram by replacing the dots by numbers $1, ..., s$ bijectively. A tableau is called standard if its rows and columns increase.

Example 3.1.6. An example of a (non-standard) Young tableau of shape $\lambda = (4, 2, 1)$:

1	4	3	6
7	2		
5			

We say that two Young tableaux are row-equivalent if their corresponding rows contain the same numbers. An equivalence class of this relation is called tabloid of shape λ. We denote tabloids by $\{t\}$. The symmetric group on s points acts on the set of Young tableaux of shape λ. Consequently, it acts on tabloids, and on the free k-module M^λ with the tabloids as basis. Given a Young tableau T of shape λ, let

$$E_T = \sum_{\sigma \in Q_T} sign(\sigma)\{\sigma(T)\} \in M^\lambda$$

where Q_T is the subgroup of permutations, preserving (as sets) all columns of T, $sign(\sigma)$ is the sign of the permutation σ and $\{\sigma(T)\}$ is the tabloid class of the permuted tableau $\sigma(T)$.

Definition 3.1.7. The Specht module of the partition λ is the module W_λ generated by the elements E_T as T runs through all tableaux of shape λ.

Whenever we have a S_n-representation V, we can decompose it using Specht Modules as $V = \bigoplus_{\lambda \vdash n} W_\lambda^{\oplus a_\lambda}$, were a_λ is the dimension of the Specht module associated to λ. Then one can show that $ch_n(V) = \sum_{\lambda \vdash n} a_\lambda s_\lambda$.

Remark 3.1.8. Observe that the homogeneous symmetric functions h_n verify $h_n = ch_n(\mathrm{Triv}_n)$, where Triv_n is the trivial S_n-representation of dimension one.

3.1.3 The ring of bisymmetric functions

All of what we made so far generalizes to $(S_m \times S_n)$-representations. We set $\Lambda^{(2)} := \Lambda \otimes \Lambda$, and we call $\Lambda^{(2)}$ the ring of bisymmetric functions. Given $f \in \Lambda$, we write $f^{(j)}$ for the inclusion of f into the jth tensor factor, $j = 1, 2$. Then we have

$$\Lambda^{(2)} = \mathbb{Q}[[p_1^{(1)}, p_1^{(2)}, p_2^{(1)}, p_2^{(2)}, \ldots]].$$

Given an $(S_m \times S_n)$-representation V, its Frobenius characteristic is the bisymmetric function

$$ch_{m,n}(V) := \frac{1}{m! \cdot n!} \sum_{(\sigma,\tau)\in S_m\times S_n} \mathrm{Tr}_V(\sigma, \tau) p^{(1)}_{\lambda(\sigma)} p^{(2)}_{\lambda(\tau)}.$$

Just as in the single variable case, the bisymmetric function $ch_{m,n}(V)$ completely determines the $(S_m \times S_n)$-representation V: if $V = \bigoplus_{\substack{\lambda\vdash m\\ \mu\vdash n}} (W_\lambda \otimes W_\mu)^{\oplus a_{\lambda\mu}}$ is its decomposition into Specht modules, then $ch_{m,n}(V) = \sum_{\lambda,\mu} a_{\lambda\mu} s^{(1)}_\lambda s^{(2)}_\mu$.

The ring $\Lambda^{(2)}$ has two plethysm operations $\circ_1$ and $\circ_2$, characterized by:

(i) for all g, the map $f \mapsto f \circ_i g$ is an algebra homomorphism $\Lambda^{(2)} \to \Lambda^{(2)}$;

(ii) for all n, the map $f \mapsto p_n^{(i)} \circ_i f$ is an algebra homomorphism $\Lambda^{(2)} \to \Lambda^{(2)}$;

(iii) $p_n^{(i)} \circ_i p_m^{(j)} = p_{nm}^{(j)}$ for any $i, j \in \{1, 2\}$;

(iv) $p_n^{(i)} \circ_j f = p_n^{(i)}$ if $i \neq j$;

see [Cha16]. Clearly those two operations generalize the plethysm defined for Λ. Since the ring Λ is also a Hopf algebra, it comes with a coproduct $\Delta : \Lambda \to \Lambda^{(2)}$ defined by $p_i \mapsto p_i^{(1)} + p_i^{(2)}$. On the level of Frobenius characteristic, we have

$$\Delta(ch_n(V)) = \sum_{k=0}^{n} ch_{k,n-k} \left(Res^{S_n}_{S_k\times S_{n-k}} V \right), \tag{3.1}$$

where $Res^G_K(V)$ indicates the restriction of the representation of the group G on a subgroup K. There is also a rank homomorphism defined on the ring of symmetric functions

$$\mathrm{rk} : \Lambda \to \mathbb{Q}[[x]], \tag{3.2}$$

determined by $ch_n(V) \mapsto \dim(V) \cdot \frac{x^n}{n!}$, or equivalently, $p_1 \mapsto x$ and $p_n \mapsto 0$ for $n > 1$. This takes plethysm into composition of power series. We use the same notation for the analogous morphism defined on bisymmetric functions:

$$\mathrm{rk} : \Lambda^{(2)} \to \mathbb{Q}[[x, y]] \tag{3.3}$$

determined by $ch_{m,n}(V) \mapsto \dim(V) \cdot \frac{x^m y^n}{m!n!}$, or $p_1^{(1)} \mapsto x$, $p_1^{(2)} \mapsto y$, and $p_n^{(j)} \mapsto 0$ for $n > 1$. In this case, the two plethysm operations $\circ_1$ and $\circ_2$ are carried into composition in x and y, respectively, i.e., $\mathrm{rk}(s \circ_1 t) = rk(s)(\mathrm{rk}(t), y)$ and $\mathrm{rk}(s \circ_2 t) = rk(s)(x, \mathrm{rk}(t))$ for $s, t \in \Lambda^{(2)}$.

3.2 The generating functions

Let X be a variety, and consider the Deligne (increasing) weight filtration

$$W_1 \subset \cdots \subset W_{2d} = H_c^*(X;\mathbb{Q}),$$

where $H_c^*(X;\mathbb{Q})$ is the compactly supported cohomology introduced in Definition 2.2.3, such that for each j, the quotient

$$Gr_i^W H_c^j(X;\mathbb{Q}) := W_i \cap H_c^j(X;\mathbb{Q})/W_{i-1} \cap H_c^j(X;\mathbb{Q})$$

Consider also the (decreasing) Hodge Filtration F on $Gr_i^W H_c^j(X,\mathbb{Q})$ induced by the one on the whole cohomology. Denote its generic piece by $Gr_k^F Gr_i^W H_c^j(X,\mathbb{Q})$. If X comes with an action of S_n, its complex cohomology groups are S_n-representations in the category of Mixed Hodge Structures. We set

$$h_X^{S_n}(u,v) := \sum_{i,p,q=0}^{2d} (-1)^i ch_n\left(\mathrm{Gr}_p^F \mathrm{Gr}_{p+q}^W H_c^i(X;\mathbb{C})\right) u^p v^q \in \Lambda[u,v] \tag{3.4}$$

for the S_n-equivariant Hodge–Deligne polynomial of X. Analogously, if X has an action of $S_m \times S_n$, its complex cohomology groups are $S_m \times S_n$-representations in the category of Mixed Hodge Structures. The $(S_m \times S_n)$-equivariant Hodge–Deligne polynomial of X is given by the formula

$$h_X^{S_m\times S_n}(u,v) := \sum_{i,p,q=0}^{2d} (-1)^i ch_{m,n}\left(\mathrm{Gr}_p^F \mathrm{Gr}_{p+q}^W H_c^i(X;\mathbb{C})\right) u^p v^q \in \Lambda^{(2)}[u,v]. \tag{3.5}$$

If X is proper, and the Mixed Hodge Structure on each cohomology group is pure, the Hodge–Deligne polynomial specializes to the usual Hodge polynomial

$$\sum_{p,q=0}^{2d} (-1)^{p+q} ch_n\left(H^{p,q}(X;\mathbb{C})\right) u^p v^q.$$

in the case of S_n-representations, or

$$\sum_{p,q=0}^{2d} (-1)^{p+q} ch_{m,n}\left(H^{p,q}(X;\mathbb{C})\right) u^p v^q.$$

in the case of $S_m \times S_n$-representations (for more details on Mixed Hodge Structures, see [PS08] or [CLNS18]).

We know that the Coarse Moduli Spaces $\overline{M}_{g,\mathcal{A}}$ are proper for any weight datum $\mathcal{A}$, so in particular they are for the weight data we take in account in this section, i.e., $1^{(n)}$ and $(1^{(m)}|\varepsilon^{(n)})$. We assemble all of the equivariant Hodge–Deligne polynomials for Heavy/Light Hassett spaces with fixed genus into series with coefficients in $\Lambda^{(2)}$:

$$\mathsf{a}_g := \sum_{m,n} h^{S_m\times S_n}_{\mathcal{M}_{g,(1^{(m)}|\varepsilon^{(n)})}}(u,v), \quad \overline{\mathsf{a}}_g := \sum_{m,n} h^{S_m\times S_n}_{\overline{\mathcal{M}}_{g,(1^{(m)}|\varepsilon^{(n)})}}(u,v) \in \Lambda^{(2)}[[u,v]].$$

We also define

$$\mathsf{b}_g := \sum_n h^{S_n}_{\mathcal{M}_{g,n}}(u,v), \quad \overline{\mathsf{b}}_g := \sum_n h^{S_n}_{\overline{\mathcal{M}}_{g,n}}(u,v) \in \Lambda[[u,v]].$$

Notice again that we defined the polynomials in the series for the Stacks, intending that they agree with the ones of the Coarse Moduli Spaces.

For $f \in \Lambda$, the inclusions of f into the jth tensor factor, $j \in \{1,2\}$, extend to maps $\Lambda[[u,v]] \to \Lambda^{(2)}[[u,v]]$. This is true also for the coproduct $\Delta : \Lambda \to \Lambda^{(2)}$. For $f \in \Lambda^{(2)}[[u,v]]$ set also

$$Exp^{(1)}(f) = \sum_{n>0} h^{(1)}_n \circ_1 f.$$

and

$$Exp^{(2)}(f) = \sum_{n>0} h^{(2)}_n \circ_2 f.$$

The main result we want to show in this Chapter is the following:

Theorem 3.2.1. *In $\Lambda^{(2)}[[u,v]]$ we have*

$$\mathsf{a}_g = \Delta(\mathsf{b}_g) \circ_2 Exp^{(2)}\left(p^{(2)}_1\right)$$

and

$$\overline{\mathsf{a}}_g = \Delta(\overline{\mathsf{b}}_g) \circ_2 \left(p^{(2)}_1 - \frac{\partial \mathsf{b}^{(2)}_0}{\partial p^{(2)}_1}\right) \circ_2 Exp^{(2)}\left(p^{(2)}_1\right).$$

Informally, Theorem 3.2.1 determines a_g and $\overline{\mathsf{a}}_g$ in terms of b_g and $\overline{\mathsf{b}}_g$. Moreover, this transformation is invertible, as $Exp^{(i)}$ has a plethystic inverse $Log^{(i)}$ and $p^{(i)}_1 - \partial \mathsf{b}^{(i)}_0 / \partial p^{(i)}_1$ is inverse to $p^{(i)}_1 + \partial \overline{\mathsf{b}}^{(i)}_0 / \partial p^{(i)}_1$, for $i \in \{1,2\}$. There is a numerical analogue of Theorem 3.2.1 which deals with the non-equivariant Hodge–Deligne polynomials, defined by the assignment

$$h_X(u,v) := \sum_{i,p,q=0}^{2d} (-1)^i \dim\left(\mathrm{Gr}^F_p \mathrm{Gr}^W_{p+q} H^i_c(X;\mathbb{C})\right) u^p v^q \in \mathbb{Q}[u,v].$$

Set

$$a_g := \sum_{m,n} h_{\mathcal{M}_{g,((1^{(m)}|\varepsilon^{(n)}))}}(u,v) \frac{x^m y^n}{m!n!}, \quad \overline{a}_g := \sum_{m,n} h_{\overline{\mathcal{M}}_{g,((1^{(m)}|\varepsilon^{(n)}))}}(u,v) \frac{x^m y^n}{m!n!} \in \mathbb{Q}[[u,v,x,y]],$$

and similarly put

$$b_g := \sum_n h_{\mathcal{M}_{g,n}}(u,v) \frac{x^n}{n!}, \quad \overline{b}_g := \sum_n h_{\overline{\mathcal{M}}_{g,n}}(u,v) \frac{x^n}{n!} \in \mathbb{Q}[[u,v,x]].$$

Corollary 3.2.2. *We have*

$$a_g = b_g|_{x\to w}$$

where $w = x + e^y - 1$, *and*

$$\overline{a}_g = \overline{b}_g|_{x\to z},$$

where

$$z = x + e^y + \frac{e^{uvy} - uv\cdot e^y + uv - 1}{uv - u^2v^2} - 1.$$

and by $f|_{x\to y}$ *we denote a change of variables.*

In order to show Theorem 3.2.1, we need to introduce the main tool for its proof, which is a generalization of Getzler–Pandharipande's Grothendieck Ring of $\mathbb{S}$-spaces. The techniques of the proof are based on prior work on the operad structure of moduli of stable curves and maps ([Get95], [Get98], [GK98] and [GP06]). Proofs of Theorem 3.2.1 and Corollary 3.2.2 are in Section 3.2.2.

3.2.1 The Grothendieck Ring of $\mathbb{S}^2$-spaces

We say that a G-variety is a variety with an action of a group G. Inside the Grothendieck Group of varieties $K_0(\mathrm{Var}/k)$ (see Definition 2.2.1) we can define the subgroup $K_0(\mathrm{Var}/k, G)$ of isomorphism classes of G-varieties, were we consider the cut and paste relations only for G-subvarieties. We define an $\mathbb{S}$-space as follows:

Definition 3.2.3. ([GP06]) An $\mathbb{S}$-space $\mathcal{W}$ is a collection of S_n-varieties $W(n)$ for $n \geq 0$, i.e., $\mathcal{W} = \{W(n)|W(n) \text{ is a } S_n\text{-variety}\}_{n\in\mathbb{N}}$

Example 3.2.4. We define the $\mathbb{S}$-space ζ_r, which contains $Spec(\mathbb{C})$ with trivial S_n-action in arity r, and $\varnothing$ elsewhere:

$$\zeta_r(n) := \begin{cases} Spec(\mathbb{C}) & \text{if } n = r; \\ \varnothing & \text{else.} \end{cases}$$

We can define the Grothendieck Group of $\mathbb{S}$-spaces as the product

$$K_0(\mathrm{Var}/k, \mathbb{S}) := \prod_{n\geq 0} K_0(\mathrm{Var}/k, S_n).$$

We can make $K_0(\mathrm{Var}/k, \mathbb{S})$ into a ring using the following $\boxtimes$-product, which we can define it for each n as

$$(\mathcal{X} \boxtimes \mathcal{Y})(n) = \coprod_{i=0}^{n} Ind_{S_i\times S_{n-i}}^{S_n} \mathcal{X}(i) \times \mathcal{Y}(n-i),$$

where $Ind_K^G(V)$ denotes the induced representation of the subgroup K of G (see [Die11], Chapter 4). The fact that this makes $K_0(\mathrm{Var}/k, \mathbb{S})$ a ring is shown in [GP06]. This notion can be generalized to the case of $(S_m \times S_n)$-varieties, using Grothendieck Groups $K_0(\mathrm{Var}/k, S_m \times S_n)$ of $(S_m \times S_n)$-varieties.

Definition 3.2.5. We define an $\mathbb{S}^2$-space $\mathcal{X}$ to be a collection of varieties $\mathcal{X}(m,n)$ together with an action of $S_m \times S_n$ for each pair (m,n) with $m,n \geq 0$, i.e.,

$$\mathcal{X} = \{X(m,n) | X(m,n) \text{ is a } S_m \times S_n\text{-variety}\}_{m,n\in\mathbb{N}}.$$

We refer to $\mathcal{X}(m,n)$ as the arity (m,n) component of $\mathcal{X}$. We define the Grothendieck Group of $\mathbb{S}^2$-spaces as the product $K_0(\mathrm{Var}/k,\mathbb{S}^2) := \prod_{m,n\geq 0} K_0(\mathrm{Var}/k, S_m\times S_n)$. It is clearly a commutative group with the sum, since the operation works independently on each component of the disjoint union, and there the sum is the one of $K_0(\mathrm{Var}/k, S_m \times S_n)$ as a subgroup of $K_0(\mathrm{Var}/k)$. Analogously to what we did before, we can make $K_0(\mathrm{Var},\mathbb{S}^2)$ into a ring using the following $\boxtimes$-product on $\mathbb{S}^2$-spaces:

$$(\mathcal{X}\boxtimes\mathcal{Y})(m,n) = \coprod_{i=0}^{m}\coprod_{j=0}^{n} Ind^{S_m\times S_n}_{S_i\times S_{m-i}\times S_j\times S_{n-j}}\mathcal{X}(i,j)\times\mathcal{Y}(m-i,n-j).$$

We use the same notation as it will be clear by the context which of the two products we are using.

Proposition 3.2.6. *The group $K_0(\mathrm{Var}/k,\mathbb{S}^2)$ is a ring with the $\boxtimes$-product.*

Proof. We have to show that $K_0(\mathrm{Var}/k,\mathbb{S}^2)$ is a monoid with respect to $\boxtimes$, so we have to show that there is a unit and that associativity holds. The unit of the box product is easily shown to be the $\mathbb{S}^2$-space id defined by

$$id(m,n) = \begin{cases} Spec(\mathbb{C}) & \text{if } (m,n) = (0,0); \\ \varnothing & \text{otherwise.} \end{cases}$$

We check each property in a fixed arity (m,n). For the associativity, we have

$$\mathcal{X}\boxtimes(\mathcal{Y}\boxtimes\mathcal{Z})(m,n) =$$

$$= \coprod_{i=0}^{m}\coprod_{j=0}^{n} Ind^{S_m\times S_n}_{S_i\times S_{m-i}\times S_j\times S_{n-j}}\mathcal{X}(i,j)\times(\mathcal{Y}\boxtimes\mathcal{Z})(m-i,n-j) =$$

$$= \coprod_{i=0}^{m}\coprod_{j=0}^{n} Ind^{S_m\times S_n}_{S_i\times S_{m-i}\times S_j\times S_{n-j}}\mathcal{X}(i,j)\times(\coprod_{k=0}^{m-i}\coprod_{l=0}^{n-j} Ind^{S_{m-i}\times S_{n-j}}_{S_k\times S_{m-i-k}\times S_l\times S_{n-j-l}}\mathcal{Y}(k,l)\times\mathcal{Z}(m-i-k,n-j-l)) =$$

$$= \coprod_{i=0}^{m}\coprod_{j=0}^{n} Ind^{S_m\times S_n}_{S_i\times S_{m-i}\times S_j\times S_{n-j}}(\coprod_{k=0}^{m-i}\coprod_{l=0}^{n-j} Ind^{S_{m-i}\times S_{n-j}}_{S_k\times S_{m-i-k}\times S_l\times S_{n-j-l}}\mathcal{X}(i,j)\times\mathcal{Y}(k,l)\times\mathcal{Z}(m-i-k,n-j-l)).$$

Analogously $(\mathcal{X}\boxtimes\mathcal{Y})\boxtimes\mathcal{Z}(m,n) =$

$$= \coprod_{i=0}^{m}\coprod_{j=0}^{n} Ind^{S_m\times S_n}_{S_i\times S_{m-i}\times S_j\times S_{n-j}}(\coprod_{k=0}^{i}\coprod_{l=0}^{j} Ind^{S_i\times S_j}_{S_k\times S_{i-k}\times S_l\times S_{j-l}}\mathcal{X}(k,l)\times\mathcal{Y}(i-k,j-l)\times\mathcal{Z}(m-i,n-j)),$$

so we just have to observe that up to shifting the indices, these two disjoint unions are the same.

Lastly, we have to show the distributive property, i.e., that $\mathcal{X}\boxtimes(\mathcal{Y}+\mathcal{Z}) = \mathcal{X}\boxtimes\mathcal{Y}+\mathcal{X}\boxtimes\mathcal{Z}$ and it is enough since the product is commutative. We have

$$\mathcal{X}\boxtimes(\mathcal{Y}+\mathcal{Z})(m,n) = \coprod_{i=0}^{m}\coprod_{j=0}^{n} Ind_{S_i\times S_{m-i}\times S_j\times S_{n-j}}^{S_m\times S_n}\mathcal{X}(i,j)\times(\mathcal{Y}+\mathcal{Z})(m-i,n-j)$$

where

$$(\mathcal{Y}+\mathcal{Z})(m-i,n-j) = \mathcal{W}(m-i,n-j)\in K_0(\mathsf{Var}, S_{m-i}\times S_{n-j}).$$

Now the sum $\mathcal{X}\boxtimes\mathcal{Y}+\mathcal{X}\boxtimes\mathcal{Z}$ gives:

$$\mathcal{X}\boxtimes\mathcal{Y}+\mathcal{X}\boxtimes\mathcal{Z} =$$

$$= \coprod_{i=0}^{m}\coprod_{j=0}^{n} Ind_{S_i\times S_{m-i}\times S_j\times S_{n-j}}^{S_m\times S_n}\mathcal{X}(i,j)\times\mathcal{Y}(m-i,n-j)+\coprod_{i=0}^{m}\coprod_{j=0}^{n} Ind_{S_i\times S_{m-i}\times S_j\times S_{n-j}}^{S_m\times S_n}\mathcal{X}(i,j)\times\mathcal{Z}(m-i,n-j),$$

which for each i and j is

$$Ind_{S_i\times S_{m-i}\times S_j\times S_{n-j}}^{S_m\times S_n}\mathcal{X}(i,j)\times\mathcal{Y}(m-i,n-j)+Ind_{S_i\times S_{m-i}\times S_j\times S_{n-j}}^{S_m\times S_n}\mathcal{X}(i,j)\times\mathcal{Z}(m-i,n-j) =$$

$$= Ind_{S_i\times S_{m-i}\times S_j\times S_{n-j}}^{S_m\times S_n}\mathcal{X}(i,j)\times\mathcal{W}(m-i,n-j),$$

which gives back the i, j piece of $\mathcal{X}\boxtimes(\mathcal{Y}+\mathcal{Z})(m,n)$. □

The ring $K_0(\mathrm{Var}/k,\mathbb{S}^2)$ is an algebra over the subring $K_0(\mathrm{Var}/k)$. Given an $\mathbb{S}$-space $\mathcal{W}$, we define its Hodge–Deligne series as

$$\mathsf{e}(\mathcal{W}) := \sum_{n\geq 0} h_{\mathcal{W}(n)}^{S_n}(u,v)\in\Lambda[[u,v]].$$

Analogously, given an $\mathbb{S}^2$-space $\mathcal{X}$, we define its Hodge–Deligne series by

$$\mathsf{e}(\mathcal{X}) := \sum_{m,n\geq 0} h_{\mathcal{X}(m,n)}^{S_m\times S_n}(u,v)\in\Lambda^{(2)}[[u,v]].$$

For $\mathcal{X},\mathcal{Y}\in K_0(\mathrm{Var}/k,\mathbb{S}^2)$ with $\mathcal{Y}(0,0)=\varnothing$, we can define two composition operations $\circ_1$ and $\circ_2$, which we call 1-plethysm and 2-plethysm, as follows:

$$(\mathcal{X}\circ_1\mathcal{Y})(m,n) = \coprod_{i=0}^{\infty}\coprod_{j=0}^{m} Ind_{S_m\times S_j\times S_{n-j}}^{S_m\times S_n}(\mathcal{X}(i,j)\times\mathcal{Y}^{\boxtimes i}(m,n-j))/S_i, \tag{3.6}$$

and

$$(\mathcal{X}\circ_2\mathcal{Y})(m,n) = \coprod_{j=0}^{\infty}\coprod_{i=0}^{m} Ind_{S_i\times S_{m-i}\times S_n}^{S_m\times S_n}(\mathcal{X}(i,j)\times\mathcal{Y}^{\boxtimes j}(m-i,n))/S_j. \tag{3.7}$$

We called the operations (3.6) and (3.7) plethysms in analogy with the plethysms of $\Lambda^{(2)}$ because of the following property: if $\mathcal{X}, \mathcal{Y}$ are $\mathbb{S}^2$-spaces, then

$$\mathsf{e}(\mathcal{X} \circ_i \mathcal{Y}) = \mathsf{e}(\mathcal{X}) \circ_i \mathsf{e}(\mathcal{Y}) \tag{3.8}$$

for $i = 1, 2$. This holds because the Hodge–Deligne polynomial is a motivic invariant (See Getzler–Pandharipande [GP06] for details, we already described motivic invariants in Section 2.2.2).

Example 3.2.7. We will put $\zeta_r^{(1)}$ for the $\mathbb{S}^2$-space given by the class of $Spec(\mathbb{C})$ with trivial action of $S_r \times S_0$ in arity $(r, 0)$ and $\varnothing$ for every other pair, i.e.:

$$\zeta_r^{(1)}(m,n) := \begin{cases} Spec(\mathbb{C}) & \text{if } (m,n) = (r,0); \\ \varnothing & \text{else.} \end{cases}$$

Analogously $\zeta_r^{(2)}$ is defined as

$$\zeta_r^{(2)}(m,n) := \begin{cases} Spec(\mathbb{C}) & \text{if } (m,n) = (0,r); \\ \varnothing & \text{else.} \end{cases}$$

Note that $\mathsf{e}\left(\zeta_r^{(j)}\right) = h_r^{(j)}$.

We have two analogues of the exponential function, given by

$$Exp^{(i)}(\mathcal{X}) = \sum_{n>0} \zeta_n^{(i)} \circ_i \mathcal{X}$$

for $i = 1, 2$ and $\mathcal{X}(0,0) = \varnothing$. Note that this exponential map commutes with e, i.e.,

$$\mathsf{e}(Exp^{(i)}(\mathcal{X})) = Exp^{(i)}(\mathsf{e}(\mathcal{X})), \tag{3.9}$$

where on the right hand side we have the exponential defined in $\Lambda^{(2)}[[u,v]]$.

Finally, given an $\mathbb{S}$-space $\mathcal{W}$, there are at least three natural ways to view $\mathcal{W}$ as an $\mathbb{S}^2$-space. First, we define

$$\Delta\mathcal{W}(m,n) := Res_{S_m \times S_n}^{S_{m+n}} \mathcal{W}(m+n).$$

By Equation 3.1, we have

$$\mathsf{e}(\Delta\mathcal{W}) = \Delta(\mathsf{e}(\mathcal{W})). \tag{3.10}$$

This is the analogous of the coproduct map coming from the Hopf Algebra structure in Λ.

Next, we put

$$I_1\mathcal{W}(m,n) := \begin{cases} \mathcal{W}(m) & \text{if } n = 0, \\ \varnothing & \text{else} \end{cases} \quad \text{and} \quad I_2\mathcal{W}(m,n) := \begin{cases} \mathcal{W}(n) & \text{if } m = 0, \\ \varnothing & \text{else.} \end{cases},$$

so $\mathsf{e}(I_j\mathcal{W}) = \mathsf{e}(\mathcal{W})^{(j)}$ for $j = 1, 2$. These are the analogous of the inclusions in the first, respectively second component of the tensor product from Λ in $\Lambda^{(2)}$.

Remark 3.2.8. The $\mathbb{S}$-space ζ_r satisfies $I_j(\zeta_r) = \zeta_r^{(j)}$.

For an $\mathbb{S}$-space $\mathcal{W}$, we define also another $\mathbb{S}$-space given by

$$\delta\mathcal{W}(n) := Res_{S_n}^{S_{n+1}} \mathcal{W}(n+1).$$

Note that $\mathsf{e}(\delta\mathcal{W}) = \frac{\partial \mathsf{e}(\mathcal{W})}{\partial p_1}$.

3.2.2 Operations with $\mathbb{S}^2$-spaces

Our main theorem is proven doing computation in the Grothendieck Ring of $\mathbb{S}^2$-spaces and consequently applying the map e. First, for each $g \geq 0$, we define two $\mathbb{S}^2$-spaces as follows:

$$\mathcal{M}_g^{\mathrm{hl}}(m,n) = \begin{cases} M_{g,(1^{(m)}|\varepsilon^{(n)})} & \text{if } 2g-2+m+\min(n,1) > 0, \\ \varnothing & \text{else,} \end{cases}$$

and

$$\overline{\mathcal{M}}_g^{\mathrm{hl}}(m,n) = \begin{cases} \overline{M}_{g,(1^{(m)}|\varepsilon^{(n)})} & \text{if } 2g-2+m+\min(n,1) > 0, \\ \varnothing & \text{else.} \end{cases}$$

We will also make use of the $\mathbb{S}$-spaces $\mathcal{M}_g^{DM}$ and $\overline{\mathcal{M}}_g^{DM}$ given by

$$\mathcal{M}_g(n)^{DM} = \begin{cases} M_{g,n} & \text{if } 2g-2+n > 0, \\ \varnothing & \text{else,} \end{cases}$$

and

$$\overline{\mathcal{M}}_g^{DM}(n) = \begin{cases} \overline{M}_{g,n} & \text{if } 2g-2+n > 0, \\ \varnothing & \text{else.} \end{cases}$$

Remark 3.2.9. Observe that $\mathsf{e}(\overline{\mathcal{M}}_g^{\mathrm{hl}}) = h^{S_m \times S_n}_{\overline{\mathcal{M}}_{g,(1^{(m)}|\varepsilon^{(n)})}}(u,v) = \overline{\mathsf{a}}_g$ is the generating function of the Heavy/Light Hodge–Deligne polynomials, as by definition e applied to an $\mathbb{S}^2$-space gives the series of the Hodge-Deligne polynomials of varieties in each arity (m,n). Analogously, $\mathsf{e}(\mathcal{M}_g^{\mathrm{hl}}) = \mathsf{a}_g$, and for the $\mathbb{S}$-spaces we have $\mathsf{e}(\mathcal{M}_g^{DM}) = \mathsf{b}_g$ and $\mathsf{e}(\overline{\mathcal{M}}_g^{DM}) = \overline{\mathsf{b}}_g$.

Proposition 3.2.10. *We have* $\mathcal{M}_g^{\mathrm{hl}} = \Delta \mathcal{M}_g^{DM} \circ_2 Exp^{(2)}\left(\zeta_1^{(2)}\right)$

Proof. We set $\mathcal{Y} = Exp^{(2)}\left(\zeta_1^{(2)}\right)$. Note that $\zeta_n^{(2)} \circ_2 \zeta_1^{(2)} = \zeta_n^{(2)}$, so $\mathcal{Y}$ is nonempty in arity (m,n) if and only if $m = 0$ and $n \geq 1$, in which case it is equal to $\zeta_n^{(2)}$. We have that $\mathcal{Y}^{\boxtimes j}(m-i,n) = \varnothing$ unless $i = m$ so for any $\mathbb{S}^2$-space $\mathcal{X}$ we have

$$\begin{aligned} (\mathcal{X} \circ_2 \mathcal{Y})(m,n) &= \coprod_{i=0}^{m} \coprod_{j=0}^{\infty} Ind_{S_i \times S_{m-i} \times S_n}^{S_m \times S_n} (\mathcal{X}(i,j) \times \mathcal{Y}^{\boxtimes j}(m-i,n))/S_j \\ &= \coprod_{j=0}^{\infty} \left(\mathcal{X}(m,j) \times \mathcal{Y}^{\boxtimes j}(0,n)\right)/S_j \\ &= \coprod_{j=0}^{\infty} \left(\mathcal{X}(m,j) \times \coprod_{\substack{k_1+\cdots+k_j=n \\ k_r>0\,\forall r}} Ind_{S_{k_1}\times\cdots\times S_{k_j}}^{S_n} Spec(\mathbb{C}) \right) /S_j. \end{aligned}$$

Now consider the $S_m \times S_n$-variety $M_{g,(1^{(m)}|\varepsilon^{(n)})}$, and its stratification deduced in the proof of Proposition 2.2.1. For $1 \leq j \leq n$, let $\mathcal{Z}_{m,j} \subset M_{g,(1^{(m)}|\varepsilon^{(n)})}$ denote the locally closed stratum in which there are precisely j distinct marked points among the last n. Then we can write

$$\mathcal{Z}_{m,j} \cong \left(\coprod_{\substack{k_1+\cdots+k_j=n \\ k_r>0\,\forall r}} Res^{S_{m+j}}_{S_m \times S_j} M_{g,m+j} \times Ind^{S_n}_{S_{k_1}\times\cdots\times S_{k_j}} Spec(\mathbb{C}) \right) /S_j. \tag{3.11}$$

Since $\mathcal{M}^{\mathrm{hl}}_g(m,n) = \sum_{j=1}^n \mathcal{Z}_{m,j}$, we see that $\mathcal{M}^{\mathrm{hl}}_g = \Delta\mathcal{M}^{DM}_g \circ_2 Exp^{(2)}\left(s_1^{(2)}\right)$ upon summing over j, m, and n on both sides of (3.11). □

In order to prove our main theorem for the series of $\overline{\mathcal{M}}^{\mathrm{hl}}_g$, it is useful to introduce an auxiliary space. First, we define what we intend by rational tails on a curve.

Definition 3.2.11. For $(C, p_1, \ldots, p_{m+n}) \in \overline{\mathcal{M}}_{g,m+n}$ (or $\overline{M}_{g,m+n}$), let $T \subset C$ be a union of irreducible components of C. We say T is a rational tail if T is a connected curve of arithmetic genus zero, and T meets $\overline{C \setminus T}$ in a single point.

Definition 3.2.12. Let $S \subseteq \{1, \ldots, m+n\}$. Given $(C, p_1, \ldots, p_{m+n}) \in \overline{\mathcal{M}}_{g,m+n}$ (or $\overline{M}_{g,m+n}$), we say a rational tail $T \subset C$ supports S if for each $i \in S$, we have $p_i \in T$.

Definition 3.2.13. We set $\overline{M}^{(k)}_{g,n} \subset \overline{M}_{g,n}$ to be the locus of curves which have no rational tails whose support consists of any subset of the last k markings, $k \leq n$. We define an $\mathbb{S}^2$-space $\overline{\mathcal{M}}^{\star}_g$ by setting $\overline{\mathcal{M}}^{\star}_g(m,n) := \overline{M}^{(n)}_{g,m+n}$.

The following proposition expresses the $\mathbb{S}^2$-space $\Delta\overline{\mathcal{M}}^{DM}_g$ in terms of $\overline{\mathcal{M}}^{\star}_g$ and the composition operation. The basic idea has appeared in the literature before, in the main theorem of [Get98]; see also [Pet12].

Proposition 3.2.14. *We have* $\Delta\overline{\mathcal{M}}^{DM}_g = \overline{\mathcal{M}}^{\star}_g \circ_2 \left(\zeta_1^{(2)} + I_2\delta\overline{\mathcal{M}}^{DM}_0\right)$.

Proof. Let $\mathcal{X}$ denote the $\mathbb{S}^2$-space on the right-hand side of the claimed equality, and set $\mathcal{Y} = \zeta_1^{(2)} + I_2\delta\overline{\mathcal{M}}^{DM}_0 = I_2\left(\zeta_1 + \delta\overline{\mathcal{M}}^{DM}_0\right)$. Consider first the $\mathbb{S}$-space $\zeta_1 + \delta\overline{\mathcal{M}}^{DM}_0$: this is $Spec(\mathbb{C})$ in arity 1, $\varnothing$ in arity 2, and $\overline{M}_{0,n}$ in arity $n \geq 3$. Then by definition $\mathcal{Y}(m,n) = \varnothing$ unless $m = 0$, and $\mathcal{Y}(0,1) = Spec(\mathbb{C})$, $\mathcal{Y}(0,2) = \varnothing$, and $\mathcal{Y}(0,n) = \overline{M}_{0,n}$ for every $n \geq 3$. Hence, for any $j > 0$, we have that $\mathcal{Y}^{\boxtimes j}(m,n) = \varnothing$ unless $m = 0$. Moreover, a point of the S_n-space $\mathcal{Y}^{\boxtimes j}(0,n)$ corresponds to an ordered tuple of varieties $(X_1, \ldots, X_j)$ such that:

1. for all i, X_i is either $Spec(\mathbb{C})$ or a pointed stable curve of arithmetic genus zero whose marked points are labelled by $\{0, \ldots, r_i\}$ for some $r_i \geq 2$;

2. there is a chosen bijection:

$$\{X_i \mid X_i = Spec(\mathbb{C})\} \cup \{p \mid p \text{ is a nonzero marked point of } X_j \text{ for some } j\} \to \{1, \ldots, n\}.$$

The group S_n acts on the chosen bijection, and S_j acts by reordering the tuple. Our goal is to show that

$$\mathcal{X}(m,n) = \coprod_{j=0}^{\infty} \left(\overline{\mathcal{M}}_g^{\star}(m,j) \times \mathcal{Y}^{\boxtimes j}(0,n) \right) / S_j,$$

where on the right we put the component in arity (m,n) of the 2-plethysm of $\mathbb{S}^2$-spaces. By definition of the map Δ, $\mathcal{X}(m,n)$ is $Res^{S_{m+n}}_{S_m \times S_n} \overline{M}_{g,m+n}$. Now take an ordered tuple $(X_1,\ldots,X_j)$ represented by a point of $\mathcal{Y}^{\boxtimes j}(0,n)$, and a pointed curve in $\overline{\mathcal{M}}_g^{\star}(m,j)$. We glue each element of the tuple X_i to the i-th distinguished marked point among the j's on the chosen curve in $\overline{\mathcal{M}}_g^{\star}(m,j)$. If X_i is a point, this remains a marked point with a label, while if X_i is a rational curve, this has the effect of adding a rational tail which support is a subsets of the final n markings. The quotient by the diagonal action of S_j makes this a point in $Res^{S_{m+n}}_{S_m \times S_n} \overline{M}_{g,m+n}$, thus giving the isomorphism. □

Before proceeding, we now characterize $(1^{(m)}|\varepsilon^{(n)})$-stability in combinatorial terms. Given a rational tail T, we say that an irreducible component E of T is a middle component if $|E \cap \overline{C \setminus E}| = 2$, and we say it is terminal if $|E \cap \overline{C \setminus E}| = 1$. The following lemma determines $(1^{(m)}|\varepsilon^{(n)})$-stability in terms of rational tails.

Lemma 3.2.15. *Let $(C, p_1, \ldots, p_{m+n}) \in \overline{\mathcal{M}}_{g,m+n}$ (or $\overline{M}_{g,m+n}$). Then C is $(1^{(m)}|\varepsilon^{(n)})$-stable if and only if C does not have any rational tails which support only markings with indices in $\{m+1,\ldots,m+n\}$.*

Proof. First assume $(C, p_1, \ldots, p_{m+n})$ is $(1^{(m)}|\varepsilon^{(n)})$-stable. Then each of its irreducible components E satisfies

$$2g(E) - 2 + |(E \cap \overline{C \setminus E}) \cup \mathrm{Sing}(E)| + \sum_{i|p_i \in E} a_i > 0.$$

If T is a rational tail, it must have at least one terminal component. The above inequality reduces to $\sum_{i|p_i\in E} a_i > 1$ for such a component, so the marked points on T cannot be only subset of the last n, since the sum of their weights would be at most 1.

Now assume $(C, p_1, \ldots, p_{m+n})$ is not $(1^{(m)}|\varepsilon^{(n)})$-stable. Let E be a component of C. The inequality

$$2g(E) - 2 + |(E \cap \overline{C \setminus E}) \cup \mathrm{Sing}(E)| + \sum_{i|p_i \in E} a_i > 0,$$

is satisfied if $g(E) \geq 1$ or $g(E) = 0$ with $|(E \cap \overline{C \setminus E})| \geq 2$. Therefore, there must be a rational tail T consisting of a single component such that

$$\sum_{i|p_i \in T} a_i \leq 1,$$

and this can happen only if T supports S with $S \subseteq \{m+1,\ldots m+n\}$. □

The final ingredient of the proof of Theorem 3.2.1 is the following formula, analogous to Proposition 3.2.10.

Proposition 3.2.16. *We have* $\overline{\mathcal{M}}_g^\star \circ_2 Exp^{(2)}\left(\zeta_1^{(2)}\right) = \overline{\mathcal{M}}_g^{\mathrm{hl}}$

Proof. The proof follows the same argument as that of Proposition 3.2.10. We stratify $\overline{M}_{g,(1^{(m)}|\varepsilon^{(n)})}$ by $\mathcal{W}_{m,j} = \{(C,p_1,\ldots,p_{n+m}) \mid$ there are j distinct points among the last $m\}$, and observe that $\mathcal{W}_{m,j} \cong \coprod_{\substack{k_1+\cdots+k_j=n\\ k_r>0\,\forall r}} \left(\overline{M}^{(j)}_{g,m+j} \times Ind^{S_n}_{S_{k_1}\times\cdots\times S_{k_j}} Spec(\mathbb{C})\right)/S_j$, by Lemma 3.2.15. The proof is complete upon summing over m and j. □

We can now prove the main Theorem.

Proof of Theorem 3.2.1. Recall that we want to prove the following two formulas in $\Lambda^2[[u,v]]$:

$$\mathsf{a}_g = \Delta(\mathsf{b}_g) \circ_2 Exp^{(2)}\left(p_1^{(2)}\right)$$

and

$$\bar{\mathsf{a}}_g = \Delta(\bar{\mathsf{b}}_g) \circ_2 \left(p_1^{(2)} - \frac{\partial \mathsf{b}_0^{(2)}}{\partial p_1^{(2)}}\right) \circ_2 Exp^{(2)}\left(p_1^{(2)}\right).$$

For the first part, consider Proposition 3.2.10: we have $\mathcal{M}_g^{\mathrm{hl}} = \Delta\mathcal{M}_g \circ_2 Exp^{(2)}\left(\zeta_1^{(2)}\right)$. We apply $\mathsf{e}(\cdot)$ on both sides: on the right hand side, by Remark 3.2.9 we get a_g. On the left-hand side, we get $\mathsf{e}(\Delta\mathcal{M}_g \circ_2 Exp^{(2)}\left(\zeta_1^{(2)}\right))$, which by Equations (3.8) and (3.10) becomes $\Delta\mathsf{e}(\mathcal{M}_g)\circ_2 \mathsf{e}(Exp^{(2)}\left(\zeta_1^{(2)}\right))$. Again by Remark 3.2.9 and by Equation 3.9 we obtain $\Delta(\mathsf{b}_g) \circ_2 Exp^{(2)}\left(p_1^{(2)}\right)$, which is exactly what we want.

For the second formula, we can apply $\mathsf{e}(\cdot)$ on both sides of Proposition 3.2.14 and apply the same reasoning plus (3.10) to see that

$$\Delta(\bar{\mathsf{b}}_g) = \mathsf{e}(\overline{\mathcal{M}}_g^\star) \circ_2 \left(p_1^{(2)} + \frac{\partial \bar{\mathsf{b}}_0^{(2)}}{\partial p_1^{(2)}}\right), \tag{3.12}$$

as $\mathsf{e}(\zeta_1) = p_1$ and $\mathsf{e}(\delta\overline{\mathcal{M}}_0) = \partial\bar{\mathsf{b}}_0/\partial p_1$. Now as shown in [Get95] the symmetric functions $p_1 + \frac{\partial \bar{\mathsf{b}}_0}{\partial p_1}$ and $p_1 - \frac{\partial \mathsf{b}_0}{\partial p_1}$ are plethystic inverses, i.e.,

$$s \circ (p_1 + \frac{\partial \bar{\mathsf{b}}_0}{\partial p_1}) \circ (p_1 - \frac{\partial \mathsf{b}_0}{\partial p_1}) = s$$

for every $s \in \Lambda$. So this continues to be true for the j-th inclusions in their tensor product $\Lambda^{(2)}$ with respect tothe j-plethysm, $j = 1,2$. We thus perform the operation $\circ_2 \left(p_1^{(2)} - \frac{\partial \mathsf{b}_0^{(2)}}{\partial p_1^{(2)}}\right)$ on both sides of (3.12) to get

$$\Delta(\bar{\mathsf{b}}_g) \circ_2 \left(p_1^{(2)} - \frac{\partial \mathsf{b}_0^{(2)}}{\partial p_1^{(2)}}\right) = \mathsf{e}(\overline{\mathcal{M}}_g^\star).$$

Now consider Proposition 3.2.16: we have $\overline{\mathcal{M}}_g^{\mathrm{hl}} = \overline{\mathcal{M}}_g^{\star} \circ_2 Exp^{(2)}\left(\zeta_1^{(2)}\right)$, so applying $\mathsf{e}(\cdot)$ we have $\mathsf{e}(\overline{\mathcal{M}}_g^{\mathrm{hl}}) = \bar{\mathsf{a}}_g$ on the right by Remark 3.2.9, and on the left we have

$$\mathsf{e}(\overline{\mathcal{M}}_g^{\star} \circ_2 Exp^{(2)}\left(\zeta_1^{(2)}\right)) =$$

$$= \mathsf{e}(\overline{\mathcal{M}}_g^{\star}) \circ_2 Exp^{(2)}(\mathsf{e}\left(\zeta_1^{(2)}\right)) =$$

$$= \Delta(\bar{\mathsf{b}}_g) \circ_2 \left(p_1^{(2)} - \frac{\partial \mathsf{b}_0^{(2)}}{\partial p_1^{(2)}}\right) \circ_2 Exp^{(2)}\left(p_1^{(2)}\right),$$

as we wanted to show. □

Proof of Corollary 3.2.2. To prove the Corollary, one uses the rank morphisms (3.2) and (3.3). We apply rk to both sides of Theorem 3.2.1, and use that

$$\mathrm{rk}\left(Exp^{(2)}\left(p_1^{(2)}\right)\right) = e^y - 1.$$

The corollary follows from the formula

$$\begin{aligned}\mathrm{rk}\left(p_1^{(2)} - \frac{\partial \mathsf{b}_0}{\partial p_1^{(2)}}\right) &= y - \sum_{n\geq 2} h_{\mathcal{M}_{0,n+1}}(u,v)\cdot \frac{y^n}{n!}\\ &= y + \frac{(y+1)^{uv} - uvy - 1}{uv - u^2v^2},\end{aligned}$$

due to Getzler ([Get95]). □

Remark 3.2.17. The first part of Corollary 3.2.2 follows from Corollary 2.2.7, where we show the following decomposition in the Grothendieck Group of varieties:

$$[M_{g,((1^{(m)}|\varepsilon^{(n)}))}] = \sum_{k=1}^{n} S(n,k)[M_{g,m+k}],$$

where $S(n,k)$ is the Stirling number of the second kind. It follows that the generating function a_g can be obtained from $b_g(x+w)$ by the substitution $w = e^y - 1$.

The Euler characteristic of $\overline{M}_{1,\varepsilon^{(n)}}$

In this section we include two byproduct results about the topological Euler characteristic $\chi(\overline{M}_{1,\varepsilon^{(n)}})$, where $\overline{M}_{1,\varepsilon^{(n)}}$ is defined as the space where $m = 0$, i.e were all the points are light. These results can be viewed as corollaries of Theorem 3.2.1 and Getzler's semi-classical approximation, see [Get98]. We compare the space $\overline{M}_{1,\varepsilon^{(n)}}$ with $\overline{M}_{1,n}$, as it parameterizes curves that have no rational tails. The first result determines the generating function for the numbers $\chi(\overline{M}_{1,\varepsilon^{(n)}})$.

Proposition 3.2.18. *Define*

$$f(y) := \sum_{n\geq 1} \chi(\overline{M}_{1,\varepsilon(n)})\frac{y^n}{n!}.$$

Then

$$f(y) = -\frac{y}{12} - \frac{1}{2}log(1-y) + \varepsilon \circ (e^y - 1),$$

where

$$\varepsilon(y) := \frac{1}{12}(19y + 23y^2/2 + 10y^3/3 + y^4/2).$$

Proof. Apply rk to both sides of Theorem 3.2.1 and consider the x-degree 0 part. We obtain an equality

$$\sum_n h_{\overline{\mathcal{M}}_{g,\varepsilon(n)}}(u,v)\frac{y^n}{n!} = \left(\sum_n h_{\overline{\mathcal{M}}_{g,n}}(u,v)\frac{y^n}{n!}\right) \circ \left(y - \sum_{n\geq 2} h_{\mathcal{M}_{0,n+1}}(u,v)\frac{y^n}{n!}\right) \circ (e^y - 1).$$

We substitute $u = v = 1$ and $g = 1$:

$$\sum_n \chi(\overline{M}_{1,\varepsilon(n)})\frac{y^n}{n!} = \left(\sum_n \chi(\overline{M}_{1,n})\frac{y^n}{n!}\right) \circ \left(y - \sum_{n\geq 2} \chi(M_{0,n+1})\frac{y^n}{n!}\right) \circ (e^y - 1).$$

Let

$$g(y) := y + \sum_{n\geq 2} \chi(\overline{M}_{0,n+1})\frac{y^n}{n!},$$

so by [Get95],

$$g \circ \left(y - \sum_{n\geq 2} \chi(M_{0,n+1})\frac{y^n}{n!}\right) = \left(y - \sum_{n\geq 2} \chi(M_{0,n+1})\frac{y^n}{n!}\right) \circ g = y$$

By [Get98, Theorem 4.1] we have

$$\sum_n \chi(\overline{M}_{1,n})\frac{y^n}{n!} = -\frac{1}{12}log(1+g(y)) - \frac{1}{2}log(1 - log(1+g(y))) + \varepsilon(g(y)),$$

so we derive

$$\begin{aligned}
\sum_n \chi(\overline{M}_{1,\varepsilon(n)})\frac{y^n}{n!} &= \left(\sum_n \chi(\overline{M}_{1,n})\frac{y^n}{n!}\right) \circ \left(y - \sum_{n\geq 2} \chi(M_{0,n+1})\frac{y^n}{n!}\right) \circ (e^y - 1) \\
&= \left(-\frac{1}{12}log(1+y) - \frac{1}{2}log(1 - log(1+y)) + \varepsilon(y)\right) \circ (e^y - 1) \\
&= -\frac{y}{12} - \frac{1}{2}log(1-y) + \varepsilon \circ (e^y - 1),
\end{aligned}$$

as claimed. $\square$

The following corollary says informally that eliminating rational tails greatly reduces the topological complexity of the Moduli Space.

Corollary 3.2.19. *We have the asymptotic formula*

$$\chi(\overline{M}_{1,\varepsilon^{(n)}}) \sim \frac{(n-1)!}{2},$$

and

$$\lim_{n\to\infty} \frac{\chi(\overline{M}_{1,\varepsilon^{(n)}})}{\chi(\overline{M}_{1,n})} = 0.$$

Proof. Consider y as a complex variable, then the function $f(y) - (-log(1-y)/2)$ is an entire function. By [Wil94, Theorem 2.4.3], the values $\chi(\overline{M}_{1,\varepsilon^{(n)}})/n!$ are approximated by the power series coefficients of the function $-\frac{1}{2}log(1-y)$ about the origin. Therefore, we have

$$\chi(\overline{M}_{1,\varepsilon^{(n)}}) \sim \frac{(n-1)!}{2}.$$

By [Get98, Corollary 4.2],

$$\chi(\overline{M}_{1,n}) \sim \frac{(n-1)!}{4(e-2)^n}(1 + Cn^{-1/2} + O(n^{-3/2})) \sim \frac{(n-1)!}{4(e-2)^n},$$

where C is a constant. By comparing $\chi(\overline{M}_{1,\varepsilon^{(n)}})$ and $\chi(\overline{M}_{1,n})$ we have:

$$\lim_{n\to\infty} \frac{\chi(\overline{M}_{1,\varepsilon^{(n)}})}{\chi(\overline{M}_{1,n})} = \lim_{n\to\infty} \frac{(n-1)!}{2} \frac{4(e-2)^n}{(n-1)!} = 0,$$

as we wanted to show. □

The above proof indicates that the Euler characteristic of $\overline{M}_{1,n}$ is much bigger than that of $\overline{M}_{1,\varepsilon^{(n)}}$. Indeed, the proof shows that we have

$$\chi(\overline{M}_{1,n}) \sim \frac{1}{2}\left(\frac{1}{e-2}\right)^n \cdot \chi(\overline{M}_{1,\varepsilon^{(n)}}) \approx \frac{(1.3922)^n}{2} \cdot \chi(\overline{M}_{1,\varepsilon^{(n)}}). \tag{3.13}$$

3.2.3 Computations

We conclude the chapter with sample calculations, done with SageMath, obtained using 3.2.1.[1] Table 3.1 contains the $(S_m \times S_n)$-equivariant Hodge polynomials of $\overline{M}_{1,(1^{(m)}|\varepsilon^{(n)})}$ for $m+n \leq 5$. These rely on the calculation of the series $\bar{\mathsf{b}}_1$ by Getzler [Get98]. For $n \leq 10$, the Mixed Hodge Structures on the cohomology groups of the Moduli Space $\overline{M}_{1,n}$ are polynomials in $\mathsf{L} := H^2_c(\mathbb{A}^1;\mathbb{C})$, the Mixed Hodge Structure of the affine line. A consequence is that $\overline{M}_{1,n}$ has only even dimensional cohomology for $n \leq 10$, and only the diagonal Hodge numbers $\dim H^{p,p}$ are nonzero. By Theorem 3.2.1, the same is true for $\overline{M}_{1,(1^{(m)}|\varepsilon^{(n)})}$ for $m+n \leq 10$.

[1] Sage code for these computations is available at `https://drive.google.com/file/d/1ejCm2uu5KzqEfOKsud7pXkc0OYdd5P5g/view`

Therefore, Table 3.1 displays the equivariant Poincaré polynomial $h^{S_m\times S_n}_{\overline{\mathcal{M}}_{g,(1^{(m)}|\varepsilon^{(n)})}}(t,t)$, and the Hodge polynomial can be recovered by setting $t^2 = uv$.

Table 3.2 contains the non-equivariant Hodge polynomials of $\overline{M}_{1,\varepsilon^{(n)}}$ for $n \leq 11$, computed with Corollary 3.2.2 and Getzler's calculation of $\bar{b}_1$. By Corollary 3.2.19 and Equation (3.13), one might expect $\overline{M}_{1,\varepsilon^{(n)}}$ to have less cohomology than $\overline{M}_{1,n}$ (this is not a direct consequence; both spaces may have odd cohomology). Indeed, comparing with the Table [Get98, p.491], we observe that $\dim H^*(\overline{M}_{1,\varepsilon^{(10)}}) = 232,076$ while $\dim H^*(\overline{M}_{1,10}) = 16,275,872$. One also notes that just as in the case of $\overline{M}_{1,11}$, the space $\overline{M}_{1,\varepsilon^{(11)}}$ has odd-dimensional cohomology; this is true of $\overline{M}_{1,1^{(m)}|\varepsilon^{(n)}}$ whenever $m+n=11$.

Table 3.3 contains the $(S_m \times S_n)$-equivariant compactly supported weight zero Euler characteristic of $\mathcal{M}_{2,1^{(m)}|\varepsilon^{(n)}}$ for $m+n \leq 6$, which is equal to $h^{S_m\times S_n}_{\mathcal{M}_{2,(1^{(m)}|\varepsilon^{(n)})}}(0,0)$, the constant term of the Hodge–Deligne polynomial. We also include the numerical weight zero Euler characteristic. This table was computed using the first part of Theorem 3.2.1, together with the formula of Chan, Faber, Galatius and Payne for $h^{S_n}_{\mathcal{M}_{g,n}}(0,0)$ [CFGP19]. We also note that this table and our techniques apply to compute the equivariant Euler characteristic $\chi^{S_m\times S_n}(\Delta_{g,(1^{(m)}|\varepsilon^{(n)})}) := \sum_i (-1)^i ch_{m,n}(H^i(\Delta_{g,(1^{(m)}|\varepsilon^{(n)})};\mathbb{Q})) \in \Lambda^{(2)}$. Indeed, one has $\chi^{S_m\times S_n}(\Delta_{g,(1^{(m)}|\varepsilon^{(n)})}) = s_m^{(1)} s_n^{(2)} - h^{S_m\times S_n}_{\mathcal{M}_{g,(1^{(m)}|\varepsilon^{(n)})}}(0,0)$ when $\Delta_{g,(1^{(m)}|\varepsilon^{(n)})}$ is connected, which holds when $g \geq 1$, and when $g=0$ and $m+n>4$.

We exclude the case $n=1$ from Tables 3.1 and 3.3, as

$$h^{S_m\times S_1}_{\overline{\mathcal{M}}_{g,1(m)|\varepsilon^{(1)}}}(u,v) = \left(\frac{\partial h^{S_{m+1}}_{\overline{M}_{g,m+1}}(u,v)}{\partial p_1}\right)^{(1)} \cdot s_1^{(2)},$$

and the analogous formula holds for the open Moduli Spaces.

(m,n)	$h^{S_m\times S_n}_{\overline{M}_{1,()}}(t,t)$
$(0,2)$	$(t^4+2t^2+1)\,s_2^{(2)}$
$(0,3)$	$(t^4+t^2)\,s_{2,1}^{(2)}+(t^6+2t^4+2t^2+1)\,s_3^{(2)}$
$(1,2)$	$(t^4+t^2)\,s_1^{(1)}s_{1,1}^{(2)}+(t^6+4t^4+4t^2+1)\,s_1^{(1)}s_2^{(2)}$
$(0,4)$	$(t^6+2t^4+t^2)\,s_{2,2}^{(2)}+(t^6+2t^4+t^2)\,s_{3,1}^{(2)}+(t^8+2t^6+3t^4+2t^2+1)\,s_4^{(2)}$
$(1,3)$	$(3t^6+6t^4+3t^2)\,s_1^{(1)}s_{2,1}^{(2)}+(t^8+5t^6+9t^4+5t^2+1)\,s_1^{(1)}s_3^{(2)}$
$(2,2)$	$\left((t^6+2t^4+t^2)\,s_{1,1}^{(1)}+(2t^6+4t^4+2t^2)\,s_2^{(1)}\right)s_{1,1}^{(2)}+\left((2t^6+4t^4+2t^2)\,s_{1,1}^{(1)}+(t^8+7t^6+13t^4+7t^2+1)\,s_2^{(1)}\right)s_2^{(2)}$
$(0,5)$	$(t^6+t^4)\,s_{2,2,1}^{(2)}+(t^8+3t^6+3t^4+t^2)\,s_{3,2}^{(2)}+(t^8+3t^6+3t^4+t^2)\,s_{4,1}^{(2)}+(t^{10}+2t^8+3t^6+3t^4+2t^2+1)\,s_5^{(2)}$
$(1,4)$	$(2t^6+2t^4)\,s_1^{(1)}s_{2,1,1}^{(2)}+(2t^8+8t^6+8t^4+2t^2)\,s_1^{(1)}s_{2,2}^{(2)}$ $+(4t^8+14t^6+14t^4+4t^2)\,s_1^{(1)}s_{3,1}^{(2)}+(t^{10}+6t^8+15t^6+15t^4+6t^2+1)\,s_1^{(1)}s_4^{(2)}$
$(2,3)$	$(t^6+t^4)\,s_{1,1}^{(1)}s_{1,1,1}^{(2)}+(t^6+t^4)\,s_2^{(1)}s_{1,1,1}^{(2)}+\left((2t^8+9t^6+9t^4+2t^2)\,s_{1,1}^{(1)}+(5t^8+20t^6+20t^4+5t^2)\,s_2^{(1)}\right)s_{2,1}^{(2)}$ $+\left((3t^8+11t^6+11t^4+3t^2)\,s_{1,1}^{(1)}+(t^{10}+9t^8+26t^6+26t^4+9t^2+1)\,s_2^{(1)}\right)s_3^{(2)}$
$(3,2)$	$t^{10}s_3^{(1)}s_2^{(2)}+t^8\left(2s_{2,1}^{(1)}+3s_3^{(1)}\right)s_{1,1}^{(2)}+\left(5s_{2,1}^{(1)}+10s_3^{(1)}\right)s_2^{(2)}$ $+t^6\left(\left(s_{1,1,1}^{(1)}+9s_{2,1}^{(1)}+12s_3^{(1)}\right)s_{1,1}^{(2)}+\left(s_{1,1,1}^{(1)}+20s_{2,1}^{(1)}+30s_3^{(1)}\right)s_2^{(2)}\right)+\cdots$

Table 3.1: The $(S_m \times S_n)$-equivariant Poincaré polynomials of $\overline{M}_{1,(1^{(m)}|\varepsilon^{(n)})}$ for $m+n\le 5$. The omitted terms are determined by Poincaré duality.

n	$h_{\overline{M}_{1,\varepsilon(n)}}(u,v)$
1	$uv+1$
2	$u^2v^2+2uv+1$
3	$u^3v^3+4u^2v^2+4uv+1$
4	$u^4v^4+7u^3v^3+13u^2v^2+7uv+1$
5	$u^5v^5+11u^4v^4+35u^3v^3+35u^2v^2+11uv+1$
6	$u^6v^6+16u^5v^5+81u^4v^4+140u^3v^3+81u^2v^2+16uv+1$
7	$u^7v^7+22u^6v^6+168u^5v^5+476u^4v^4+476u^3v^3+168u^2v^2+22uv+1$
8	$u^8v^8+29u^7v^7+323u^6v^6+1456u^5v^5+2458u^4v^4+1456u^3v^3+323u^2v^2+29uv+1$
9	$u^9v^9+37u^8v^8+591u^7v^7+4201u^6v^6+11901u^5v^5+11901u^4v^4+4201u^3v^3+591u^2v^2+37uv+1$
10	$u^{10}v^{10}+46u^9v^9+1051u^8v^8+11850u^7v^7+55975u^6v^6+94230u^5v^5+55975u^4v^4+11850u^3v^3+1051u^2v^2+46uv+1$
11	$u^{11}v^{11}+56u^{10}v^{10}+1848u^9v^9+33451u^8v^8+258940u^7v^7+710512u^6v^6-u^{11}-v^{11}+710512u^5v^5+258940u^4v^4+\cdots$

Table 3.2: The Hodge polynomial of $\overline{M}_{1,\varepsilon(n)}$ for $n \leq 11$.

(m,n)	$h^{S_m\times S_n}_{\mathcal{M}_{2,(1^{(m)}\vert\varepsilon^{(n)})}}(0,0)=\sum_{k=0}^{6+2(m+n)}(-1)^k\,\mathrm{ch}_{m,n}\left(W_0H_c^k\left(M_{2,(1^{(m)}\vert\varepsilon^{(n)})};\mathbb{C}\right)\right)$	$h_{\mathcal{M}_{2,(1^{(m)}\vert\varepsilon^{(n)})}}(0,0)$
$(0,2)$	$-s_2^{(2)}$	-1
$(0,3)$	$-s_{2,1}^{(2)}-s_3^{(2)}$	-3
$(1,2)$	$-s_1^{(1)}s_2^{(2)}$	-1
$(0,4)$	$-s_{2,1,1}^{(2)}-s_{2,2}^{(2)}-s_{3,1}^{(2)}-s_4^{(2)}$	-9
$(1,3)$	$-s_1^{(1)}\left(s_{1,1,1}^{(2)}+s_{2,1}^{(2)}\right)$	-3
$(2,2)$	$\left(-s_{1,1}^{(1)}-s_2^{(1)}\right)s_{1,1}^{(2)}+\left(-s_{1,1}^{(1)}+s_2^{(1)}\right)s_2^{(2)}$	-2
$(0,5)$	$-s_{2,1,1,1}^{(2)}-s_{2,2,1}^{(2)}-s_{3,1,1}^{(2)}-s_{3,2}^{(2)}-s_{4,1}^{(2)}-s_5^{(2)}$	-25
$(1,4)$	$-s_1^{(1)}s_{2,1,1}^{(2)}$	-3
$(2,3)$	$s_2^{(1)}\left(s_{1,1,1}^{(2)}+s_3^{(2)}\right)+\left(2s_2^{(1)}-s_{1,1}^{(1)}\right)s_{2,1}^{(2)}$	4
$(3,2)$	$\left(s_{2,1}^{(1)}+2s_3^{(1)}\right)s_{1,1}^{(2)}+\left(s_{2,1}^{(1)}+2s_3^{(1)}\right)s_2^{(2)}$	8
$(0,6)$	$-s_{1,1,1,1,1,1}^{(2)}-s_{2,1,1,1,1}^{(2)}-s_{2,2,1,1}^{(2)}-s_{2,2,2}^{(2)}-s_{3,1,1,1}^{(2)}-s_{3,2,1}^{(2)}-s_{4,1,1}^{(2)}-s_{4,2}^{(2)}-s_{5,1}^{(2)}-s_6^{(2)}$	-71
$(1,5)$	$-s_1^{(1)}\left(2s_{1,1,1,1,1}^{(2)}+s_{2,1,1,1}^{(2)}+s_{2,2,1}^{(2)}\right)$	-11
$(2,4)$	$\left(-3s_{1,1}^{(1)}-s_2^{(1)}\right)s_{1,1,1,1}^{(2)}+\left(-4s_{1,1}^{(1)}+2s_2^{(1)}\right)s_{2,1,1}^{(2)}+\left(-s_{1,1}^{(1)}-s_2^{(1)}\right)s_{2,2}^{(2)}+\left(-s_{1,1}^{(1)}+s_2^{(1)}\right)s_{3,1}^{(2)}$	-14
$(3,3)$	$\left(-5s_{1,1,1}^{(1)}-3s_{2,1}^{(1)}+s_3^{(1)}\right)s_{1,1,1}^{(2)}+\left(-2s_{1,1,1}^{(1)}-4s_{2,1}^{(1)}\right)s_{2,1}^{(2)}+\left(s_{1,1,1}^{(1)}-s_{2,1}^{(1)}-s_3^{(1)}\right)s_3^{(2)}$	-32
$(4,2)$	$\left(-3s_{1,1,1,1}^{(1)}-5s_{2,1,1}^{(1)}-3s_{2,2}^{(1)}-2s_{3,1}^{(1)}\right)s_{1,1}^{(2)}+\left(-s_{1,1,1,1}^{(1)}-5s_{2,2}^{(1)}-2s_{3,1}^{(1)}-3s_4^{(1)}\right)s_2^{(2)}$	-50

Table 3.3: The $(S_m\times S_n)$-equivariant and numerical weight zero compactly supported Euler characteristics of $\mathcal{M}_{2,(1^{(m)}\vert\varepsilon^{(n)})}$.

Chapter 4

Wall Crossing and filtrations of Tropical Moduli Spaces

Here we describe the results of [Ser22] concerning the wall-crossing study of stability conditions defining Tropical Moduli Spaces, their Graph Complexes and how this phenomenon affects the Top Weight Cohomology of $\mathcal{M}_{g,\mathcal{A}}$.

4.1 Wall crossing for weight data

4.1.1 Chambers decompositions

In this section we study the chamber decompositions of the space of weight data $\mathcal{D}_{g,n}$ (defined on page 8) and how it influences the behaviour of the Moduli Spaces of Tropical curves, taking inspiration on the analogous study pursued by Hassett in [Has03] for Moduli Spaces of Algebraic Curves.

A chamber decomposition of $\mathcal{D}_{g,n}$ consists of a finite set W of hyperplanes of $\mathbb{R}^n$, called walls of the decomposition. The chambers of the decomposition are the connected components of

$$\mathcal{D}_{g,n} \setminus \bigcup_{w \in W} w.$$

Two chamber decomposition give the same chambers if we can obtain one from the other by excluding the hyperplanes which are tangent or does not intersect $\mathcal{D}_{g,n}$. Indeed, we will work only with chamber decomposition where each wall intersects properly $\mathcal{D}_{g,n}$.

There are two meaningful chamber decompositions for our problems defined in [Has03]. The first is the fine chamber decomposition:

$$W_f = \left\{ \sum_{j \in S} a_j = 1 : S \subset \{1, ..., n\}, 2 \leq |S| \leq n - 2\delta_{g,0} \right\}$$

where $\delta_{i,j}$ is the the Kronecker delta, and we will denote the set of chambers of this decomposition with $\mathbf{K}_f$, referring to them as fine chambers (note that W_f and $\mathbf{K}_f$ both depend on g and n, but we avoid to recall this to lighten the notation).

The second is the coarse chamber decomposition, defined as

$$W_c = \left\{ \sum_{j \in S} a_j = 1 : S \subset \{1, ..., n\}, 2 < |S| \leq n - 2\delta_{g,0} \right\}.$$

We will denote the set of the chambers of this decomposition with $\mathbf{K}_c$, referring to them as coarse chambers. Notice that $W_c \subset W_f$.

Consider $\{a_1, ..., a_n\}$ as variables. A wall in W_f or W_c with equation $a_{i_1} + ... + a_{i_m} = 1$ divides $\mathcal{D}_{g,n}$ in two connected components defined by inequalities

$$a_{i_1} + ... + a_{i_m} > 1 \text{ and } a_{i_1} + ... + a_{i_m} < 1.$$

Given a chamber Ch in one of the two chamber decompositions we are considering, we can observe that it is defined by a set of inequalities: each inequality is associated to a wall, and indicates in which of the two components determined by that wall the elements of Ch lie.

Example 4.1.1. Let $g \geq 1$ and $n = 2$. We have

$$\mathcal{D}_{g,2} = \{(a_1, a_2) \in ((0, 1] \cup \mathbb{Q})^2 : 0 < a_i \leq 1\} \subset (0, 1]^2.$$

Since $n = 2$, we have only a possible $w \in W_f$, i.e., only a wall given by $a_1 + a_2 = 1$, and we get only two chambers as shown in Figure 4.1. In this case W_c is empty.

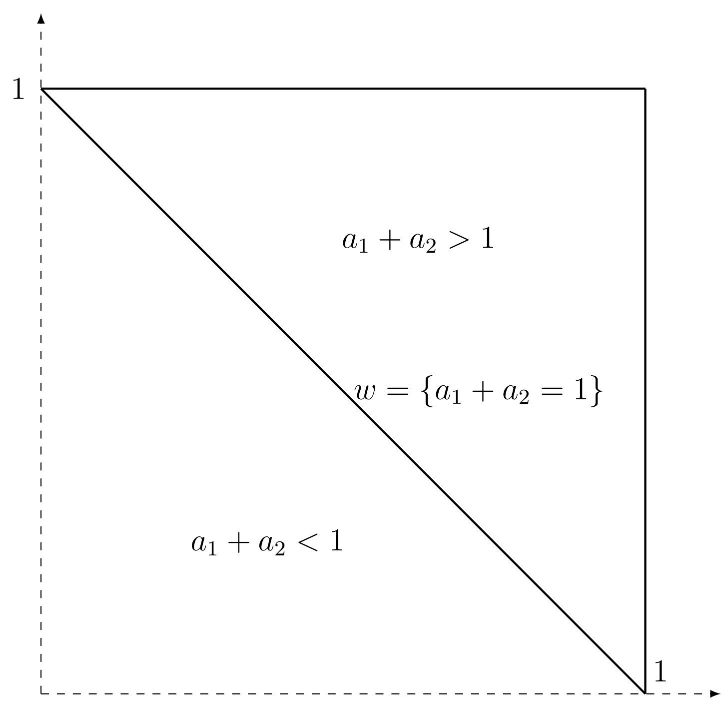

Figure 4.1: The space $\mathcal{D}_{g,2}$ for $g \geq 1$ is contained in $(0, 1]^2$, and has two fine chambers, while since W_c is empty, it as a unique coarse chamber made by itself.

When $g = 0$, we have the condition $a_1 + a_2 > 2$, which is impossible since the a_i are smaller or equal than 1, so there are no admissible input data. This reflects the fact that there are no stable rational curves (either tropical or algebraic) with only two marked points.

Example 4.1.2. Let $g \geq 1$ and $n = 3$. Here $\mathcal{D}_{g,3} \subset (0,1]^3$. We have

$$W_f = \{\{a_1 + a_2 = 1\}, \{a_1 + a_3 = 1\}, \{a_2 + a_3 = 1\}, \{a_1 + a_2 + a_3 = 1\}\},$$

which are all planes in $\mathbb{R}^3$. The chambers of the fine decomposition are defined by the following sets of inequalities:

$$Ch_1 := \begin{cases} a_1 + a_2 > 1 \\ a_1 + a_3 > 1 \\ a_2 + a_3 > 1 \\ a_1 + a_2 + a_3 > 1 \text{(which is implied by the three above)} \end{cases}$$

$$Ch_2 := \begin{cases} a_1 + a_2 < 1 \\ a_1 + a_3 > 1 \\ a_2 + a_3 > 1 \\ a_1 + a_2 + a_3 > 1 \end{cases} ; Ch_3 := \begin{cases} a_1 + a_2 > 1 \\ a_1 + a_3 < 1 \\ a_2 + a_3 > 1 \\ a_1 + a_2 + a_3 > 1 \end{cases}$$

$$Ch_4 := \begin{cases} a_1 + a_2 > 1 \\ a_1 + a_3 > 1 \\ a_2 + a_3 < 1 \\ a_1 + a_2 + a_3 > 1 \end{cases} ; Ch_5 := \begin{cases} a_1 + a_2 < 1 \\ a_1 + a_3 < 1 \\ a_2 + a_3 > 1 \\ a_1 + a_2 + a_3 > 1 \end{cases}$$

$$Ch_6 := \begin{cases} a_1 + a_2 < 1 \\ a_1 + a_3 > 1 \\ a_2 + a_3 < 1 \\ a_1 + a_2 + a_3 > 1 \end{cases} ; Ch_7 := \begin{cases} a_1 + a_2 > 1 \\ a_1 + a_3 < 1 \\ a_2 + a_3 < 1 \\ a_1 + a_2 + a_3 > 1 \end{cases}$$

$$Ch_8 := \begin{cases} a_1 + a_2 < 1 \\ a_1 + a_3 < 1 \\ a_2 + a_3 < 1 \\ a_1 + a_2 + a_3 > 1 \end{cases} ; Ch_9 := \begin{cases} a_1 + a_2 + a_3 < 1 \text{(which implies the three below)} \\ a_1 + a_2 < 1 \\ a_1 + a_3 < 1 \\ a_2 + a_3 < 1 \end{cases} .$$

When $g = 0$, we have

$$\mathcal{D}_{0,3} = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 0 < a_i \leq 1, a_1 + a_2 + a_3 > 2\},$$

and W_f becomes the empty set, since the condition $2 \leq |S| \leq 1$ is impossible. So there is only a non-empty chamber without walls. Observe that with this parameters

$$W_c = \{\{a_1 + a_2 + a_3 = 1\}\},$$

so there are only two chambers in $\mathcal{D}_{g,3}$ for $g \geq 1$ similarly to what happened in the previous example for the fine decomposition.

From Example 4.1.2, we can notice that there are inequalities which are not independent. Namely if $\sum_{j\in S} a_j < 1$ then it has to be that $\sum_{j\in S'} a_j < 1$ for every $S' \subset S$. At the same time if there is $S' \subset S$ such that $\sum_{j\in S'} a_j > 1$, then $\sum_{j\in S} a_j > 1$. Moreover, whenever we have two sets S and T such that $S \cap T = \varnothing$ and both $\sum_{j\in S} a_j > 1$ and $\sum_{j\in T} a_j > 1$, then clearly $\sum_{j\in S\cup T} a_j > 2$, so if there is a $T' \subset S \cup T$ such that $\sum_{j\in T'} a_j < 1$, then $\sum_{j\in (S\cup T)-T'} a_j > 1$.

Example 4.1.3. Suppose we have $n = 4$ and the defining inequalities $a_1+a_3 > 1$, $a_2+a_4 > 1$, and $a_1 + a_2 < 1$. Then $a_1 + a_2 + a_3 + a_4 > 2$, and so it must be that $a_3 + a_4 > 1$, i.e., this inequality is forced by the others, otherwise we obtain an empty chamber.

By the work of Hassett [Has03] we already know some wall-crossing properties for his moduli spaces, summarized in the following Theorem.

Proposition 4.1.4. *([Has03], Proposition 5.1) The coarse chamber decomposition is the coarsest decomposition of $\mathcal{D}_{g,n}$ such that $\mathcal{M}_{g,\mathcal{A}}$ is constant on each chamber. The fine chamber decomposition is the coarsest decomposition of $\mathcal{D}_{g,n}$ such that the universal curve $\mathcal{C}_{g,\mathcal{A}}$ is constant on each chamber.*

Chambers up to symmetry

Let us now consider the natural action of S_n on $\mathcal{D}_{g,n}$ given by permuting the entries of a weight datum $\mathcal{A} \in \mathcal{D}_{g,n}$:

$$\begin{aligned} S_n \times \mathcal{D}_{g,n} &\longrightarrow \mathcal{D}_{g,n} \\ (\sigma, (a_1, ..., a_n)) &\longmapsto (a_{\sigma(1)}, ..., a_{\sigma(n)}). \end{aligned}$$

Let us take $\sigma \in S_n$, and denote by $\sigma(\mathcal{A})$ the weight datum obtained applying the permutation σ to the entries of $\mathcal{A}$. The action of S_n on $\mathcal{D}_{g,n}$ induces an action on $\mathbf{K}_f$ and $\mathbf{K}_c$, since a chamber is sent into another chamber by permutation of the coordinates. Under this action, two chambers are in the same orbit if there is a permutation in S_n which sends all the inequalities defining the first chamber in the inequalities defining the second one by permuting the indices of the variables. In particular, note that for a given Ch in one of the two chamber sets $\mathbf{K}_f$ and $\mathbf{K}_c$, we have $\sigma(Ch_{\mathcal{A}}) = Ch_{\sigma(\mathcal{A})}$. We call the orbits of this action chambers up to symmetry, we denote by $[Ch]$ the chamber up to symmetry of the chamber Ch, and we denote the set of all the chambers up to symmetry by $[\mathbf{K}_f]$ (respectively $[\mathbf{K}_c]$).

Example 4.1.5. Let $g \geq 1$, $n = 3$. There are five orbits in the fine chamber decomposition, namely

$$[\mathbf{K}_f] = \{\{Ch_1\}, \{Ch_2, Ch_3, Ch_4\}, \{Ch_5, Ch_6, Ch_7\}, \{Ch_8\}, \{Ch_9\}\}.$$

To see how to get an orbit, let us compute for example $\{Ch_2, Ch_3, Ch_4\}$. By the inequalities of Example 4.1.2, we can see that $\sigma(Ch_2) = Ch_3$ for $\sigma = (2\ \ 3), (1\ \ 3\ \ 2)$ and $\tau(Ch_2) = Ch_4$ for $\tau = (1\ \ 3), (1\ \ 2\ \ 3)$, while id and $(1\ \ 2)$ fix it. Analogously, we can see that $\theta(Ch_3) = Ch_4$ for $\theta = (1\ \ 2), (1\ \ 3\ \ 2)$. To find a permutation σ' such that $\sigma'(Ch_3) = Ch_2$ one can just pick the inverse of a permutation of one of the σ's, and analogously for τ' and θ'.

Definition 4.1.6. Let $\mathcal{A} = (a_1, ..., a_n)$ and $\mathcal{B} = (b_1, ..., b_n) \in \mathcal{D}_{g,n}$ be two weight data such that $a_i \leq b_i$ for every i from 1 to n. Then we write $\mathcal{A} \leq \mathcal{B}$.

We now focus on some technical results concerning weight data and chambers. We say that two chambers are adjacent if there is only a wall dividing them: in terms of inequalities, this means that there is only a subset $S \subset \{1, ..., n\}$ such that weight data in the first chamber satisfy $\sum_{i\in S} a_i < 1$, and weight data in the second chamber satisfy the opposite inequality $\sum_{i\in S} a_i > 1$, while for every other $S' \neq S$ the induced defining inequality has the same direction for the two chambers or, in other words, they lie in the same half space induced by S' on $\mathcal{D}_{g,n}$. We denote by $Ch_1|Ch_2$ the portion of the wall $w_S := \{\sum_{i\in S} a_i = 1\} \subset \overline{\mathcal{D}}_{g,n}$ dividing these two chambers, i.e., the subset of w_S which verifies all the common defining inequalities of the two chambers. Here with $\overline{\mathcal{D}}_{g,n}$ we denote its closure with respect to the real topology as subspace of $\mathbb{R}^n$, i.e., $\overline{\mathcal{D}}_{g,n} = (0, 1]^n$ when $g \neq 0$, or

$$\overline{\mathcal{D}}_{0,n} = \{(a_1, \ldots, a_n) \in (0, 1]^n | a_1 + \cdots + a_n > 2\}.$$

Lemma 4.1.7. *Let Ch_1 and Ch_2 be two adjacent chambers in either $\mathbf{K}_f$ or $\mathbf{K}_c$. The set $Ch_1|Ch_2$ is not empty.*

Proof. Let S be the subset of $\{1, ..., n\}$ indexing the variables appearing in the defining inequality which differ between Ch_1 and Ch_2. Suppose that weight data in Ch_1 verify $\sum_{i\in S} a_i < 1$ and weight data in Ch_2 verify $\sum_{i\in S} a_i > 1$. Let $\mathcal{X} = (x_1, ..., x_n)$ in Ch_1 and $\mathcal{Y} = (y_1, ..., y_n)$ in Ch_2. Consider the segment between $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{D}_{g,n}$: each point of the segment can be described by $P_t = (1 - t)\mathcal{X} + t\mathcal{Y}$ for $t \in [0, 1]$.

Now consider the function $f : [0, 1] \to \mathbb{R}$ sending t into $(1 - t)\sum_{i\in S} x_i + t\sum_{i\in S} y_i - 1$: we have

$$f(0) = \sum_{i\in S} x_i - 1 < 0$$

since $\mathcal{X}$ is in Ch_1, while

$$f(1) = \sum_{i\in S} y_i - 1 > 0$$

since $\mathcal{Y} = (y_1, ..., y_n)$ is in Ch_2. Then there must be a $t_0 \in [0, 1]$ such that $f(t_0) = 0$, and this implies that P_{t_0} belongs to the wall w_S.

Now let $T \neq S$ be a subset of $\{1, ..., n\}$, and suppose $\sum_{i\in T} a_i > 1$ for each point in Ch_1 and Ch_2. Then

$$(1 - t)\sum_{i\in T} x_i + t\sum_{i\in T} y_i > (1 - t) + t = 1,$$

for every t from 0 to 1.

Analogously if we pick $T' \neq S$ such that $\sum_{i\in T'} a_i < 1$ for each point in Ch_1 and Ch_2, then

$$(1 - t)\sum_{i\in T} x_i + t\sum_{i\in T} y_i < (1 - t) + t = 1,$$

again for every t from 0 to 1.

So in particular P_{t_0} verifies all the common defining inequalities of the two chambers and belongs to the wall w_S, hence it lies in $Ch_1|Ch_2$. $\square$

Proposition 4.1.8. *Let Ch_1 and Ch_2 be two adjacent chambers in either $\mathbf{K}_f$ or $\mathbf{K}_c$, with $S \subset \{1, ..., n\}$ such that data in Ch_1 satisfy $\sum_{i \in S} a_i < 1$, while data in Ch_2 satisfy $\sum_{i \in S} a_i > 1$, and for every other $T \neq S$ the corresponding inequalities agree for both chambers. Then there are $\mathcal{A} \in Ch_1$ and $\mathcal{B} \in Ch_2$ such that $\mathcal{A} \leq \mathcal{B}$.*

Proof. By Lemma 4.1.1 the set $Ch_1|Ch_2$ is non empty, so we can choose

$$\mathcal{X} := (x_1, ..., x_n) \in Ch_1|Ch_2.$$

Let $\varepsilon \in \mathbb{Q}$ be such that

$$\varepsilon < \min_{T \neq S, T \subset [n]} |\sum_{i \in T} x_i - 1|.$$

Then we can pick $\mathcal{A} = (x_1, ..., x_i - \varepsilon, ..., x_n)$ and $\mathcal{B} = (x_1, ..., x_i + \varepsilon, ..., x_n)$ for some $i \in S$.

Assume both have all rational components, then these two weight data verify all the common defining inequalities of Ch_1 and Ch_2. Indeed, if we choose a subset $T \neq S$ of $\{1, ..., n\}$ such that $\sum_{i \in T} x_i < 1$, then

$$\sum_{i \in T} a_i = \sum_{i \in S} x_i - \varepsilon < 1 \text{ and } \sum_{i \in T} b_i = \sum_{i \in T} x_i + \varepsilon < 1$$

since $\sum_{i \in T} x_i < 1 - \varepsilon$ by how we pick ε.

Analogously if we choose $T \neq S$ such that $\sum_{i \in T} x_i > 1$, then

$$\sum_{i \in T} b_i = \sum_{i \in T} x_i + \varepsilon > 1 \text{ and } t \sum_{i \in S} a_i = \sum_{i \in T} x_i - \varepsilon > 1$$

since $\sum_{i \in T} x_i > 1 + \varepsilon$, so $\mathcal{A}$ and $\mathcal{B}$ verify all the common defining inequalities of Ch_1 and Ch_2. Moreover $\sum_{i \in S} a_i < 1$ and $\sum_{i \in S} b_i > 1$, so it follows that $\mathcal{A} \in Ch_1$, $\mathcal{B} \in Ch_2$, and $\mathcal{A} \leq \mathcal{B}$ by construction.

If $\mathcal{A} = (x_1, ..., x_i - \varepsilon, ..., x_n)$ has irrational components, we can find a weight datum $\mathcal{A}'$ in Ch_1 round down each the irrational components x_i with a rational number x_i' such that $x_i - x_i' < \varepsilon$. The same can be done with $\mathcal{B}$ rounding up, so that we have $\mathcal{B}' \in Ch_2$ and by construction $\mathcal{A}' \leq \mathcal{B}'$, hence the assumption we made before can always be performed, so the proof follows. □

Definition 4.1.9. Let $Ch_1, Ch_2 \in \mathbf{K}_f$. We say that $Ch_1 \leq Ch_2$ if they are equal or there are $S_1, ..., S_t \subset \{1, ..., n\}$ such that for every $\mathcal{A} \in Ch_1$ we have $\sum_{i \in S_j} a_i < 1$ and for every $\mathcal{B} \in Ch_2$ we have $\sum_{i \in S_j} b_i > 1$ for every j from 1 to t, and given any $S' \neq S_j$ for every j from 1 to t, the two chambers are in the same half-space induced by the wall

$$\{\sum_{i \in S'} a_i = 1\}$$

on $\mathcal{D}_{g,n}$, i.e., all the other defining inequalities agree.

Informally we are asking that to move from Ch_1 to Ch_2 we need to cross t walls changing the direction of t inequalities from left (< 1) to right (> 1), without changing others.

Proposition 4.1.10. *The relation defined in Definition 4.1.9 is a partial order on* $\mathbf{K}_f$ *(respectively* $\mathbf{K}_c$*).*

Proof. The relation is reflexive by definition. It is also clearly transitive: let $Ch_1 \leq Ch_2$ and $Ch_2 \leq Ch_3$, and let $S_1, ..., S_t$ be the subsets of $\{1, ..., n\}$ on which the defining inequalities of Ch_1 and Ch_3 disagree. Fix a j from 1 to t. If the inequalities corresponding to S_j agree for Ch_1 and Ch_2, then it has to be that $\sum_{i \in S_j} a_i < 1$ for both, since it has to disagree with the inequality corresponding to S_j of Ch_3 and $Ch_2 \leq Ch_3$ by hypothesis. If the inequalities corresponding to S_j disagree for Ch_1 and Ch_2, then it has to be that $\sum_{i \in S_j} a_i < 1$ for Ch_1 and $\sum_{i \in S_j} a_i > 1$ for Ch_2, since we have $Ch_1 \leq Ch_2$. But then it has to be that $\sum_{i \in S_j} a_i > 1$ also for Ch_3, otherwise it can not be that $Ch_2 \leq Ch_3$, and so we have $Ch_1 \leq Ch_3$.

For antisymmetry, let $Ch_1 \leq Ch_2$ and $Ch_2 \leq Ch_1$, and suppose they are different. By the first relation we get that there are $S_1, ..., S_q$ such that their inequalities agree for $S' \neq S_j$, for $j = 1, ..., q$ and for them $\sum_{i \in S_j} a_i < 1$ in Ch_1 and $\sum_{i \in S_j} a_i > 1$ in Ch_2. On the other hand, the second equation gives that there are $T_1, ..., T_r$ such that their inequalities agree for $S' \neq T_j$, and for them $\sum_{i \in T_j} a_i < 1$ in Ch_2 and $\sum_{i \in T_j} a_i > 1$ in Ch_1. But this is impossible, since such T_j's can not exist by the first relation, so they must be the same chamber. $\square$

The partial order we just defined on $\mathbf{K}_f$ and $\mathbf{K}_c$ naturally induces a partial order on $[\mathbf{K}_f]$ and $[\mathbf{K}_c]$.

Definition 4.1.11. Let $[Ch_1], [Ch_2] \in [\mathbf{K}_f]$ (respectively $[\mathbf{K}_c]$), we say $[Ch_1] \leq [Ch_2]$ if there are two chambers $Ch_1 \in [Ch_1]$ and $Ch_2 \in [Ch_2]$ such that $Ch_1 \leq Ch_2$.

Then by Proposition 4.1.10 we have the following.

Corollary 4.1.12. *The relation defined in Definition 4.1.11 is a partial order on the set* $[\mathbf{K}_f]$ *(respectively* $[\mathbf{K}_c]$*).*

Both the number of chambers and the number of chambers up to symmetry are finite, as shown in [ADGH20]. For every g, there is a unique maximal chamber, namely the one given by the weight datum $1^{(n)}$. It is easy to see that it is the only element in the orbit of the action of S_n, and consequently $[Ch_{1^{(n)}}]$ is also the maximum with respect to the partial order on chambers up to symmetry. When $g \geq 1$, there is also a minimal chamber (up to symmetry) which contains all the admissible weight data $\mathcal{A} \leq \frac{1}{n}^{(n)}$. Also in this case, the orbit of the minimal chamber is made only by itself. From now on when $g \geq 1$, we denote by $\mathcal{E}$ a generic weight datum in the minimal chamber, which is denoted by $Ch_{\mathcal{E}}$.

4.1.2 An algorithm to compare weight data

Given two arbitrary weight data $\mathcal{A} = (a_1, ..., a_n)$ and $\mathcal{B} = (b_1, ..., b_n)$ in $\mathcal{D}_{g,n}$, it is possible to describe an algorithm which compares their chambers up to symmetry with respect to the order we put on them, and also says whenever they are not comparable. We describe the algorithm over the set $\mathbf{K}_f$: of course, the same algorithm works for $\mathbf{K}_c$ by considering the condition $2 < |S| \leq n - 2\delta_{g,0}$ instead of $2 \leq |S| \leq n - 2\delta_{g,0}$ each time we do an iteration over the set.

The procedure is the following:

1) For a given input n, consider the group of permutations

$$S_n = \{\sigma_1 = id, ..., \sigma_k, ..., \sigma_{n!}\}.$$

Consider also the inputs $\mathcal{A}$ and $\mathcal{B}$, and denote by $\mathcal{A}_k = (a_{1,k}, ..., a_{n,k})$ the datum $\sigma_k(\mathcal{A})$.

2) For each set $S \subset \{1, ..., n\}$ such that $2 \leq |S| \leq n - 2\delta_{g,0}$, we compute the sums $\sum_{i\in S} b_i$.

The index k will count the iterations of the algorithm. So to start the iteration here we set $k = 1$ and $\mathcal{A}_1 = \mathcal{A}$.

3) For each set $S \subset \{1, ..., n\}$ such that $2 \leq |S| \leq n - 2\delta_{g,0}$, we compute the sums $\sum_{i\in S} a_{i,k}$.

Then we can have the following outputs:

3.1) If the condition $\sum_{i\in S} a_{i,k} \leq 1$ holds if and only if the condition $\sum_{i\in S} b_i \leq 1$ holds (and of course $\sum_{i\in S} a_{i,k} > 1$ if and only if $\sum_{i\in S} b_i > 1$), then $[Ch_{\mathcal{A}}] = [Ch_{\mathcal{B}}]$;

3.2) If there are $S_1, ..., S_d$ such that $\sum_{i\in S_j} a_{i,k} \leq 1$ and $\sum_{i\in S_j} b_i > 1$, while for every other $S \neq S_j$ for every $j = 1, ..., d$ we have $\sum_{i\in S} a_{i,k} \leq 1$ if and only if $\sum_{i\in S} b_i \leq 1$, then $[Ch_{\mathcal{A}}] \leq [Ch_{\mathcal{B}}]$. If the same happens but with the roles of $\mathcal{A}_k$ and $\mathcal{B}$ are reversed then $[Ch_{\mathcal{B}}] \leq [Ch_{\mathcal{A}}]$

3.3) If there are S, T such that $\sum_{i\in S} a_{i,k} \leq 1$ and $\sum_{i\in T} b_i > 1$ while $\sum_{i\in T} a_{i,k} > 1$ and $\sum_{i\in T} b_i \leq 1$, we let the index k grow by one.

   If $k \leq n!$ we restart from the point (3) of the algorithm.

   When $k = n! + 1$, we can conclude that $[Ch_{\mathcal{A}}]$ and $[Ch_{\mathcal{B}}]$ are not comparable in the partial order.

When n grows, this algorithm is not really efficient as it needs an extremely large number of operations: in the worst case, if $g \geq 1$ we have to compute $(2^n - n)(n! + 1)$ sums and make $(2^n - n)n!$ comparisons of results, while if $g = 0$ the number of sums is $(2^n - 2n - 1)(n! + 1)$, with $(2^n - 2n - 1)n!$ comparisons.

Remark 4.1.13. Notice that in the case of comparable chambers up to symmetry a smarter choice of the permutation can reduce the number of the operations, but they will never fall below the number of operations of the best case, which is when we need a single iteration. In this case the number of sums is $2(2^n - n)$ if $g \geq 1$ and $2(2^n - 2n - 1)$ if $g = 0$, while the number of comparisons is $(2^n - n)$ if $g \geq 1$ and $(2^n - 2n - 1)$ if $g = 0$.

4.2 Filtration Theorem

First of all we remark the following property:

Remark 4.2.1. Let $(g, \mathcal{A} = (a_1, ...a_n))$ and $(g, \mathcal{B} = (b_1, ...b_n))$ be two input data such that $\mathcal{A} \leq \mathcal{B}$. Then a $(g, \mathcal{A})$-stable graph is always $(g, \mathcal{B})$-stable, because for every vertex $v \in V(G)$ we have

$$0 < 2w(v) - 2 + |v|_E + |v|_{\mathcal{A}} \leq 2w(v) - 2 + |v|_E + |v|_{\mathcal{B}}.$$

In particular, for any weight data $\mathcal{A}$, a graph that is $(g, \mathcal{A})$-stable is always stable in the standard sense, since $\mathcal{A} \leq 1^{(n)}$ for every weight datum $\mathcal{A}$.

Here in this Section we prove the following Theorem

Theorem 4.2.2. *Let $g \geq 0$, $n \geq 1$ be two integers, and $\mathcal{A} \in \mathcal{D}_{g,n}$ a weight datum. There are filtrations of $M^{trop}_{g,\mathcal{A}}$ given by embeddings induced by the partial order on the set of chambers up to symmetry $[\mathbf{K}_f]$ (respectively $\mathbf{K}_c$). Namely, a ordered sequence*

$$[Ch_{\mathcal{A}_1}] \leq [Ch_{\mathcal{A}_2}] \leq ... \leq [Ch_{\mathcal{A}_p}] \leq ... \leq [Ch_{\mathcal{A}_{N-1}}] \leq [Ch_{\mathcal{A}}]$$

induces a filtration

$$M^{trop}_{g,\mathcal{A}_1} \hookrightarrow M^{trop}_{g,\mathcal{A}_2} \hookrightarrow ... \hookrightarrow M^{trop}_{g,\mathcal{A}_p} \hookrightarrow ... \hookrightarrow M^{trop}_{g,\mathcal{A}_{N-1}} \hookrightarrow M^{trop}_{g,\mathcal{A}}.$$

The same result holds if we replace $M^{trop}_{g,\mathcal{A}}$ with the moduli space of extended weighted tropical $(g, \mathcal{A})$-stable curves $\overline{M}^{trop}_{g,\mathcal{A}}$ or the moduli space of $(g, \mathcal{A})$-stable tropical curves with volume 1, $\Delta_{g,\mathcal{A}}$.

The strategy of the proof is to show that there are filtrations of the graph categories $\mathcal{G}_{g,\mathcal{A}}$ induced by the partial order on the chambers up to symmetry. Then the direct limit description of the moduli spaces of tropical curves will give us the result.

Notice that we have an ordered sequence of chambers up to symmetry $[Ch_{\mathcal{A}_1}] \leq \cdots \leq [Ch_{\mathcal{A}_N}]$ induced by any analogous sequence of chambers $Ch_{\mathcal{A}_1} \leq \cdots \leq Ch_{\mathcal{A}_N}$, and also if we have a sequence of weight data $\mathcal{A}_1 \leq \cdots \leq \mathcal{A}_N$. We exploit these properties in the next sub-section.

4.2.1 Wall-Crossing properties for graphs

Consider the graph categories $\mathcal{G}_{g,\mathcal{A}}$ defined in Remark 1.2.16. We start with the following property.

Proposition 4.2.3. *Consider the map*

$$\Psi : \mathcal{D}_{g,n} \setminus \bigcup_{w \in W_f} w \to \{ \text{ Graph Categories } \mathcal{G}_{g,\mathcal{A}} \}$$

sending a weight datum $\mathcal{A}$ in the graph category $\mathcal{G}_{g,\mathcal{A}}$. Then

1)The map Ψ is constant on each chamber of the fine chamber decomposition.

2)The fine chamber decomposition is the coarsest one with the above property, i.e., if $\mathcal{A}$ and $\mathcal{B}$ are in two different chambers of the fine chamber decomposition, their image under the above map is different.

Proof. We follow the lines of Proposition 6.2 of [Uli15], since the proof relies only on graphs. The map is clearly constant on the fine chambers of $\mathcal{D}_{g,n}$: in fact, pick $\mathcal{A}$ and $\mathcal{A}'$ in the same chamber. Then a n-marked weighted graph G of genus g is $\mathcal{A}$-stable if and only if for every vertex v the condition $2w(v) - 2 + val(v) + |v|_{\mathcal{A}} > 0$ is satisfied. Now if $w(v) \geq 1$ and/or $val(v) \geq 3$ this condition does not depend on $\mathcal{A}$, so assume $w(v) = 0$ and $1 \leq val(v) \leq 2$. Then $2w(v) - 2 + val(v) + |v|_{\mathcal{A}} > 0$ becomes $|v|_{\mathcal{A}} > 2 - val(v)$. If $val(v) = 2$, then we are asking that $|v|_{\mathcal{A}} > 0$, but this is true for every weight datum in $\mathcal{D}_{g,n}$. If $val(v) = 1$, then saying that the graph is $\mathcal{A}$-stable implies that $|v|_{\mathcal{A}} = \sum\limits_{x_i \in L(v)} a_i > 1$. But now since $\mathcal{A}$ and $\mathcal{A}'$ are in the same chamber, $\sum\limits_{x_i \in L(v)} a'_i > 1$ too, so the graph is also $\mathcal{A}'$-stable. By interchanging the role of $\mathcal{A}$ and $\mathcal{A}'$ we show also that a graph which is $\mathcal{A}'$-stable is also $\mathcal{A}$-stable.

We are left to show that $\mathcal{G}_{g,\mathcal{A}}$ changes whenever we cross a wall. Let $S \subset \{1, ..., n\}$ with $2 \leq |S| \leq n$ be the subset indexing the equation of the wall $w = \{\sum_{i \in S} a_i = 1\}$. Suppose first that $g \geq 1$. Let $S \subset \{1, ..., n\}$ with $2 \leq |S| \leq n$ be the subset indexing the equation of the wall $w = \{\sum_{i \in S} a_i = 1\}$, and consider the graph containing one edge between two vertices, one with g loops on it and one with all the legs with index in S (see Figure 4.2). Then the graph is stable of type $(g, \mathcal{A})$ if $\sum_{i \in S} a_i$ is greater than 1, otherwise it is not, i.e., changing the half-space of $\mathcal{D}_{g,n}$ we are also changing the category $\mathcal{G}_{g,\mathcal{A}}$, since the graphs are not the same.

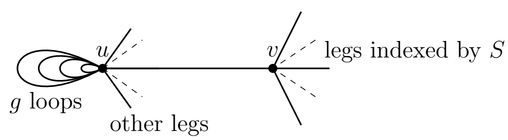

Figure 4.2: This graph is stable unless $\sum_{i \in S} a_i \leq 1$.

In the case $g = 0$, let $S \subset \{1, ..., n\}$ with $2 \leq |S| \leq n - 2$ be the subset indexing the equation of the wall $w = \{\sum_{i \in S} a_i = 1\}$, and consider the graph with two vertices of weight 0 connected by an edge and legs incident to the first vertex having indices in S, while the others are incident to the second vertex (Figure 4.3).

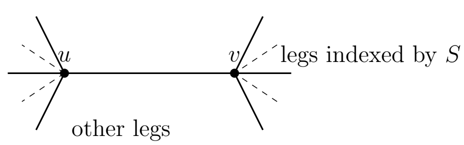

Figure 4.3: This rational graph becomes stable only whenever $\sum_{i \in S} a_i \geq 1$; this is enough since the stability condition on the vertex adjacent to legs indexed by S^c is automatically verified due to the conditions imposed in the start.

Suppose that $\sum_{i \in S} a_i \leq 1$, then the condition $\sum_{i=1}^{n} a_i > 2$ implies $\sum_{i \notin S} a_i > 1$. So when crossing the wall $\sum_{i \in S} a_i = 1$ without changing the a_i such that $i \notin S$ we obtain that the described graph is stable of type $(0, \mathcal{A})$ if $\sum_{i \in S} a_i > 1$, and it is not otherwise, and again we conclude that changing the half-space of $\mathcal{D}_{g,n}$ the category $\mathcal{G}_{g,\mathcal{A}}$ is also different. $\square$

Remark 4.2.4. There is an analogous result by Hassett, namely Proposition 5.1 of [Has03], on Hassett Moduli Spaces of curves. Here Hassett shows that the spaces $\overline{\mathcal{M}}_{g,\mathcal{A}}$ are invariant in the same chamber whenever we pick it from the coarse chamber decomposition , while the same property holds for Universal curves $\overline{\mathcal{C}}_{g,\mathcal{A}}$ if we consider the fine chamber decomposition.

In the following lemma we will consider the case of weight data lying on walls.

Lemma 4.2.5. *Let $d \geq 1$ be an integer and $S_1, ..., S_d \subset \{1, ..., n\}$. Let Ch be a chamber (either of $\mathbf{K}_f$ or $\mathbf{K}_c$) such that $\sum_{i \in S_j} a_i < 1$ for every j from 1 to d. Let $\mathcal{R}$ be a weight datum such that $\sum_{i \in S_j} r_i = 1$, for every j from 1 to d, while for every $S \neq S_j$ the weight datum $\mathcal{R}$ belongs to the half-space induced by S containing Ch. Then $\mathcal{G}_{g,\mathcal{A}}$ is equal to $\mathcal{G}_{g,\mathcal{R}}$, for every $\mathcal{A} \in Ch$.*

Proof. By definition, for all S and for every $\mathcal{A} \in Ch$ we have $\sum_{i \in S} a_i < 1$ if and only if $\sum_{i \in S} r_i \leq 1$. Indeed since $\mathcal{A}$ belongs to a chamber $\sum_{i \in S} a_i \neq 1$ for any $S \subset \{1, ..., n\}$. Moreover, we can find $\mathcal{A} \in Ch$ with the property that $\mathcal{A} \leq \mathcal{R}$. Indeed, let $\mathcal{S} = \bigcup_{j=1}^{d} S_i$ and let $\varepsilon := (\min\limits_{S' \neq S} |\sum_{i \in S'} x_i - 1|)/2|\mathcal{S}|$. We can consider $\mathcal{A} = (\overline{r}_1, ..., \overline{r}_n)$ where:

$$\overline{r}_i := \begin{cases} r_i \text{ if } i \notin \mathcal{S} \\ r_i - \varepsilon \text{ if } i \in \mathcal{S} \end{cases}$$

By construction if $\sum_{i \in S_j} r_i = 1$ then $\sum_{i \in S_j} \overline{r}_i < 1$ for every $j = 1, ..., d$. Moreover if $\sum_{i \in S} r_i < 1$ then $\sum_{i \in S} \overline{r}_i < 1$ and if $\sum_{i \in S} r_i > 1$ then $\sum_{i \in S} \overline{r}_i > 1$. Moreover, $\mathcal{A} \leq \mathcal{R}$, so the category $\mathcal{G}_{g,\mathcal{A}}$ is a full subcategory of $\mathcal{G}_{g,\mathcal{R}}$ by Remark 4.2.1. As in the proof of Proposition 4.1.8 we can assume $\mathcal{A}$ to have all rational components, so that $\mathcal{A} \in Ch$. Now by contradiction suppose there is $G \in Ob(\mathcal{G}_{g,\mathcal{R}}) \setminus Ob(\mathcal{G}_{g,\mathcal{A}})$. If this happens, there is $v \in V(G)$ such that

$$2w(v) - 2 + |v|_E + |v|_{\mathcal{A}} < 0 < 2w(v) - 2 + |v|_E + |v|_{\mathcal{R}}.$$

This implies that there is an S such that $|v|_{\mathcal{A}} = \sum_{i \in S} a_i < 1$ while $|v|_{\mathcal{R}} = \sum_{i \in S} r_i > 1$, indeed $2w(v) - 2 + |v|_E + |v|_{\mathcal{A}} < 0$ implies $2w(v) - 2 + |v|_E < -|v|_{\mathcal{A}}$, so $w(v) = 0$ and $|v|_E \leq 1$, since they are all integers. But this is a contradiction by our hypothesis on Ch and $\mathcal{R}$, so the result follows. □

Proposition 4.2.6. *Let $Ch_1, Ch_2 \in \mathbf{K}_f$ (respectively $\mathbf{K}_c$) be two chambers, $\mathcal{A} \in Ch_1$, $\mathcal{B} \in Ch_2$ two weight data. Then, if $Ch_1 \leq Ch_2$, the category $\mathcal{G}_{g,\mathcal{A}}$ is a full subcategory of $\mathcal{G}_{g,\mathcal{B}}$.*

Proof. Suppose Ch_1 and Ch_2 are different, otherwise the result is trivial by Lemma 4.2.3. Since $Ch_1 \leq Ch_2$, there are $S_1, ..., S_d \subseteq \{1, ..., n\}$ such that $\sum_{i \in S} a_i < 1$ for $(a_1, ..., a_n) \in Ch_1$ and $\sum_{i \in S} b_i > 1$ for $(b_1, ..., b_n) \in Ch_2$, for $j = 1, ..., d$, while for $S \neq S_j$ the defining inequalities of Ch_1 and Ch_2 have the same direction.

Suppose first $d = 1$. Since $\mathcal{G}_{g,\mathcal{A}}$'s are constant on each chamber, by Proposition 4.1.8 we can find $\mathcal{A}' \in Ch_1$, $\mathcal{B}' \in Ch_2$ such that $\mathcal{A}' \leq \mathcal{B}'$ and we have $\mathcal{G}_{g,\mathcal{A}} = \mathcal{G}_{g,\mathcal{A}'}$ and $\mathcal{G}_{g,\mathcal{B}} = \mathcal{G}_{g,\mathcal{B}'}$. The inclusion follows from Remark 4.2.1.

Let $d \geq 2$, and suppose by induction that for every $d' < d$, given two chambers Ch_1' and Ch_2' whose inequalities agree for all but d' sets of indices $S_1', ..., S_{d'}'$, and for every $\mathcal{A} \in Ch_1$

we have $\sum_{i\in S'_j} a_i < 1$ while for every $\mathcal{B} \in Ch_2$ we have $\sum_{i\in S'_j} a_i > 1$, then we can find a weight datum in $\mathcal{A}' \in Ch'_1$ and $\mathcal{B}' \in Ch'_2$ such that $\mathcal{A}' \leq \mathcal{B}'$.

First, suppose there is a chamber Ch_3 whose defining inequalities agree with the one of Ch_1 except for $S_1, ..., S_c$, for a number $c < d$. Then by induction there is $\mathcal{A} \in Ch_1$, $\mathcal{P}, \mathcal{Q} \in Ch_3$ and $\mathcal{B} \in Ch_2$ such that $\mathcal{A} \leq \mathcal{P}$ and $\mathcal{Q} \leq \mathcal{B}$. Then we can induce inclusions

$$\mathcal{G}_{g,\mathcal{A}} \hookrightarrow \mathcal{G}_{g,\mathcal{P}} = \mathcal{G}_{g,\mathcal{Q}} \hookrightarrow \mathcal{G}_{g,\mathcal{B}},$$

so by composition we get the desired inclusion.

Suppose now there is no chamber Ch_3, i.e., for every weight datum $\mathcal{Q}$ not belonging to walls such that $\sum_{i\in S_j} q_i > 1$ for some $i = 1, ..., d$, then $\sum_{i\in S_k} q_i > 1$ for every $k = 1, ..., d$ and the same holds picking the symbol $<$ instead of the symbol $>$.

Let $\mathcal{X} \in Ch_1$, $\mathcal{Y} \in Ch_2$ and consider the segment which goes from $\mathcal{X}$ and $\mathcal{Y}$: then every point on the segment $P_t = (1-t)\mathcal{X} + t\mathcal{Y}$ for $t \in [0,1]$ verifies all the common inequalities, and there is at least a $t' \in [0,1]$ such that $\mathcal{P}_{t'} = (p_1, ..., p_n)$ belongs to a wall w_{S_k}, for some $k = 1, ..., d$. Let $\varepsilon = \min_{\{T | \sum_{i\in T} p_i \neq 1\}} |1 - \sum_{i\in T} p_i|$ and let $\delta < \varepsilon$. Define $\mathcal{B} = (b_1, ..., b_n) = (p_1 + \frac{\delta}{n}, ..., p_n + \frac{\delta}{n})$. It follows by construction that $\mathcal{B}$ verifies all the inequalities verified by $\mathcal{P}_t$, and since $\sum_{i\in S_k} b_i > 1$ and it can not be on a wall it belongs to Ch_2. Analogously we can define $\mathcal{A} = (a_1..., a_n) = (p_1 - \frac{\delta}{n}, ..., p_n - \frac{\delta}{n})$. Since $\sum_{i\in S_k} a_i < 1$ while all other equalities agree with the one of $\mathcal{P}_{t'}$ then it belongs to Ch_1, and $\mathcal{A} \leq \mathcal{B}$ by construction, so we conclude by observing that we can suppose them to have all rational components as in the proof of 4.1.8. $\square$

Remark 4.2.7. We shown also that for any two chambers $Ch_1 \leq Ch_2$ we can find a weight datum in $\mathcal{A} \in Ch_1$ and $\mathcal{B} \in Ch_2$ such that $\mathcal{A} \leq \mathcal{B}$.

Now consider the action of S_n on the sets of chambers. This action induces an analogous action on the set of categories $\mathcal{G}_{g,\mathcal{A}}$ which permutes the label of the marked points on graphs, as shown in the following Example.

Example 4.2.8. Consider $g = 1$, $n = 3$, let $\mathcal{A} = (\varepsilon, \frac{2}{3}, \frac{2}{3}) \in \mathcal{D}_{1,3}$, for some $\varepsilon > 0$. Then the graph G in Figure 4.4 is $(1, \mathcal{A})$-stable:

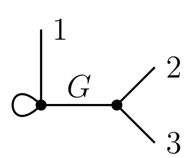

Figure 4.4: A $(1, (\varepsilon, \frac{2}{3}, \frac{2}{3}))$-stable graph.

Let now $\sigma = (1 \quad 2 \quad 3) \in S_3$, then $\sigma(\mathcal{A}) = (\frac{2}{3}, \varepsilon, \frac{2}{3})$, and one can observe that G is not anymore stable with respect to the new weight datum. But changing the label of the legs of G according to the same permutation σ gives the new graph G' in Figure 4.5, which is $(1, \sigma(\mathcal{A}))$-stable.

By Example 4.2.8 we can easily deduce that each time we have a weight datum $\mathcal{A}$ and a permutation $\sigma \in S_n$, the two categories $\mathcal{G}_{g,\mathcal{A}}$ and $\mathcal{G}_{g,\sigma(\mathcal{A})}$ are isomorphic, since it is enough

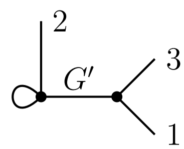

Figure 4.5: A $(1, (\frac{2}{3}, \varepsilon, \frac{2}{3}))$-stable graph.

to send each graph to the one obtained relabeling its legs according to σ, without changing morphisms. This isomorphism becomes an equality if the chosen permutation acts trivially on the chamber. So, up to isomorphism, there is only a category $\mathcal{G}_{g,\mathcal{A}}$ for each chamber up to symmetry, which is the one containing the chamber to which $\mathcal{A}$ belongs. Therefore, the latter Proposition can be rephrased including this symmetry property, giving the following:

Proposition 4.2.9. *Let $[Ch_1]$ and $[Ch_2]$ be two chambers up to symmetry, $\mathcal{A} \in Ch_1 \in [Ch_1]$, $\mathcal{B} \in Ch_2 \in [Ch_2]$. There is an inclusion as a full subcategory $\mathcal{G}_{g,\mathcal{A}} \hookrightarrow \mathcal{G}_{g,\mathcal{B}}$ each time $[Ch_1] \leq [Ch_2]$. It is an isomorphism if the two chambers up to symmetry are the same.*

Since the stability conditions on tropical curves are defined on their underlying graphs, everything we shown so far can be easily generalized for $M^{trop}_{g,\mathcal{A}}$, $\overline{M}^{trop}_{g,\mathcal{A}}$ and $\Delta_{g,\mathcal{A}}$. We can resume everything in the following Proposition.

Proposition 4.2.10. *Let $g, n \geq 1$ be two integers and $\mathcal{A}$ and $\mathcal{B}$ two weight data in $\mathcal{D}_{g,n}$.*

1) If $\mathcal{A} \leq \mathcal{B}$, then $M^{trop}_{g,\mathcal{A}} \subset M^{trop}_{g,\mathcal{B}}$;

2) If $\mathcal{A}$ and $\mathcal{B}$ are in the same chamber, then $M^{trop}_{g,\mathcal{A}} = M^{trop}_{g,\mathcal{B}}$;

3) If $\mathcal{A}$ and $\mathcal{B}$ are obtained one from the other through a permutation of coordinates, then $M^{trop}_{g,\mathcal{A}}$ is homeomorphic to $M^{trop}_{g,\mathcal{B}}$ through a relabeling homeomorphism;

4) If $\mathcal{A}$ and $\mathcal{B}$ are in chambers which belong to the same orbit, then $M^{trop}_{g,\mathcal{A}}$ is homeomorphic to $M^{trop}_{g,\mathcal{B}}$ through a relabeling homeomorphism.

The same results hold if we replace the $M^{trop}_{g,\mathcal{A}}$'s with the $\overline{M}^{trop}_{g,\mathcal{A}}$'s or the $\Delta_{g,\mathcal{A}}$'s.

Proof. It is enough to observe that a point in $M^{trop}_{g,\mathcal{A}}$ is identified by a graph $G \in \mathcal{G}_{g,\mathcal{A}}$ and a edge length function. Then 1) follows by 4.2.1, 2) follows from 4.2.6, 3) and 4) follows from 4.2.9. The result for $\overline{M}^{trop}_{g,\mathcal{A}}$ and $\Delta_{g,\mathcal{A}}$ holds in the same way by considering the extended edge lengths and edge lengths making the volume being one, respectively. $\square$

Remark 4.2.11. A priori, given $\mathcal{A}$ and $\mathcal{B}$ in different chambers, we can not say if two moduli spaces $M^{trop}_{g,\mathcal{A}}$ and $M^{trop}_{g,\mathcal{B}}$ are not homeomorphic as topological space. The point on using morphisms of Proposition 4.2.10 is that they preserve the moduli space structure.

We can now conclude the proof of the main Theorem of this section.

Proof of Theorem 4.2.2. Consider an ordered sequence of chambers up to symmetry

$$[Ch_{\mathcal{A}_1}] \leq [Ch_{\mathcal{A}_2}] \leq ... \leq [Ch_{\mathcal{A}_p}] \leq ... \leq [Ch_{\mathcal{A}_{N-1}}] \leq [Ch_{\mathcal{A}}].$$

At each step of the filtration $[Ch_{\mathcal{A}_p}] \leq [Ch_{\mathcal{A}_{p+1}}]$ we can find two weight data $\mathcal{A} \leq \mathcal{B}$ and two chambers $Ch_1 \leq Ch_2$ such that $\mathcal{A} \in Ch_1 \in [Ch_{\mathcal{A}_p}]$ and $\mathcal{B} \in Ch_2 \in [Ch_{\mathcal{A}_{p+1}}]$. Then $M^{trop}_{g,\mathcal{A}_p} \approx M^{trop}_{g,\mathcal{A}} \subset M^{trop}_{g,\mathcal{B}} \approx M^{trop}_{g,\mathcal{A}_{p+1}}$, where by $\approx$ we indicate homeomorphisms, and the homeomorphism and inclusion are the one of 4.2.10. This induces the desired inclusion map $M^{trop}_{g,\mathcal{A}_p} \hookrightarrow M^{trop}_{g,\mathcal{A}_{p+1}}$. The result of the Theorem then follows repeating this reasoning for each step of the sequence. □

Example 4.2.12. Consider the case $g \geq 1$, $n = 3$. We saw the chamber decomposition in Example 4.1.2, and the chamber decomposition up to symmetry in Example 4.1.5. We choose a weight datum for each chamber, and for symmetric chambers we choose data obtained after a permutation:

- $(1,1,1) \in Ch_1$;
- $(\frac{12}{27}, \frac{14}{27}, 1-\varepsilon) \in Ch_2$;
- $(\frac{14}{27}, 1-\varepsilon, \frac{12}{27}) \in Ch_3$;
- $(1-\varepsilon, \frac{12}{27}, \frac{14}{27}) \in Ch_4$;
- $(\frac{12}{27}, \frac{14}{27}, \frac{14}{27}-\varepsilon) \in Ch_5$,
- $(\frac{14}{27}-\varepsilon, \frac{12}{27}, \frac{14}{27}) \in Ch_6$;
- $(\frac{14}{27}, \frac{14}{27}-\varepsilon, \frac{12}{27}) \in Ch_7$;
- $(\frac{4}{9}-\varepsilon, \frac{4}{9}-\varepsilon, \frac{4}{9}-\varepsilon) \in Ch_8$;
- $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}-\varepsilon) \in Ch_9$,

for $0 < \varepsilon < \frac{1}{27}$ rational. We pick these ε perturbations in order to have our weight data in the interior of chambers. Since moduli spaces are constant on each chamber, for every weight datum $\mathcal{A} \in \mathcal{D}_{g,3}$, the moduli space $M^{trop}_{g,\mathcal{A}}$ is the same as the space obtained picking one of the nine data above (the one which lies in the same chamber of $\mathcal{A}$). So by the partial order on the set of the chambers, we get the following diagram:

$$\begin{tikzcd}
 & & M^{trop}_{g,3} = M^{trop}_{g,(1,1,1)} & & \\
M^{trop}_{g,(\frac{12}{27},\frac{14}{27},1-\varepsilon)} \arrow[urr, hook] & \approx & M^{trop}_{g,(\frac{14}{27},1-\varepsilon,\frac{12}{27})} \arrow[u, hook] & \approx & M^{trop}_{g,(1-\varepsilon,\frac{12}{27},\frac{14}{27})} \arrow[ull, hook'] \\
M^{trop}_{g,(\frac{12}{27},\frac{14}{27},\frac{14}{27}-\varepsilon)} \arrow[u, hook] & \approx & M^{trop}_{g,(\frac{14}{27},\frac{14}{27}-\varepsilon,\frac{12}{27})} \arrow[u, hook] & \approx & M^{trop}_{g,(\frac{14}{27}-\varepsilon,\frac{12}{27},\frac{14}{27})} \arrow[u, hook] \\
 & & M^{trop}_{g,(\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon)} \arrow[ull, hook'] \arrow[u, hook] \arrow[urr, hook] & & \\
 & & M^{trop}_{g,(\frac{1}{3},\frac{1}{3},\frac{1}{3}-\varepsilon)} \arrow[u, hook] & &
\end{tikzcd}$$

In the diagram, we indicated the relabeling homeomorphism of Proposition 4.2.10 with $\approx$.

For example, the following filtration

$$M^{trop}_{g,(\frac{1}{3},\frac{1}{3},\frac{1}{3}-\varepsilon)} \subset M^{trop}_{g,(\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon)} \subset M^{trop}_{g,(\frac{14}{27}-\varepsilon,\frac{12}{27},\frac{14}{27})} \subset M^{trop}_{g,(1-\varepsilon,\frac{12}{27},\frac{14}{27})} \subset M^{trop}_{g,3},$$

of the space $M^{trop}_{g,3}$ can be obtained by picking the right column of the diagram.

Clearly, the same diagram and the same filtrations also work for $\overline{M}^{trop}_{g,\mathcal{A}}$ and $\Delta_{g,\mathcal{A}}$. Notice that in this case the partial order induced on the chambers up to symmetry becomes total, since we can say for each couple of chambers up to symmetry which of them is greater or equal than the other. This is not true in general, as shown in the following example.

Example 4.2.13. Let $g = 0$, $n = 8$. Consider the chamber Ch_1 defined by the following set of inequalities:

$$\begin{cases} a_1 + a_2 > 1 \\ a_1 + a_3 > 1 \\ a_1 + a_4 > 1 \\ a_1 + a_5 > 1 \\ a_1 + a_6 > 1 \\ a_1 + a_7 > 1 \\ a_i + a_j < 1 \text{ for any other couple of indices} \\ \sum_{i \in S} a_i > 1 \text{ for any } S \text{ such that } |S| \geq 3 \end{cases}$$

This is not empty, for example the datum

$$\mathcal{A}_1 = (\frac{1}{2} + 2\varepsilon, \frac{1}{2} - \varepsilon, \frac{1}{2} - \varepsilon, \frac{1}{2} - \varepsilon, \frac{1}{2} - \varepsilon, \frac{1}{2} - \varepsilon, \frac{1}{2} - \varepsilon, 2\varepsilon)$$

belongs to Ch_1 for a sufficiently small ε. Consider now the chamber Ch_2 defined by

$$\begin{cases} a_1 + a_2 > 1 \\ a_1 + a_3 > 1 \\ a_1 + a_4 > 1 \\ a_2 + a_3 > 1 \\ a_2 + a_4 > 1 \\ a_3 + a_4 > 1 \\ a_i + a_j < 1 \text{ for any other couple of indices} \\ \sum_{i \in S} a_i > 1 \text{ only if } |S| \geq 3 \text{ and } |S \cap \{1,2,3,4\}| \geq 2. \end{cases}$$

This also is not empty since

$$\mathcal{A}_2 = (\frac{1}{2} + \varepsilon, \frac{1}{2} + \varepsilon, \frac{1}{2} + \varepsilon, \frac{1}{2} + \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon)$$

belongs to Ch_2 for a sufficiently small ε. We can see that Ch_1 and Ch_2 are not comparable with respect to the partial order on $\mathbf{K}_f$ by looking at their defining inequalities. Indeed, by definition, given Ch_a and Ch_b in $\mathbf{K}$ we say that $Ch_a \leq Ch_b$ if they are equal or, informally, all of their defining inequalities which have different direction are such that $\sum_{i\in S} a_i < 1$ for every $\mathcal{A} \in Ch_a$ and $\sum_{i\in S} b_i > 1$ for every $\mathcal{B} \in Ch_b$. But here we have that in Ch_1, $a_1 + a_5 > 1$ while $a_1 + a_5 < 1$ in Ch_2. Meanwhile, $a_2 + a_4 < 1$ in Ch_1, but $a_2 + a_4 > 1$ in Ch_2 so the above condition is not satisfied.

Moreover these two chambers are in different orbits of the action of S_8 on $\mathbf{K}$ since the inequalities with two variables and right direction defining Ch_1 all contains the variable a_1, while in Ch_2 there is no variable repeated in all the inequalities with two variables and right direction. So there is no permutation σ in S_8 such that $\sigma(Ch_1) = Ch_2$, and this implies that $[Ch_1]$ and $[Ch_2]$ are not in $[\mathbf{K}_f]$. At the level of moduli spaces of tropical curves, it means that we can find points in $M^{trop}_{0,\mathcal{A}_1}$ with combinatorial types which are not in $M^{trop}_{0,\mathcal{A}_2}$ and viceversa, asshown in the Figure 4.2.13.

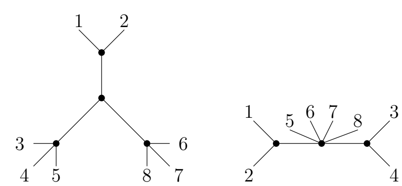

Figure 4.6: The graph on the left is a combinatorial type of points in $M^{trop}_{0,\mathcal{A}_1}$ but not in $M^{trop}_{0,\mathcal{A}_2}$, while the graph on the right is a combinatorial type of points in $M^{trop}_{0,\mathcal{A}_2}$ but not in $M^{trop}_{0,\mathcal{A}_1}$.

4.2.2 Examples of filtrations

In this section we construct some sequences of weight data, chambers and chambers up to symmetry which define interesting filtrations of moduli spaces.

Example 4.2.14. Recall that for any $g, n \geq 1$ there are two special chambers up to symmetry: the maximal chamber $[Ch_{1^{(n)}}]$, which is greater than any other chamber up to symmetry, and the minimal chamber $[Ch_{\varepsilon^{(n)}}] := [Ch_{\mathcal{E}}]$, for $\varepsilon < \frac{1}{n}$, which is smaller than any other chamber up to symmetry. Moreover, both these orbits are made by a single chamber.

Let $\mathcal{F} = (\frac{1}{n} + \varepsilon, \frac{1}{n}, ..., \frac{1}{n})$. This datum belongs to the chamber $Ch_{\mathcal{F}}$ described by the inequalities $\sum_{i\in S} a_i < 1$ for every $S \neq \{1, ..., n\}$ and $\sum_{i=1}^{n} a_i > 1$. It is clearly invariant by the action of S_n and so its orbit $[Ch_{\mathcal{F}}]$ is made only by itself. In particular, $\sigma(\mathcal{F}) \in Ch_{\mathcal{F}}$ for every $\sigma \in S_n$. Moreover, by the inequalities defining $Ch_{\mathcal{F}}$ it is clear that $Ch_{\varepsilon^{(n)}} \leq Ch_{\mathcal{F}} \leq Ch$ for any chamber $Ch \neq Ch_{\varepsilon^{(n)}}$.

Suppose we have two chambers up to symmetry $[Ch_1]$ and $[Ch_2]$ such that $[Ch_1] \leq [Ch_2]$: then it is always possible to construct a five term sequence

$$[Ch_{\mathcal{E}}] \leq [Ch_{\mathcal{F}}] \leq [Ch_1] \leq [Ch_2] \leq [Ch_{1^{(n)}}]$$

inducing a filtration of moduli spaces

$$M^{trop}_{g,\mathcal{E}} \hookrightarrow M^{trop}_{g,\mathcal{F}} \hookrightarrow M^{trop}_{g,\mathcal{A}} \hookrightarrow M^{trop}_{g,\mathcal{B}} \hookrightarrow M^{trop}_{g,n},$$

where $\mathcal{A} \in Ch_1$ and $\mathcal{B} \in Ch_2$.

Remark 4.2.15. In general, given a sequence $[Ch_{\mathcal{A}_1}] \leq ... \leq [Ch_{\mathcal{A}_p}]$ which does not contain already $[Ch_{\mathcal{E}}]$, $[Ch_{\mathcal{F}}]$ and $[Ch_{1_{(n)}}]$, it is always possible to extend it by two terms $[Ch_{\mathcal{E}}] \leq [Ch_{\mathcal{F}}]$ on the left and by $[Ch_{1^{(n)}}]$ on the right.

Example 4.2.16. Let $g \geq 1$, $n \geq 2$, and $\varepsilon < \frac{1}{n}$, and consider the Heavy/Light space $(1^{(m)}|\varepsilon^{(n-m)})$ with n entries, $m \leq n$. Notice that if $m = 0$ we get the weight datum $\varepsilon^{(n)}$, while if $m = n$ we get the weight datum $1^{(n)}$.

By construction, we have

$$(1^{(m)}, \varepsilon^{(n-m)}) \leq (1^{(m+1)}, \varepsilon^{(n-m-1)})$$

for any $m = 0, ..., n-1$, and passing to their chambers it is easy to see that we have

$$Ch_{\mathcal{E}} \leq Ch_{(1^{(1)},\varepsilon^{(n-1)})} \leq ... \leq Ch_{(1^{(m)},\varepsilon^{(n-m)})} \leq ... \leq Ch_{(1^{(n-1)},\varepsilon^{(1)})} \leq Ch_{1^{(n)}}.$$

It is easy to check that all of these inequalities are strict except the last one, since

$$Ch_{(1^{(n-1)},\varepsilon^{(1)})} = Ch_{1^{(n)}}.$$

So there is an induced filtration of spaces $M^{trop}_{g,\mathcal{E}} \hookrightarrow M^{trop}_{g,(1^{(1)},\varepsilon^{(n-1)})} \hookrightarrow ... \hookrightarrow M^{trop}_{g,(1^{(m)},\varepsilon^{(n-m)})} \hookrightarrow ... \hookrightarrow M^{trop}_{g,n}$. Of course the same holds also if we consider the chambers up to symmetry.

Remark 4.2.17. The Heavy/Light filtration exists also for $g = 0$, but in this case we have to start by the case $m = 2$.

Example 4.2.18. Fix $g \geq 0$ and $l \in \{2, ..., n\}$. Suppose we have a chamber defined by the following set of inequalities: $\sum_{i \in S} a_i > 1$ if and only if $|S| \geq l$. The weight datum

$$\mathcal{H}_l := (\frac{1}{l} + \varepsilon_l, ..., \frac{1}{l} + \varepsilon_l)$$

belongs to that chamber for ε_l sufficiently small, and its chamber $Ch_{\mathcal{H}_l}$ is invariant with respect to the action of S_n. We call $[Ch_{\mathcal{H}_l}]$ the l-th floor of the chamber decomposition.

Notice that if $l = 2$, $Ch_{\mathcal{H}_2} = Ch_{1^{(n)}}$, while if $l = n$, $Ch_{\mathcal{H}_n} = Ch_{\mathcal{F}}$. Moreover by construction $Ch_{\mathcal{H}_l} \leq Ch_{\mathcal{H}_{l-1}}$ (and clearly the same holds if we pick them up to symmetry) so there is an induced filtration $M^{trop}_{g,\mathcal{F}} \hookrightarrow ... \hookrightarrow M^{trop}_{g,\mathcal{H}_l} \hookrightarrow M^{trop}_{g,\mathcal{H}_{l-1}} ... \hookrightarrow M^{trop}_{g,n}$, eventually extendable on the left with $M^{trop}_{g,\mathcal{E}}$. Notice that being stable for a curve in the $l-$th floor means that each valence one vertex is adjacent to at least l markings.

4.3 Graph Complexes

4.3.1 Definition of Graph Complexes

Let $(g, \mathcal{A})$ be an input datum. The chain complex $G^{(g,\mathcal{A})}$ is a complex of rational vector spaces generated by elements $[G, \omega]$ where G is an n-marked $(g, \mathcal{A})$-stable pure graph and ω is a total order on the set of the edges of G. Generators are subject to the relation

$$[G, \omega] = sgn(\sigma)[G', \omega']$$

if there is an isomorphism of n-marked graphs $G \cong G'$ under which the orders ω and ω' are related by the permutation $\sigma \in S_{|E(G)|}$.
This forces $[G, \omega] = 0$ when G admits an automorphism that induces an odd permutation on the edges. The homological degree of G is $|E(G)| - 2g$. So $G^{(g,\mathcal{A})} = \bigoplus G_j^{(g,\mathcal{A})}$, where $G_j^{(g,\mathcal{A})}$ is the Rational vector space spanned by elements $[\mathbf{G}, \omega]$ where $\mathbf{G}$ has $j + 2g$ edges. The differential on $[G, \omega] \neq 0$ is defined as

$$\partial[G, \omega] = \sum_{i=0}^{|E(G)|} (-1)^i [G/e_i, \omega|_{E(G)\setminus\{e_i\}}],$$

where G/e_i indicates the contraction and $\omega|_{E(G)\setminus\{e_i\}}$ is the induced order. If e_i is a loop, we interpret the corresponding term in the formula of the differential as zero.

Remark 4.3.1. This generalizes the notions and the theory developed in [CGP21] and [CGP22]. In fact, when $\mathcal{A} = 1^{(n)}$, we recover the definition of the Graph Complex $G^{(g,n)}$ we have in [CGP22].

Once we have Graph Complexes,we can prove the following Theorem we already announced in Section 1.3.4.

Theorem 4.3.2. *Let $g \geq 1$ and $\mathcal{A} \in \mathcal{D}_{g,n}$ a weight datum. There is a natural injection of chain complexes*

$$G^{(g,\mathcal{A})} \to C_*(\Delta_{g,\mathcal{A}}, \mathbb{Q})$$

decreasing degrees by $2g - 1$, inducing isomorphisms on homology

$$\widetilde{H}_{k+2g-1}(\Delta_{g,\mathcal{A}}; \mathbb{Q}) \to H_k(G^{(g,\mathcal{A})})$$

for all k's.

Proof. We prove this theorem following the lines of Theorem 1.4 of [CGP22]. Consider the cellular chain complex $C_*(\Delta_{g,\mathcal{A}}, \mathbb{Q})$. It is generated in degree p by $[G, \omega]$ where $G \in \mathcal{G}_{g,\mathcal{A}}$ is a graph and $\omega : E(G) \to [p]$ is a bijection, with the relations $[G, \omega] = sgn(\sigma)[G', \omega']$ if there is an isomorphism $G \to G'$ of graphs inducing the permutation σ of the set $[p]$. We claim that the complex $C_*(\Delta_{g,\mathcal{A}}, \mathbb{Q})$ splits into the direct sum of two subcomplexes $A^{(g,\mathcal{A})} \bigoplus B^{(g,\mathcal{A})}$, $A^{(g,\mathcal{A})}$ being spanned by the generators where G has no loops and whose vertices have weight zero, and $B^{(g,\mathcal{A})}$ is spanned by the generators such that G has at least one loop or one nonzero vertex weight. These are in fact subcomplexes: for $B^{(g,\mathcal{A})}$ it is clear, while for $A^{(g,\mathcal{A})}$, we need just to observe that if G has no loops and all vertices have weight zero, and $[G, \omega] \neq 0$, then G has no parallel edges. Therefore, every contraction G/e also has no loops and also has all vertices of weight zero.

Now we note that $A^{(g,\mathcal{A})}$ is isomorphic to $G^{(g,\mathcal{A})}$ up to shifting degrees by $2g - 1$, and $B^{(g,\mathcal{A})}$ is the cellular chain complex associated to the subcomplex of tropical curves with underlying graphs having at least a loop and/or a vertex with positive weight $\Delta_{g,\mathcal{A}}^{lw}$, which is contractible whenever it is nonempty by Theorem 2.1.1. Therefore $B^{(g,\mathcal{A})}$ is an acyclic complex, so the result follows. ∎

So the Graph Complexes $G^{(g,\mathcal{A})}$ compute the reduced rational homology of the $\Delta_{g,\mathcal{A}}$. This Theorem generalizes Theorem 1.4 of [CGP22], and gives the following Corollary.

Corollary 4.3.3. *There is a natural isomorphism*

$$Gr^W_{6g-6+2n} H^{4g-6+2n-k}(\mathcal{M}_{g,\mathcal{A}};\mathbb{Q}) \to H_k(G^{(g,\mathcal{A})}).$$

Proof. Using the language of [CGP21] and [CGP22], the proof follows by Theorem 4.3.2 and Theorem 5.8 of [CGP21] just observing that $\Delta(\mathcal{M}_{g,\mathcal{A}} \subset \overline{\mathcal{M}}_{g,\mathcal{A}}) = \Delta_{g,\mathcal{A}}$. Indeed, we already observed that by Theorem 1.1 of [Uli15], we have $D := \overline{\mathcal{M}}_{g,\mathcal{A}} \setminus \mathcal{M}_{g,\mathcal{A}}$ to be a divisor with normal crossing (stack theoretically), so by Corollary 5.6 of [CGP21] $\Delta(D)$ is the symmetric Δ-complex associated to the smooth generalized cone complex $\mathfrak{S}(\overline{\mathcal{M}}_{g,\mathcal{A}})$. The rest of the proof is analogous to the one of Corollary 5.6 of [CGP21] and Theorem 6.1 of [CGP22]. $\square$

Example 4.3.4. Suppose we have $g = 1$, $n = 3$. We want to compute $\widetilde{H}_i(\Delta_{g,\mathcal{A}},\mathbb{Q})$, for different $\mathcal{A} \in \mathcal{D}_{1,3}$. We have 5 different chambers up to symmetry as seen in Example 4.1.5. To represent them we can pick the weight data at the end of the example 4.2.12:

$$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}-\varepsilon\right) \le \left(\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon\right) \le \left(\frac{14}{27}-\varepsilon,\frac{12}{27},\frac{14}{27}\right) \le \left(1-\varepsilon,\frac{12}{27},\frac{14}{27}\right) \le (1,1,1)\,.$$

We want to compute the homology of all of the five possible chain complexes by Theorem 4.3.2. For simplicity, let us call the weight data above respectively $\mathcal{E}$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $1^{(3)}$. We compute the vector spaces $G_i^{(1,\mathcal{A})}$ for all these $\mathcal{A}$'s in all degrees. The space $G_i^{(1,\mathcal{A})}$ is generated by pairs $[G,\omega]$, where

$$i = |E(G)| - 2g = |E(G)| - 2$$

and G is a pure $(1,\mathcal{A})$-stable graph, modulo the relation $[G,\omega] = sgn(\sigma)[G',\omega']$, which we refer to as the sign relation. For our situation, the number of edges goes from 1 to 3, so we have three vector spaces for each $\mathcal{A}$. We draw all the stable graphs of genus 1 with three legs in Figures from 4.7 to 4.15.

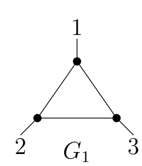

Figure 4.7: The graph G_1 has no automorphisms, and there are six different possible orderings of its edges. This leads to six different generators $[G_1,\omega_{1,i}]$ in degree 1, for $i = 1,...,6$, which are all the same after the sign relation.

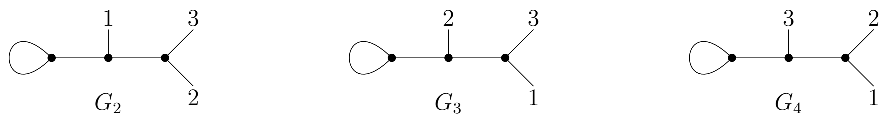

Figure 4.8: These graphs with this only an automorphism which flips the loop. This does not produce any odd permutation on the set of edges, so these classes are non-zero.

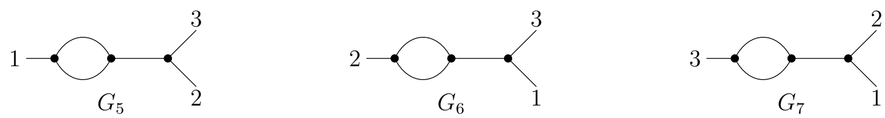

Figure 4.9: All the graphs with this combinatorial type admits the automorphism switching the two parallel edges, which induces odd permutations on the set of edges. So each class coming from these three graphs is zero.

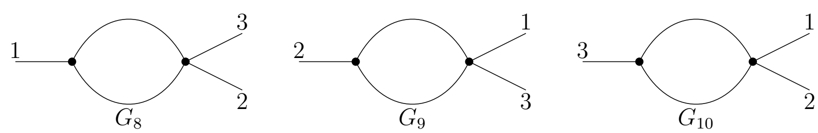

Figure 4.10: These combinatorial types give zero classes, since they have the automorphism which switches the parallel edges.

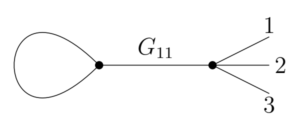

Figure 4.11: The graph G_{11} has no automorphisms, and since it has only two edges there are only two possible orderings. It gives two generators $[G_{11}, \omega_i]$ in degree 0, for $i = 1, 2$, becoming one the opposite of the other after the sign relation.

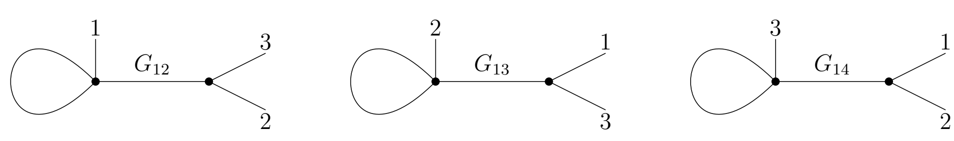

Figure 4.12: Automorphisms of graphs with this combinatorial type just flip the loop, so they do not induce odd permutations. Each of these graphs then gives generators $[G_j, \omega_{j,i}]$ in degree 0, with $j = 12, 13, 14$ and $i = 1, 2$, with $[G_j, \omega_{j,1}] = -[G_j, \omega_{j,2}]$.

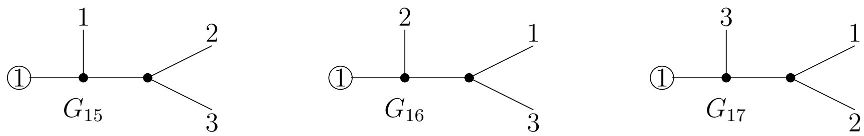

Figure 4.13: These graphs are not pure, so they do not contribute with generators, and we can think to them as 0 classes when they come after a contraction.

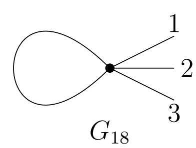

Figure 4.14: This graph as only the loop-flipping automorphism, and only a possible order on its set of edges. The resulting generator $[G_{18}, \omega]$ has degree -1.

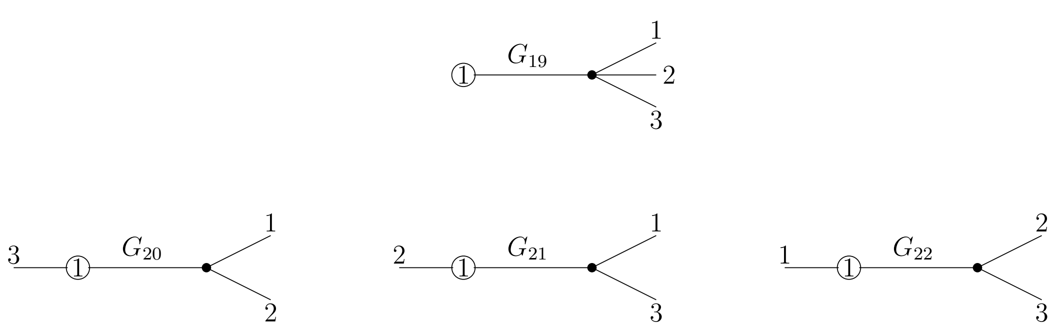

Figure 4.15: All the other graph with a single edge are non-pure, so they do not give generators and are zero if obtained after a contraction.

Now we compute the boundary map on each meaningful generator.

$$\partial[G_1,\omega_i] = -[G_{10},\omega_i|_{G_{10}}] + [G_8,\omega_i|_{G_8}] - [G_9,\omega_i|_{G_9}] = 0$$

$$\partial[G_2,\omega_i] = 0 + [G_{12},\omega_i|_{G_{12}}] - [G_{11},\omega_i|_{G_{11}}] = [G_{12},\omega_i|_{G_{12}}] - [G_{11},\omega_i|_{G_{11}}]$$

$$\partial[G_3,\omega_i] = 0 + [G_{13},\omega_i|_{G_{13}}] - [G_{11},\omega_i|_{G_{11}}] = [G_{13},\omega_i|_{G_{13}}] - [G_{11},\omega_i|_{G_{11}}]$$

$$\partial[G_4,\omega_i] = 0 + [G_{14},\omega_i|_{G_{14}}] - [G_{11},\omega_i|_{G_{11}}] = [G_{14},\omega_i|_{G_{14}}] - [G_{11},\omega_i|_{G_{11}}]$$

$$\partial[G_{11},\omega_i] = 0 + [G_{18},\omega_i|_{G_{18}}] = [G_{18},\omega_i|_{G_{18}}]$$

$$\partial[G_{12},\omega_i] = 0 + [G_{18},\omega_i|_{G_{18}}] = [G_{18},\omega_i|_{G_{18}}]$$

$$\partial[G_{13},\omega_i] = 0 + [G_{18},\omega_i|_{G_{18}}] = [G_{18},\omega_i|_{G_{18}}]$$

$$\partial[G_{14},\omega_i] = 0 + [G_{18},\omega_i|_{G_{18}}] = [G_{18},\omega_i|_{G_{18}}]$$

Of course, $\partial[G_{18},\omega] = 0$ by definition. By this, we can compute the chain complexes and the maps between them. For each graph, we say for which weight data it is stable:

- The only two $(1,\mathcal{E})$-stable graphs which give nonzero generators are G_1 and G_{18}.
- The only a $(1,\mathcal{A}_2)$-stable graph which is not $(1,\mathcal{E})$-stable and gives nonzero generators is G_{11}.
- The $(1,\mathcal{A}_3)$-stable graph which are not $(1,\mathcal{A}_2)$-stable and give nonzero generators are G_2, G_5 and G_{12}.
- The $(1,\mathcal{A}_4)$-stable graph which are not $(1,\mathcal{A}_3)$-stable and give nonzero generators are G_3, G_6, G_9 and G_{13}.
- Lastly, stable graphs of genus g with three legs which are not stable for any other weight datum we choose are G_4, G_7 and G_{14}.

By this description, it is easy to compute the vector spaces of the chain complexes, keeping in mind the sign relation. In degree -1, since the only 1-edge significant graph is G_{18}, which is stable for each weight datum, we have

$$G_{-1}^{(1,3)} = G_{-1}^{(1,\mathcal{A}_4)} = G_{-1}^{(1,\mathcal{A}_3)} = G_{-1}^{(1,\mathcal{A}_2)} = G_{-1}^{(1,\mathcal{E})},$$

and it is isomorphic to $\mathbb{Q}$.

In degree 0, we have $G_0^{(1,\mathcal{E})} = 0$ since there are no $(1,\mathcal{E})$-stable significant graphs with two edges. Then $G_0^{(1,\mathcal{A}_2)}$ is generated by G_{11} and its two orientations which are related by the permutation (1 2) ($[G_{11},\omega_1] = -[G_{11},\omega_2]$) , so it is isomorphic to $\mathbb{Q}$. In $G_0^{(1,\mathcal{A}_3)}$ we add generators given by G_{12} and its orientations modulo the sign relation to the generators of the previous one, so it is isomorphic to $\mathbb{Q}^2$. Analogously for $G_0^{(1,\mathcal{A}_4)}$ we add generators given by G_{13} and its orientations modulo the sign relation to the generators of the previous two, so it is isomorphic to $\mathbb{Q}^3$. Lastly in $G_0^{(1,3)}$ we add generators given by G_{14} modulo the sign relation, and it is isomorphic to $\mathbb{Q}^4$.

Something similar happens in degree 1: we have $G_1^{(1,\mathcal{E})} = G_1^{(1,\mathcal{A}_2)}$ since both are spanned by G_1 and its six orientations, giving an isomorphism with $\mathbb{Q}$ since we divide by the sign relation given by S_3. Then each step of the filtration adds six generators modulo the sign relation: in particular $G_1^{(1,\mathcal{A}_3)}$ adds G_2, $G_1^{(1,\mathcal{A}_4)}$ adds G_3 and $G_1^{(1,3)}$ adds G_4, giving respectively isomorphisms with $\mathbb{Q}^2$, $\mathbb{Q}^3$ and $\mathbb{Q}^4$.

Through this description and the boundary maps computed before we can compute the homology groups in each degree at each step of the filtration. Recall that the shifting degree isomorphism from Theorem 4.3.2 gives $\widetilde{H}_{k+1}(\Delta_{1,\mathcal{A}};\mathbb{Q}) \cong H_k(G^{(g,\mathcal{A})})$. The first complex is the following:

$$G^{(1,\mathcal{E})} : G_1^{(1,\mathcal{E})} \xrightarrow{\partial_1^{\mathcal{E}}} G_0^{(1,\mathcal{E})} \xrightarrow{\partial_0^{\mathcal{E}}} G_{-1}^{(1,\mathcal{E})} \xrightarrow{\partial_{-1}^{\mathcal{E}}} 0,$$

The map $\partial_1^{\mathcal{E}}$ is the zero map since $G_0^{(1,\mathcal{E})} = \{0\}$. This together with the fact that $G_1^{(1,\mathcal{E})}$ is the first vector space of the sequence gives

$$H_1(G^{(1,\mathcal{E})}) = \frac{Ker(\partial_1^{\mathcal{E}})}{Im(\partial_2^{\mathcal{E}})} = \frac{G_1^{(1,\mathcal{E})}}{\{0\}} = G_1^{(1,\mathcal{E})} \cong \mathbb{Q}.$$

Combining with the shifting degree isomorphism, we get

$$\widetilde{H}_2(\Delta_{1,\mathcal{E}};\mathbb{Q}) \cong \mathbb{Q}.$$

In the same way we can compute

$$\widetilde{H}_1(\Delta_{1,\mathcal{E}};\mathbb{Q}) \cong H_0(G^{(1,\mathcal{E})}) = \frac{Ker(\partial_0^{\mathcal{E}})}{Im(\partial_1^{\mathcal{E}})} = \{0\}$$

and

$$\widetilde{H}_0(\Delta_{1,\mathcal{E}};\mathbb{Q}) \cong H_{-1}(G^{(1,\mathcal{E})}) = \frac{Ker(\partial_{-1}^{\mathcal{E}})}{Im(\partial_0^{\mathcal{E}})} = \frac{G_{-1}^{(1,\mathcal{E})}}{\{0\}} = G_{-1}^{(1,\mathcal{E})} \cong \mathbb{Q},$$

since both $\partial_{-1}^{\mathcal{E}}$ and $\partial_0^{\mathcal{E}}$ are the zero map, $G_0^{(1,\mathcal{E})} = \{0\}$ and $G_{-1}^{(1,\mathcal{E})} \cong \mathbb{Q}$.

Next, we have the following complex:

$$G^{(1,\mathcal{A}_2)} : G_1^{(1,\mathcal{A}_2)} \xrightarrow{\partial_1^{\mathcal{A}_2}} G_0^{(1,\mathcal{A}_2)} \xrightarrow{\partial_0^{\mathcal{A}_2}} G_{-1}^{(1,\mathcal{A}_2)} \xrightarrow{\partial_{-1}^{\mathcal{A}_2}} 0$$

First we observe that all the three spaces of the sequence are isomorphic to $\mathbb{Q}$. We observe also that the map $\partial_1^{\mathcal{A}_2}$ sends the generators $[G_1, \omega_{1,j}]$ into 0, so it is the zero map, giving $Ker(\partial_1^{\mathcal{A}_2}) = G_1^{(1,\mathcal{A}_2)}$ and $Im(\partial_1^{\mathcal{A}_2}) = \{0\}$. So we have

$$\widetilde{H}_2(\Delta_{1,\mathcal{A}_2};\mathbb{Q}) \cong H_1(G^{(1,\mathcal{A}_2)}) = \frac{Ker(\partial_1^{\mathcal{A}_2})}{Im(\partial_2^{\mathcal{A}_2})} = G_1^{(1,\mathcal{A}_2)} \cong \mathbb{Q}.$$

The map $\partial_0^{\mathcal{A}_2}$ sends generators of type $[G_{11}, \omega_{11,j}]$ (which modulo the sign relation are the same up to the sign) into $[G_{18}, \omega]$, so it is an isomorphism. We can conclude that

$$\widetilde{H}_1(\Delta_{1,\mathcal{A}_2}; \mathbb{Q}) \cong H_0(G^{(1,\mathcal{A}_2)}) = \frac{Ker(\partial_0^{\mathcal{A}_2})}{Im(\partial_1^{\mathcal{A}_2})} = \{0\}$$

since $Ker(\partial_0^{\mathcal{A}_2}) = \{0\}$ and

$$\widetilde{H}_0(\Delta_{1,\mathcal{A}_2}; \mathbb{Q}) \cong H_{-1}(G^{(1,\mathcal{A}_2)}) = \frac{Ker(\partial_{-1}^{\mathcal{A}_2})}{Im(\partial_0^{\mathcal{A}_2})} = \frac{G_{-1}^{(1,\mathcal{A}_2)}}{G_0^{(1,\mathcal{A}_2)}} = \{0\}.$$

The third complex is $G^{(1,\mathcal{A}_3)}$

$$G^{(1,\mathcal{A}_3)} : G_1^{(1,\mathcal{A}_3)} \xrightarrow{\partial_1^{\mathcal{A}_3}} G_0^{(1,\mathcal{A}_3)} \xrightarrow{\partial_0^{\mathcal{A}_3}} G_{-1}^{(1,\mathcal{A}_3)} \xrightarrow{\partial_{-1}^{\mathcal{A}_3}} 0$$

Here $G_1^{(1,\mathcal{A}_3)}$ and $G_0^{(1,\mathcal{A}_3)}$ are isomorphic to $\mathbb{Q}^2$. The Kernel of the map $\partial_1^{\mathcal{A}_3}$ is generated by the generators $[G_1, \omega_{1,j}]$, which are all equivalent, so it has dimension 1, and it is isomorphic to $\mathbb{Q}$, giving

$$\widetilde{H}_2(\Delta_{1,\mathcal{A}_3}; \mathbb{Q}) \cong H_1(G^{(1,\mathcal{A}_3)}) = \frac{Ker(\partial_1^{\mathcal{A}_3})}{Im(\partial_2^{\mathcal{A}_3})} = \frac{Ker(\partial_{-1}^{\mathcal{A}_3})}{\{0\}} \cong \mathbb{Q}.$$

Since $\partial_0^{\mathcal{A}_3}$ sends both the types of generator $[G_{11}, \omega_{11,j}]$ and $[G_{12}, \omega_{12,j}]$ into $[G_{18}, \omega]$ the map is surjective with Kernel spanned by $[G_{11}, \omega_{11,j}] - [G_{12}, \omega_{12,j}]$, and in particular it coincides with the image of $\partial_1^{\mathcal{A}_3}$, giving the following result:

$$\widetilde{H}_1(\Delta_{1,\mathcal{A}_3}; \mathbb{Q}) \cong H_0(G^{(1,\mathcal{A}_3)}) = \frac{Ker(\partial_0^{\mathcal{A}_3})}{Im(\partial_1^{\mathcal{A}_3})} = \{0\}.$$

Since $\partial_{-1}^{\mathcal{A}_3}$ is the last map we can conclude also for $\widetilde{H}_0(\Delta_{1,\mathcal{A}_3}; \mathbb{Q})$:

$$\widetilde{H}_0(\Delta_{1,\mathcal{A}_3}; \mathbb{Q}) \cong H_{-1}(G^{(1,\mathcal{A}_3)}) = \frac{Ker(\partial_{-1}^{\mathcal{A}_3})}{Im(\partial_0^{\mathcal{A}_3})} = \frac{G_{-1}^{(1,\mathcal{A}_3)}}{G_{-1}^{(1,\mathcal{A}_3)}} = \{0\}.$$

At the fourth step of the filtration we have the complex

$$G^{(1,\mathcal{A}_4)} : G_1^{(1,\mathcal{A}_4)} \xrightarrow{\partial_1^{\mathcal{A}_4}} G_0^{(1,\mathcal{A}_4)} \xrightarrow{\partial_0^{\mathcal{A}_4}} G_{-1}^{(1,\mathcal{A}_4)} \xrightarrow{\partial_{-1}^{\mathcal{A}_4}} 0$$

The space $G_1^{(1,\mathcal{A}_4)}$ has generators coming from graphs G_1, G_2 and G_3, and by the computations of ∂ we made above we have that $Ker(\partial_1^{\mathcal{A}_4})$ is spanned by $[G_1, \omega_{1,j}]$ and has dimension 1. So we can compute

$$\widetilde{H}_2(\Delta_{1,\mathcal{A}_4}; \mathbb{Q}) \cong H_1(G^{(1,\mathcal{A}_4)}) = \frac{Ker(\partial_1^{\mathcal{A}_4})}{Im(\partial_2^{\mathcal{A}_4})} = \frac{Ker(\partial_1^{\mathcal{A}_4})}{\{0\}}) \cong \mathbb{Q}.$$

We observe that $dim(Im(\partial_1^{\mathcal{A}_4})) = 2$ and by construction $Im(\partial_1^{\mathcal{A}_4}) \subseteq Ker(\partial_0^{\mathcal{A}_4})$ as a vector subspace. By the fact that $\partial_0^{\mathcal{A}_4}$ is surjective, which is evident by the computations above, we conclude again that $dim(Ker(\partial_0^{\mathcal{A}_4})) = 2$, so $Im(\partial_1^{\mathcal{A}_4}) = Ker(\partial_0^{\mathcal{A}_4})$ and

$$\widetilde{H}_1(\Delta_{1,\mathcal{A}_4};\mathbb{Q}) \cong H_0(G^{(1,\mathcal{A}_4)}) = \frac{Ker(\partial_0^{\mathcal{A}_4})}{Im(\partial_1^{\mathcal{A}_4})} = \{0\}.$$

At the same time, since $\partial_0^{\mathcal{A}_4}$ is surjective, we conclude also that

$$\widetilde{H}_0(\Delta_{1,\mathcal{A}_4};\mathbb{Q}) \cong H_{-1}(G^{(1,\mathcal{A}_4)}) = \frac{Ker(\partial_{-1}^{\mathcal{A}_4})}{Im(\partial_0^{\mathcal{A}_4})} = \frac{G_{-1}^{(1,\mathcal{A}_4)}}{G_{-1}^{(1,\mathcal{A}_4)}} = \{0\}.$$

The last complex of the filtration coincides with the standard one:

$$G^{(1,3)} : G_1^{(1,3)} \xrightarrow{\partial_1} G_0^{(1,3)} \xrightarrow{\partial_0} G_{-1}^{(1,3)} \xrightarrow{\partial_{-1}} 0$$

Here the situation is completely analogous to the one of the previous case, except with the fact that the first two spaces $G^{(1,3)}$, $G_1^{(1,3)}$ and $G_0^{(1,3)}$ have dimension 4 instead of 3. By similar computation we get

$$\widetilde{H}_2(\Delta_{1,3};\mathbb{Q}) \cong H_1(G^{(1,3)}) \cong \mathbb{Q},$$

$$\widetilde{H}_1(\Delta_{1,3};\mathbb{Q}) \cong H_0(G^{(1,3)}) = \{0\},$$

$$\widetilde{H}_0(\Delta_{1,3};\mathbb{Q}) \cong H_{-1}(G^{(1,3)}) = \{0\}.$$

We give a first general result we can obtain about the homology of $\Delta_{g,\mathcal{A}}$ using Graph Complexes.

Proposition 4.3.5. *Let $(g, \mathcal{A})$ be an input datum, $g \geq 1$. Then we have the following:*

1) The homology $\widetilde{H}_k(\Delta_{g,\mathcal{A}};\mathbb{Q}) = 0$ for any $k \leq g-1$ and $k > 3g+n-4$;

2) Fix $g = 1$. If $[Ch_{\mathcal{A}}] = [Ch_{\mathcal{E}}]$ then $\widetilde{H}_0(\Delta_{1,\mathcal{A}};\mathbb{Q}) \cong \mathbb{Q}$, otherwise it is zero.

Proof. We consider the Graph Complex $G^{(g,\mathcal{A})}$. Since in each degree its generators come from pure graphs, and we have pure graphs only when $g \leq |E(G)| \leq 3g-3+n$, (1) trivially follows.

Now for any input datum $(g, \mathcal{A})$ the only pure $(g, \mathcal{A})$-stable graph is the graph $R_{g,n}$ made by a single vertex, g loops and all the legs attached to it. If we have more than one loop $R_{g,n}$ admits multiple edges so its class is zero, otherwise if $g = 1$, $G_{-1}^{(1,\mathcal{A})} =< [R_{1,n}, \omega] >\cong \mathbb{Q}$.

Assume $\mathcal{A} = \mathcal{E}$, then $[R_{1,n}, \omega] \in Ker(\partial_{-1}^{\mathcal{E}}) - Im(\partial_0^{\mathcal{E}})$, so it generates $H_{-1}(G^{(1,\mathcal{E})}) \cong \mathbb{Q} \cong \widetilde{H}_0(\Delta_{1,\mathcal{E}};\mathbb{Q})$ (and this is true for any $\mathcal{A}$ such that $[Ch_{\mathcal{A}}] = [Ch_{\mathcal{E}}]$).

If $[Ch_{\mathcal{A}}] \neq [Ch_{\mathcal{E}}]$, then for every g we have $[R_{g,n}, \omega] \in Ker(\partial_{-g}^{\mathcal{A}}) \cap Im(\partial_{-g+1}^{\mathcal{A}})$, so $H_{-g}(G^{(g,\mathcal{A})}) \cong 0 \cong \widetilde{H}_{g-1}(\Delta_{g,\mathcal{A}};\mathbb{Q})$. $\square$

4.3.2 Filtered Chain Complexes

We introduce here Filtered Chain Complexes and some properties of these objects, as we want to see Graph Complexes as Filtered Chain Complexes using an analogous of the Filtration Theorem 4.2.2. Our references for this section are [McC01] and [Hut11].

Definition 4.3.6. A filtered module is an R-module A with an increasing sequence of submodules $F_pA \subset F_{p+1}A$ indexed by $p \in \mathbb{Z}$ such that $\bigcup_{p\in\mathbb{Z}} F_pA = A$ and $\bigcap_{p\in\mathbb{Z}} F_pA = \{0\}$. We call $\{F_pA\}_{p\in\mathbb{Z}}$ a filtration of A.

We say that the filtration $\{F_pA\}_{p\in\mathbb{Z}}$ is bounded if $F_pA = \{0\}$ for sufficiently small p and $F_pA = A$ for sufficiently large p.

Definition 4.3.7. Let A be a filtered module. The associated graded module is defined, in degree p, as $G_pA = F_pA/F_{p-1}A$.

Notice that there is a short exact sequence

$$0 \to F_{p-1}A \to F_pA \to G_pA \to 0.$$

Definition 4.3.8. A Filtered Chain Complex is a chain complex (C_*, ∂) together with a filtration $\{F_pC_i\}_{p\in\mathbb{Z}}$ on each C_i such that the differential preserves the filtration, i.e. $\partial(F_pC_i) \subset F_pC_{i-1}$.

We have a well defined induced differential $\partial : G_pC_i \to G_pC_{i-1}$, and so we can define an associated graded chain complex G_pC_*. Moreover, there is an induced filtration on the homology of C_* given by

$$F_pH_i(C_*) = \{\alpha \in H_i(C_*) | \alpha = [x], \exists x \in F_pC_i\}.$$

Again, this has associated graded pieces $G_pH_i(C_*)$ defined as before.

Definition 4.3.9. A spectral sequence consists of the following:

- An R-module $E^r_{p,q}$ defined for each $p, q \in \mathbb{Z}$ and each integer $r \geq r_0$, where r_0 is some nonnegative integer;
- Differentials $\partial_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ such that $\partial_r^2 = 0$ and E^{r+1} is the homology of (E^r, ∂_r), i.e.,
$$E^{r+1}_{p,q} = \frac{Ker(\partial_r : E^r_{p,q} \to E^r_{p-r,q+r-1})}{Im(\partial_r : E^r_{p+r,q-r+1} \to E^r_{p,q})}$$

A spectral sequence converges if for every p, q, if r is sufficiently large then ∂_r vanishes on $E^r_{p,q}$. In this case, for each p, q, the module $E^r_{p,q}$ is independent of r for r sufficiently large, and we denote this by $E^\infty_{p,q}$. For a given r, the collection of R-modules $\{E^r_{p,q}\}$, together with the differentials ∂_r between them, is called the r-th page of the spectral sequence. Each page is the homology of the previous page.

Given a Filtered Chain Complex, we have an associated spectral sequence obtained from the short exact sequences extracted from the filtrations. Namely, let $E^0_{p,q} := G_pC_{p+q}$. Then there is a well defined $\partial : E^0_{p,q} \to E^0_{p,q-1}$. We denote $E^1_{p,q} = H_{p+q}(G_pC_*)$, and we define $\partial_1 : E^1_{p,q} \to E^1_{p-1,q}$ as follows. A homology class $\alpha \in E^1_{p,q}$ can be represented by a chain $x \in F_pC_{p+q}$ such that $\partial x \in F_{p-1}C_{p+q-1}$. We set $\partial_1(\alpha) = [\partial x]$. It follows easily from $\partial_1^2 = 0$ that ∂_1 is well defined and $\partial_1^2 = 0$. We now consider the homology

$$E^2_{p,q} = \frac{Ker(\partial_1 : E^1_{p,q} \to E^1_{p-1,q})}{Im(\partial_1 : E^1_{p+1,q} \to E^1_{p,q})}.$$

We can iterate this process for every nonnegative integer r, so we can define an r-th order approximation to $G_pH_{p+q}(C_*)$ by

$$E^r_{p,q} = \frac{\{x \in F_pC_{p+q} | \partial x \in F_{p-r}C_{p+q-1}\}}{F_{p-1}C_{p+q} + \partial(F_{p+r-1}C_{p+q+1})}.$$

Here the notation indicates the quotient of the numerator by its intersection with the denominator.

4.3.3 Wall-crossing on Graph Complexes

As already mentioned, the Graph Complexes $G^{(g,\mathcal{A})}$'s are defined upon the same stability conditions that we have been considering so far on graphs, so we can establish a theorem analogous to Theorem 4.2.2 which holds for them. First, we show the analogous version of Proposition 4.2.10.

Proposition 4.3.10. *Let $g, n \geq 1$ be two integers, and $\mathcal{A}$ and $\mathcal{B}$ two weight data in $\mathcal{D}_{g,n}$.*

1) *If $\mathcal{A} \leq \mathcal{B}$, then $G^{(g,\mathcal{A})} \subset G^{(g,\mathcal{B})}$;*

2) *If $\mathcal{A}$ and $\mathcal{B}$ are in the same chamber, then $G^{(g,\mathcal{A})} = G^{(g,\mathcal{B})}$;*

3) *If $\mathcal{A}$ and $\mathcal{B}$ are obtained one from the other through a permutation of coordinates, then $G^{(g,\mathcal{A})}$ is isomorphic to $G^{(g,\mathcal{B})}$;*

4) *If $\mathcal{A}$ and $\mathcal{B}$ are in chambers Ch_1 and Ch_2 such that $[Ch_1] = [Ch_2] \in [\mathbf{K}_f]$, then $G^{(g,\mathcal{A})}$ is isomorphic to $G^{(g,\mathcal{B})}$.*

Proof. A generator of $G^{(g,\mathcal{A})}$ is a $(g, \mathcal{A})$-stable graph with an edge ordering under the relation $[\mathbf{G}, \omega] = sgn(\sigma)[\mathbf{G}', \omega']$ if there is an isomorphism of n-marked graphs $\mathbf{G} \cong \mathbf{G}'$ under which the orders ω and ω' are related by the permutation $\sigma \in S_{|E(\mathbf{G})|}$. If $\mathcal{A} \leq \mathcal{B}$, every $(g, \mathcal{A})$-stable graph is also $(g, \mathcal{B})$-stable, so to show 1) we just send a generator into itself. For 2), if $\mathcal{A}$ and $\mathcal{B}$ are in the same chamber then $\mathcal{G}_{g,\mathcal{A}} = \mathcal{G}_{g,\mathcal{B}}$, so the result follows. Proof of 3) follows by the same reasoning of 1 by sending a generator of $G^{(g,\mathcal{A})}$ into the generator of $G^{(g,\mathcal{B})}$ obtained relabeling the legs according to the permutation which sends $\mathcal{A}$ to $\mathcal{B}$. This also shows 4) as by hypothesis we can assume $\mathcal{A}$ and $\mathcal{B}$ are obtained one from the other after a permutation of coordinates. $\square$

Theorem 4.3.11. *Let $g \geq 0$, $n \geq 1$ be two integers. Fix a weight datum $\mathcal{A} \in \mathcal{D}_{g,n}$. There are filtrations of $G^{(g,\mathcal{A})}$ induced by the partial order on the set of chambers up to symmetry given by inclusions of complexes. Namely an ordered sequence*

$$[Ch_{\mathcal{A}_1}] \leq [Ch_{\mathcal{A}_2}] \leq ... \leq [Ch_{\mathcal{A}_p}] \leq ... \leq [Ch_{\mathcal{A}_{N-1}}] \leq [Ch_{\mathcal{A}}],$$

induces a filtration of chain complexes

$$G^{(g,\mathcal{A}_1)} \hookrightarrow G^{(g,\mathcal{A}_2)} \hookrightarrow ... \hookrightarrow G^{(g,\mathcal{A}_p)} \hookrightarrow ... \hookrightarrow G^{(g,\mathcal{A}_{N-1})} \hookrightarrow G^{(g,\mathcal{A})},$$

with $G^{(g,\mathcal{A}_{p-1})} \hookrightarrow G^{(g,\mathcal{A}_p)}$ being an injective map of chain complexes for every $p = 2, ..., N$.

Proof. At each step of the filtration $[Ch_{\mathcal{A}_p}] \leq [Ch_{\mathcal{A}_{p+1}}]$ we can find two weight data $\mathcal{A} \leq \mathcal{B}$ and two chambers $Ch_1 \leq Ch_2$ such that $\mathcal{A} \in Ch_1 \in [Ch_{\mathcal{A}_p}]$ and $\mathcal{B} \in Ch_2 \in [Ch_{\mathcal{A}_{p+1}}]$. Then by Proposition 4.3.10, $G^{(g,\mathcal{A}_p)} \cong G^{(g,\mathcal{A})} \subset G^{(g,\mathcal{B})} \cong G^{(g,\mathcal{A}_{p+1})}$ analogously to what we did to prove Theorem 4.2.2. □

Example 4.3.12. Consider again the situation of Example 4.3.1. We can consider an induced filtration of chain complexes

$$G^{(1,(\frac{1}{3},\frac{1}{3},\frac{1}{3}-\varepsilon))} \subset G^{(1,(\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon))} \subset G^{(1,(\frac{14}{27}-\varepsilon,\frac{12}{27},\frac{14}{27}))} \subset G^{(1,(1-\varepsilon,\frac{12}{27},\frac{14}{27}))} \subset G^{(1,3)},$$

which gives $G^{(1,3)}$ (and in general to all the chain complexes but the first) the structure of Filtered Chain Complex. The computation made in 4.3.1 shows also that the filtration does not descend to the homology, since for instance the well defined map

$$\widetilde{H}_0(\Delta_{1,\mathcal{E}};\mathbb{Q}) \to \widetilde{H}_0(\Delta_{1,\mathcal{A}_2};\mathbb{Q})$$

induced by the map on the chain complexes is not injective.

4.3.4 Decomposition of the Top Weight Cohomology

Fix $g \geq 1$, $n \geq 2$ and a sequence $[Ch_{\mathcal{A}_1}] \leq ... \leq [Ch_{\mathcal{A}_N}]$, $\mathcal{A}_i \in \mathcal{D}_{g,n}$ for every i from 1 to N. By Theorem 4.3.11 it gives a filtration of chain complexes

$$G^{(g,\mathcal{A}_1)} \hookrightarrow ... \hookrightarrow G^{(g,\mathcal{A}_p)} \hookrightarrow ... \hookrightarrow G^{(g,\mathcal{A}_N)};$$

If we set by convention $G^{(g,\mathcal{A}_p)} = \{0\}$ for every $p \leq 0$ and we let it stabilize at the last term for every $p \geq N$, we can extend the above filtration for every $p \in \mathbb{Z}$, with the trivial differential outside the bounds 0 and N. Then the induced filtration on each $G_j^{(g,\mathcal{A}_N)}$ makes $G^{(g,\mathcal{A}_N)}$ a Filtered Chain Complex.

This structure on $G^{(g,\mathcal{A}_N)}$ induces a spectral sequence as already seen, and we have

$$E^0_{p,q} = G_p G^{(g,\mathcal{A}_N)}_{p+q} = F_p G^{(g,\mathcal{A}_N)}_{p+q} / F_{p-1} G^{(g,\mathcal{A}_N)}_{p+q} = G^{(g,\mathcal{A}_p)}_{p+q} / G^{(g,\mathcal{A}_{p-1})}_{p+q} := G^{(g,\mathcal{A}_p,\mathcal{A}_{p-1})}_{p+q},$$

with $G^{(g,\mathcal{A}_p,\mathcal{A}_{p-1})}$ being the complex of $\mathcal{A}_p$ but not $\mathcal{A}_{p-1}$ stable graphs, and

$$E^1_{p,q} = H_{p+q}(G_p G^{(g,\mathcal{A}_N)}) = H_{p+q}(G^{(g,\mathcal{A}_p)}_{p+q} / G^{(g,\mathcal{A}_{p-1})}_{p+q}) = H_{p+q}(G^{(g,\mathcal{A}_p,\mathcal{A}_{p-1})}_{p+q}).$$

So we are ready to state the following Theorem.

Theorem 4.3.13. *Fix $g \geq 1$, $n \geq 2$. Assume we have a sequence of chambers up to symmetry*

$$[Ch_{\mathcal{A}_1}] \leq ... \leq [Ch_{\mathcal{A}_p}] \leq ... \leq [Ch_{\mathcal{A}_N}],$$

and let

$$G^{(g,\mathcal{A}_1)} \hookrightarrow ... \hookrightarrow G^{(g,\mathcal{A}_p)} \hookrightarrow ... \hookrightarrow G^{(g,\mathcal{A}_N)}$$

be the induced filtration on Graph Complexes. Then

$$Gr^W_{6g-6+2n} H^{4g-6+2n-k}(\mathcal{M}_{g,\mathcal{A}_N}; \mathbb{Q}) \cong \bigoplus_{p=1}^{N} E^{\infty}_{p,k-p},$$

where the terms $E^{\infty}_{p,k-p}$ are the ones to which the spectral sequence induced by the above filtration converges.

Proof. We consider the homology ring $H_*(G^{(g,\mathcal{A}_N)}) = \bigoplus_{k\in\mathbb{Z}} H_k(G^{(g,\mathcal{A}_N)})$. By construction of the Graph Complexes $H_*(G^{(g,\mathcal{A}_N)})$ is supported only in degrees $1-2g \leq k \leq g+n-3$ so the sum is finite, and since any of the $H_k(G^{(g,\mathcal{A}_N)})$'s is finite dimensional this is true also for $H_*(G^{(g,\mathcal{A}_N)})$. Moreover, each $H_k(G^{(g,\mathcal{A}_N)})$ comes with a filtration induced by the one on $G^{(g,\mathcal{A}_N)}$:

$$F_p H_k(G^{(g,\mathcal{A}_N)}) = \{\alpha \in H_k(G^{(g,\mathcal{A}_N)}) | \alpha = [x], x \in G_k^{(g,\mathcal{A}_p)}\},$$

so also $H_*(G^{(g,\mathcal{A}_N)})$ is filtered by $F_p H_*(G^{(g,\mathcal{A}_N)}) = \bigoplus_{k=1-2g}^{g+n-3} F_p H_k(G^{(g,\mathcal{A}_N)})$. This gives to $H_*(G^{(g,\mathcal{A}_N)})$ the structure of a finite dimensional filtered graded vector space.

By [McC01], Section 1 such an object can be decomposed, in each degree, as the direct sum

$$H_k(G^{(g,\mathcal{A}_N)}) = \bigoplus_{p+q=k} \frac{F_p H_{p+q}(G^{(g,\mathcal{A}_N)})}{F_{p-1} H_{p+q}(G^{(g,\mathcal{A}_N)})} = \bigoplus_{p+q=k} G_p H_{p+q}(G^{(g,\mathcal{A}_N)}).$$

Moreover, since the filtration is bounded, we can rewrite this sum taking only the significant indices:

$$H_k(G^{(g,\mathcal{A}_N)}) = \bigoplus_{p=1}^{N} \frac{F_p H_{p+q}(G^{(g,\mathcal{A}_N)})}{F_{p-1} H_{p+q}(G^{(g,\mathcal{A}_N)})} = \bigoplus_{p=1}^{N} G_p H_{p+q}(G^{(g,\mathcal{A}_N)}).$$

where $q = k - p$.

Consider now the associated spectral sequence: by construction we have

$$G_p H_{p+q}(G^{(g,\mathcal{A}_N)}) = E^{\infty}_{p,q}$$

since the spectral sequence converges to it, so

$$H_k(G^{(g,\mathcal{A}_N)}) = \bigoplus_{p=1}^{N} E^{\infty}_{p,k-p}.$$

To conclude it is enough to apply Corollary 4.3.3, so that

$$Gr^W_{6g-6+2n} H^{4g-6+2n-k}(\mathcal{M}_{g,\mathcal{A}_N}; \mathbb{Q}) \cong H_k(G^{(g,\mathcal{A}_N)}) = \bigoplus_{p=1}^{N} E^{\infty}_{p,k-p}.$$

□

We add some remarks about the proof:

Remark 4.3.14. The proof additionally shows that the same decomposition holds for $\widetilde{H}_{k-1}(\Delta_{g,\mathcal{A}_N};\mathbb{Q})$, just by applying Theorem 4.3.2 instead of the Corollary 4.3.3.

Remark 4.3.15. Since the filtration is bounded, by Lemma 3.1.d of [Hut11] the approximations of the spectral sequence stabilize after a certain r, i.e $E^\infty_{p,q} = E^r_{p,q}$ for r sufficiently large. By the description

$$E^r_{p,q} = \frac{\{x \in F_pC_{p+q} | \partial x \in F_{p-r}C_{p+q-1}\}}{F_{p-1}C_{p+q} + \partial(F_{p+r-1}C_{p+q+1})}$$

we can see that for our filtrations these terms stabilize when $r \geq max\{p, N-p+1\}$ since for these r's $F_{p-r}C_{p+q-1} = \{0\}$ and $F_{p+r-1}C_{p+q+1} = G^{(g,\mathcal{A}_N)}_{p+q}$. So $E^\infty_{p,q} = E^{max\{p,N-p+1\}}_{p,q}$.

Remark 4.3.16. The top weight cohomology $Gr^W_{6g-6+2n}H^{4g-6+2n-k}(\mathcal{M}_{g,\mathcal{A}_N})$ does not depend on the chosen filtration: a priori different filtrations give different spectral sequences and different convergence terms $E^\infty_{p,k-p}$, Nevertheless, the two direct sums will be isomorphic.

In general, some of the terms $E^\infty_{p,k-p}$ in the direct sum may be zero.

Example 4.3.17. We put ourselves in the case $g = 1, n = 3$. We fix the sequence of weight data of Example 4.2.12:

$$(\frac{1}{3},\frac{1}{3},\frac{1}{3}-\varepsilon) \leq (\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon) \leq (\frac{14}{27}-\varepsilon,\frac{12}{27},\frac{14}{27}) \leq (1-\varepsilon,\frac{12}{27},\frac{14}{27}) \leq 1^{(n)}.$$

We set also $\mathcal{A}_1 = (\frac{1}{3},\frac{1}{3},\frac{1}{3}-\varepsilon)$, $\mathcal{A}_2 = (\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon)$, $\mathcal{A}_3 = (\frac{14}{27}-\varepsilon,\frac{12}{27},\frac{14}{27})$, $\mathcal{A}_4 = (1-\varepsilon,\frac{12}{27},\frac{14}{27})$ and $\mathcal{A}_5 = 1^{(n)}$ for simplicity. The corresponding sequence of chambers up to symmetry is then

$$[Ch_{(\frac{1}{3},\frac{1}{3},\frac{1}{3}-\varepsilon)}] \leq [Ch_{(\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon)}] \leq [Ch_{(\frac{14}{27}-\varepsilon,\frac{12}{27},\frac{14}{27})}] \leq [Ch_{(1-\varepsilon,\frac{12}{27},\frac{14}{27})}] \leq [Ch_{1^{(n)}}].$$

By Theorem 4.3.13, we can compute the homology of $\Delta_{1,1^{(n)}} = \Delta_{1,3}$, which corresponds to the top weight cohomology of $\mathcal{M}_{1,3}$, computing first the terms of the spectral sequence of $G^{(1,3)}$ and then using the shifting degree isomorphism. Since we know that $G^{(1,3)}$ has homology only in degrees -1, 0, and 1, we have to compute only three direct sums, each with five terms. All the terms $E^r_{p,k-p}$ stabilize for $r \geq 5$.

- Case $k = -1$. Here we have to compute $\bigoplus_{p=1}^5 E^5_{p,-1-p}$. For each p from 1 to 5, the term $E^5_{p,-1-p}$ is equal to the quotient $G^{(1,\mathcal{A}_p)}_{-1}/(G^{(1,\mathcal{A}_{p-1})}_{-1} + \partial G^{(1,\mathcal{A}_{p-1})}_0)$. Now for every $p = 1,...5$, $G^{(1,\mathcal{A}_p)}_{-1}$ is generated by the loop graph with a single vertex and its only orientation depicted in Figure 4.14, so the quotient is zero whenever $p \geq 2$

  When $p = 1$, $\partial G^{(1,\mathcal{A}_{p-5+1})}_0 = G^{(1,\mathcal{A}_5)}_{-1} = G^{(1,\mathcal{A}_1)}_{-1}$ by what we saw, so the quotient is again zero. Hence

$$Gr^W_6 H^5(\mathcal{M}_{1,3};\mathbb{Q}) \cong \bigoplus_{p=1}^5 E^5_{p,-1-p} = \{0\}.$$

- Case $k = 0$: In this case, the terms are $E^5_{p,-p}$, for $p = 1, ..., 5$. On the numerator of the quotient which defines $E^5_{p,-1-p}$ we have $Ker(\partial : G_0^{(1,\mathcal{A}_p)} \to G_{-1}^{(1,\mathcal{A}_p)})$ for each p by construction. For $p = 1$ this is zero since $G_0^{(1,\mathcal{A}_1)}$ is already zero. When $p = 2$, $G_0^{(1,\mathcal{A}_2)}$ is generated by $[G, \omega_i]$, where G is the graph of Figure 4.11.

  All the ω_i are related by permutations of the indices, so there is only a generator class. The map ∂ sends the generators $[G, \omega_i]$ into $[L, \omega]$, so it is an isomorphism and $Ker(\partial : G_0^{(1,\mathcal{A}_2)} \to G_{-1}^{(1,\mathcal{A}_2)})$ is trivial. When $p = 3, 4, 5$, we can observe that the kernel $Ker(\partial : G_0^{(1,\mathcal{A}_p)} \to G_{-1}^{(1,\mathcal{A}_p)})$ coincides with $Im(\partial : G_0^{(1,\mathcal{A}_p)} \to G_{-1}^{(1,\mathcal{A}_p)})$, due to the generators coming from the graphs of figure 4.12. But $Im(\partial : G_0^{(1,\mathcal{A}_p)} \to G_{-1}^{(1,\mathcal{A}_p)})$ is exactly what we have in the denominator of the quotient defining $E^5_{p,-p}$, so it is zero. Hence all the terms of the direct sum are zero, so

$$Gr_6^W H^4(\mathcal{M}_{1,3}; \mathbb{Q}) = \{0\}.$$

- Case $k = 1$: Here we have to compute $\bigoplus_{p=1}^5 E^5_{p,1-p}$. By construction, we see that

$$E^5_{p,1-p} = Ker(\partial : G_1^{(1,\mathcal{A}_p)} \to G_0^{(1,\mathcal{A}_p)})/G_1^{(1,\mathcal{A}_{p-1})}.$$

  Now for every $\mathcal{A}_p$ the Kernel at the numerator has a single generator coming from the graph in Figure 4.7, but since it is $(1, \mathcal{A}_p)$-stable for any p the generators coming from it belong to $G_1^{(1,\mathcal{A}_{p-1})}$ for any $p = 2, ..., 5$, giving $E^5_{p,1-p} = 0$. When $p = 1$, $G_1^{(1,\mathcal{A}_0)} = \{0\}$ by the convention we adopted at the beginning, so

$$Gr_6^W H^3(\mathcal{M}_{1,3}; \mathbb{Q}) = \bigoplus_{p=1}^{5} E^5_{p,1-p} = E^5_{1,0} = Ker(\partial : G_1^{(1,\mathcal{A}_1)} \to G_0^{(1,\mathcal{A}_1)}) \cong \mathbb{Q}.$$

The computations made in this example give what was expected from Corollary 1.3 of [CGP22], which says that the top weight cohomology of $\mathcal{M}_{1,n}$ is supported in degree n with rank $(n-1)!/2$ for $n \geq 3$, which equals 1 when $n = 3$, and agree with the computations of Example 4.3.1.

Remark 4.3.18. We can use the Theorem to estimate the dimension of the cohomology of $\mathcal{M}_{g,n}$. Suppose we have $g, n \geq 1$, and a fixed sequence $[Ch_{\mathcal{A}_1}] \leq ... \leq [Ch_{\mathcal{A}_p}] \leq ... \leq [Ch_{\mathcal{A}_N}]$. Let $E^\infty_{p,k-p}$ be one of the pieces of the direct sum coming from the filtration, then

$$dim H^{4g-6+2n-k}(\mathcal{M}_{g,n}; \mathbb{Q}) \geq dim Gr^W_{6g-6+2n} H^{4g-6+2n-k}(\mathcal{M}_{g,n}; \mathbb{Q}) \geq dim E^\infty_{p,k-p},$$

so if we are able to estimate the dimension of one of the pieces we can give nonvanishing results for the cohomology of $\mathcal{M}_{g,n}$.

Example 4.3.19. Let $g = 2$, $n = 3$ and consider the sequence coming from the floor filtration of example 4.2.18, extended on the left by the minimal chamber:

$$Ch_{\varepsilon^{(3)}} \leq Ch_{\mathcal{H}_3} \leq Ch_{1^{(3)}}.$$

By the Theorem 4.3.13 we know that $Gr^W_{12}H^{8-k}(\mathcal{M}_{2,3};\mathbb{Q}) \cong \bigoplus_{p=1}^N E^\infty_{p,k-p}$. For $k = 2$ the last term of the sum is $E^3_{3,-1}$, which has dimension at least 1. To see this, consider the graphs in Figure 4.16, then $H_1 - H_2 + H_3 - G_1 + G_2 - G_3$ belongs to $G_2^{(2,3)}$ and one can see its differential is zero. However, its quotient by $G_2^{(2,\mathcal{H}_3)}$ is nonzero, namely it is $-G_1+G_2-G_3$, so it defines a nontrivial element of $E^3_{3,-5}$.

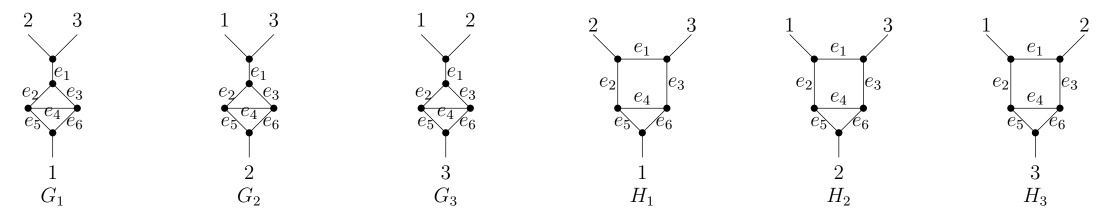

Figure 4.16: These are generators of $G_2^{(2,3)}$. The letters e_i represent the chosen order on the set of edges.

Then we conclude that

$$dimH^6(\mathcal{M}_{2,3};\mathbb{Q}) \geq dimGr^W_{12}H^6(\mathcal{M}_{2,3};\mathbb{Q}) \geq dimE^3_{3,-1} \geq 1,$$

i.e., $H^6(\mathcal{M}_{2,3};\mathbb{Q})$ does not vanish.

4.4 Relative Homology and Staircaise Diagrams

4.4.1 Relative Homology

For every weight data $\mathcal{A} \in Ch_1$ and $\mathcal{B} \in Ch_2$ such that $[Ch_1] \leq [Ch_2]$, we already know that there is an inclusion as a sub-symmetric Δ-complex $\Delta_{g,\mathcal{A}} \subset \Delta_{g,\mathcal{B}}$. There is a relation between the spectral sequence associated to a filtration and the relative homology with respect to the inclusion of $\Delta_{g,\mathcal{A}} \subset \Delta_{g,\mathcal{B}}$. Indeed, whenever $X \subset Y$ is a subcomplex, for every $p \geq -1$ one can consider the exact sequence

$$0 \to C_p(X) \to C_p(Y) \to C_p(Y,X) \to 0,$$

with $C_*(Y,X)$ computing the relative homology.

In particular, whenever we have an inclusion $\Delta_{g,\mathcal{A}} \hookrightarrow \Delta_{g,\mathcal{B}}$, the relative chain complex $C_*(\Delta_{g,\mathcal{B}}, \Delta_{g,\mathcal{A}})$ has its homology coinciding with relative rational homology

$$H_i(C_*(\Delta_{g,\mathcal{B}}, \Delta_{g,\mathcal{A}})) \cong H_i(\Delta_{g,\mathcal{B}}, \Delta_{g,\mathcal{A}};\mathbb{Q}),$$

as it has a natural isomorphism with $C_*(\Delta_{g,\mathcal{B}})/C_*(\Delta_{g,\mathcal{A}})$.

Now as in the proof of Theorem 4.3.2 we have a decomposition $C_*(\Delta_{g,\mathcal{A}}) = A^{(g,\mathcal{A})} \bigoplus B^{(g,\mathcal{A})}$, so we can decompose also $C_*(\Delta_{g,\mathcal{B}})/C_*(\Delta_{g,\mathcal{A}})$ into

$$A^{(g,\mathcal{B})}/C_*(\Delta_{g,\mathcal{A}}) \bigoplus B^{(g,\mathcal{B})}/C_*(\Delta_{g,\mathcal{A}}).$$

The term $B^{(g,\mathcal{B})}/C_*(\Delta_{g,\mathcal{A}})$ is acyclic, since also $B^{(g,\mathcal{B})}$ is. Now consider the shifting degree injection $j : G^{(g,\mathcal{A})} \to C_*(\Delta_{g,\mathcal{A}}, \mathbb{Q})$ of Theorem 4.3.2: it is clear by the definition of $A^{(g,\mathcal{B})}$ that $A^{(g,\mathcal{B})}/C_*(\Delta_{g,\mathcal{A}})$ is isomorphic to $A^{(g,\mathcal{B})}/j(G^{(g,\mathcal{A})})$ and this is isomorphic to

$$G^{(g,\mathcal{B})}/G^{(g,\mathcal{A})} := G^{(g,\mathcal{B},\mathcal{A})},$$

with the isomorphism shifting degrees by $2g-1$. The complex $G^{(g,\mathcal{B},\mathcal{A})}$ can be seen as the one generated by $(g,\mathcal{B})$-stable but not $(g,\mathcal{A})$-stable graphs, with the same conventions on the degree and the same boundary map.

Through the latter isomoprhism, we can conclude that there is an isomorphism

$$H_{k-2g+1}(G^{(g,\mathcal{B},\mathcal{A})}) \cong H_k(\Delta_{g,\mathcal{B}}, \Delta_{g,\mathcal{A}}; \mathbb{Q}).$$

Example 4.4.1. Consider the floor filtration of Example 4.2.18. A graph is $(g,\mathcal{H}_l)$-stable if and only if its leaves have at least l markings, and its vertices of valence 2 have at least one. In particular, when $l=2$ we get the usual notion of stability. Then the complex $G^{(g,\mathcal{H}_l,\mathcal{H}_{l+i})}$ is generated by stable graphs (in the standard sense) with a number of markings on each leaf between l and $l-i-1$., and by the previous computations we have

$$H_{k-2g+1}(G^{(g,\mathcal{H}_l,\mathcal{H}_{l+i})}) \cong H_k(\Delta_{g,\mathcal{H}_l}, \Delta_{g,\mathcal{H}_{l+i}}; \mathbb{Q}).$$

In particular, if $i=1$, $G^{(g,\mathcal{H}_l,\mathcal{H}_{l+1})}$ is generated by stable graphs with exactly l markings on each leaf.

When $l=2$, we have $\mathcal{H}_2 \in Ch_{1^{(n)}}$ so

$$H_{k-2g+1}(G^{(g,\mathcal{H}_2,\mathcal{H}_3)}) \cong H_k(\Delta_{g,n}, \Delta_{g,\mathcal{H}_3}; \mathbb{Q}).$$

As an example of computation, if $g=1$ and for every n we have $H_0(\Delta_{1,n}, \Delta_{1,\mathcal{H}_3}; \mathbb{Q}) = 0$, $H_1(\Delta_{1,n}, \Delta_{1,\mathcal{H}_3}; \mathbb{Q}) = 0$.

4.4.2 Staircase Diagrams of Graph Complexes

We fix $g, n \geq 1$, and a sequence of chambers up to symmetry $[Ch_1] \leq \ldots \leq [Ch_N]$, and for each chamber up to symmetry $[Ch_p]$ choose a weight datum $\mathcal{A}_p$ belonging to a chamber $Ch_p \in [Ch_p]$, for every p from 1 to N.

Consider again the filtrations of chain complexes:

$$G^{(g,\mathcal{A}_1)} \hookrightarrow \ldots \hookrightarrow G^{(g,\mathcal{A}_p)} \hookrightarrow \ldots \hookrightarrow G^{(g,\mathcal{A}_N)};$$

$$C_*(\Delta_{g,\mathcal{A}_1}) \hookrightarrow \ldots \hookrightarrow C_*(\Delta_{g,\mathcal{A}_p}) \hookrightarrow \ldots \hookrightarrow C_*(\Delta_{g,\mathcal{A}_N}).$$

We can consider the short exact sequences defined for each step of the filtrations above,

$$0 \to G^{(g,\mathcal{A}_p)} \to G^{(g,\mathcal{A}_{p+1})} \to G^{(g,\mathcal{A}_{p+1},\mathcal{A}_p)} \to 0;$$

$$0 \to C_*(\Delta_{g,\mathcal{A}_p}) \to C_*(\Delta_{g,\mathcal{A}_{p+1}}) \to C_*(\Delta_{g,\mathcal{A}_{p+1}}, \Delta_{g,\mathcal{A}_p}) \to 0,$$

leading to long exact sequences in homology:

$$
\begin{tikzcd}
\dots \arrow[r] & H_j(G^{(g,\mathcal{A}_{p-1})}) \arrow[r] \arrow[d, "\cong"] & H_j(G^{(g,\mathcal{A}_p)}) \arrow[r] \arrow[d, "\cong"] & H_{j-1}(G^{(g,\mathcal{A}_p,\mathcal{A}_{p-1})}) \arrow[r] & H_{j-1}(G^{(g,\mathcal{A}_{p-1})}) \arrow[r] \arrow[d, "\cong"] & \dots \\
\dots \arrow[r] & \widetilde{H}_l(\Delta_{g,\mathcal{A}_{p-1}};\mathbb{Q}) \arrow[r] & \widetilde{H}_l(\Delta_{g,\mathcal{A}_p};\mathbb{Q}) \arrow[r] & H_{l-1}(\Delta_{g,\mathcal{A}_p},\Delta_{g,\mathcal{A}_{p-1}};\mathbb{Q}) \arrow[r] & \widetilde{H}_{l-1}(\Delta_{g,\mathcal{A}_{p-1}};\mathbb{Q}) \arrow[r] & \dots
\end{tikzcd}
$$

where $l = j + 2g - 1$ and the vertical isomorphisms are the ones described in Theorem 4.3.2.

Since all the homology groups appear in different long exact sequences, we can rearrange them together in order to get a so called staircase diagram as follows:

$$
\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \\
\dots \arrow[r, red] & \widetilde{H}_{l+1}(\Delta_{g,\mathcal{A}_p};\mathbb{Q}) \arrow[r, red] \arrow[d] & H_{l+1}(\Delta_{g,\mathcal{A}_p},\Delta_{g,\mathcal{A}_{p-1}};\mathbb{Q}) \arrow[r, red] & \widetilde{H}_l(\Delta_{g,\mathcal{A}_{p-1}};\mathbb{Q}) \arrow[r] \arrow[d, red] & H_l(\Delta_{g,\mathcal{A}_{p-1}},\Delta_{g,\mathcal{A}_{p-2}};\mathbb{Q}) \arrow[r] & \dots \\
\dots \arrow[r] & \widetilde{H}_{l+1}(\Delta_{g,\mathcal{A}_{p+1}};\mathbb{Q}) \arrow[r] \arrow[d] & H_{l+1}(\Delta_{g,\mathcal{A}_{p+1}},\Delta_{g,\mathcal{A}_p};\mathbb{Q}) \arrow[r] & \widetilde{H}_l(\Delta_{g,\mathcal{A}_p};\mathbb{Q}) \arrow[r, red] \arrow[d] & H_l(\Delta_{g,\mathcal{A}_p},\Delta_{g,\mathcal{A}_{p-1}};\mathbb{Q}) \arrow[r, red] & \dots \\
\dots \arrow[r] & \widetilde{H}_{l+1}(\Delta_{g,\mathcal{A}_{p+2}};\mathbb{Q}) \arrow[r] \arrow[d] & H_{l+1}(\Delta_{g,\mathcal{A}_{p+2}},\Delta_{g,\mathcal{A}_{p+1}};\mathbb{Q}) \arrow[r] & \widetilde{H}_l(\Delta_{g,\mathcal{A}_{p+1}};\mathbb{Q}) \arrow[r] \arrow[d] & H_l(\Delta_{g,\mathcal{A}_{p+1}},\Delta_{g,\mathcal{A}_p};\mathbb{Q}) \arrow[r] & \dots \\
 & \vdots & & \vdots & &
\end{tikzcd}
$$

where in red we highlighted a single long exact sequence, and such that for each piece of the diagram, we have described a Graph Complex whose cohomology is isomorphic to it.

Example 4.4.2. Let $g = 1$, $n = 3$ and consider again the sequence of weight data

$$
\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}-\varepsilon\right) \leq \left(\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon,\frac{4}{9}-\varepsilon\right) \leq \left(\frac{14}{27}-\varepsilon,\frac{12}{27},\frac{14}{27}\right) \leq \left(1-\varepsilon,\frac{12}{27},\frac{14}{27}\right) \leq (1,1,1)\,,
$$

for semplicity, we refer to them again as $\mathcal{E}$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $1^{(3)}$. For every $\mathcal{A} \in \mathcal{D}_{1,3}$, we know that the $\tilde{H}_j(\Delta_{1,\mathcal{A}};\mathbb{Q})$ can be nonzero only for $j \in \{0, 1, 2\}$.

So we get the following diagram

$$
\begin{array}{ccccccccccccccc}
 & & 0 & & & & & & & & & & & & \\
 & & \downarrow & & & & & & & & & & & & \\
0 & \to & \tilde{H}_1(\Delta_{1,\mathcal{E}};\mathbb{Q}) & \longrightarrow & 0 & & & & & & & & & & \\
 & & \downarrow & & & & & & & & & & & & \\
0 & \to & \tilde{H}_2(\Delta_{1,\mathcal{A}_2};\mathbb{Q}) & \to & H_2(\Delta_{1,\mathcal{A}_2},\Delta_{1,\mathcal{E}};\mathbb{Q}) & \to & \tilde{H}_1(\Delta_{1,\mathcal{E}};\mathbb{Q}) & \longrightarrow & 0 & & & & & & \\
 & & \downarrow & & & & \downarrow & & & & & & & & \\
0 & \to & \tilde{H}_2(\Delta_{1,\mathcal{A}_3};\mathbb{Q}) & \to & H_2(\Delta_{1,\mathcal{A}_3},\Delta_{1,\mathcal{A}_2};\mathbb{Q}) & \to & \tilde{H}_1(\Delta_{1,\mathcal{A}_2};\mathbb{Q}) & \to & H_1(\Delta_{1,\mathcal{A}_2},\Delta_{1,\mathcal{E}};\mathbb{Q}) & \to & \tilde{H}_0(\Delta_{1,\mathcal{E}};\mathbb{Q}) & \longrightarrow & 0 & & \\
 & & \downarrow & & & & \downarrow & & & & \downarrow & & & & \\
0 & \to & \tilde{H}_2(\Delta_{1,\mathcal{A}_4};\mathbb{Q}) & \to & H_2(\Delta_{1,\mathcal{A}_4},\Delta_{1,\mathcal{A}_3};\mathbb{Q}) & \to & \tilde{H}_1(\Delta_{1,\mathcal{A}_3};\mathbb{Q}) & \to & H_1(\Delta_{1,\mathcal{A}_3},\Delta_{1,\mathcal{A}_2};\mathbb{Q}) & \to & \tilde{H}_0(\Delta_{1,\mathcal{A}_2};\mathbb{Q}) & \to & H_0(\Delta_{1,\mathcal{A}_2},\Delta_{1,\mathcal{E}};\mathbb{Q}) & \to & 0 \\
 & & \downarrow & & & & \downarrow & & & & \downarrow & & & & \\
0 & \to & \tilde{H}_2(\Delta_{1,3};\mathbb{Q}) & \longrightarrow & H_2(\Delta_{1,3},\Delta_{1,\mathcal{A}_4};\mathbb{Q}) & \to & \tilde{H}_1(\Delta_{1,\mathcal{A}_4};\mathbb{Q}) & \to & H_1(\Delta_{1,\mathcal{A}_4},\Delta_{1,\mathcal{A}_3};\mathbb{Q}) & \to & \tilde{H}_0(\Delta_{1,\mathcal{A}_3};\mathbb{Q}) & \to & H_0(\Delta_{1,\mathcal{A}_3},\Delta_{1,\mathcal{A}_2};\mathbb{Q}) & \to & 0 \\
 & & \downarrow & & & & \downarrow & & & & \downarrow & & & & \\
 & & 0 & & & & \tilde{H}_1(\Delta_{1,3};\mathbb{Q}) & \longrightarrow & H_1(\Delta_{1,3},\Delta_{1,\mathcal{A}_4};\mathbb{Q}) & \to & \tilde{H}_0(\Delta_{1,\mathcal{A}_4};\mathbb{Q}) & \to & H_0(\Delta_{1,\mathcal{A}_4},\Delta_{1,\mathcal{A}_3};\mathbb{Q}) & \to & 0 \\
 & & & & & & \downarrow & & & & \downarrow & & & & \\
 & & & & & & 0 & & & & \tilde{H}_0(\Delta_{1,3};\mathbb{Q}) & \longrightarrow & H_0(\Delta_{1,3},\Delta_{1,\mathcal{A}_4};\mathbb{Q}) & \to & 0 \\
 & & & & & & & & & & \downarrow & & & & \\
 & & & & & & & & & & 0 & & & &
\end{array}
$$

From Example 4.3.4, we already know the $\tilde{H}_j(\Delta_{1,\mathcal{A}};\mathbb{Q})$ that appear in the diagram, so we get the following:

$$
\begin{array}{ccccccccccccccc}
 & & 0 & & & & & & & & & & & & \\
 & & \downarrow & & & & & & & & & & & & \\
0 & \to & \mathbb{Q} & \longrightarrow & 0 & & & & & & & & & & \\
 & & \downarrow & & & & & & & & & & & & \\
0 & \to & \mathbb{Q} & \to & H_2(\Delta_{1,\mathcal{A}_2},\Delta_{1,\mathcal{E}};\mathbb{Q}) & \to & 0 & \longrightarrow & 0 & & & & & & \\
 & & \downarrow & & & & \downarrow & & & & & & & & \\
0 & \to & \mathbb{Q} & \to & H_2(\Delta_{1,\mathcal{A}_3},\Delta_{1,\mathcal{A}_2};\mathbb{Q}) & \to & 0 & \to & H_1(\Delta_{1,\mathcal{A}_2},\Delta_{1,\mathcal{E}};\mathbb{Q}) & \to & \mathbb{Q} & \longrightarrow & 0 & & \\
 & & \downarrow & & & & \downarrow & & & & \downarrow & & & & \\
0 & \to & \mathbb{Q} & \to & H_2(\Delta_{1,\mathcal{A}_4},\Delta_{1,\mathcal{A}_3};\mathbb{Q}) & \to & 0 & \to & H_1(\Delta_{1,\mathcal{A}_3},\Delta_{1,\mathcal{A}_2};\mathbb{Q}) & \to & 0 & \to & H_0(\Delta_{1,\mathcal{A}_2},\Delta_{1,\mathcal{E}};\mathbb{Q}) & \to & 0 \\
 & & \downarrow & & & & \downarrow & & & & \downarrow & & & & \\
0 & \to & \mathbb{Q} & \to & H_2(\Delta_{1,3},\Delta_{1,\mathcal{A}_4};\mathbb{Q}) & \to & 0 & \to & H_1(\Delta_{1,\mathcal{A}_4},\Delta_{1,\mathcal{A}_3};\mathbb{Q}) & \to & 0 & \to & H_0(\Delta_{1,\mathcal{A}_3},\Delta_{1,\mathcal{A}_2};\mathbb{Q}) & \to & 0 \\
 & & \downarrow & & & & \downarrow & & & & \downarrow & & & & \\
 & & 0 & & & & 0 & \to & H_1(\Delta_{1,3},\Delta_{1,\mathcal{A}_4};\mathbb{Q}) & \to & 0 & \to & H_0(\Delta_{1,\mathcal{A}_4},\Delta_{1,\mathcal{A}_3};\mathbb{Q}) & \to & 0 \\
 & & & & & & \downarrow & & & & \downarrow & & & & \\
 & & & & & & 0 & & & & 0 & \to & H_0(\Delta_{1,3},\Delta_{1,\mathcal{A}_4};\mathbb{Q}) & \to & 0. \\
 & & & & & & & & & & \downarrow & & & & \\
 & & & & & & & & & & 0 & & & &
\end{array}
$$

Appendix A

Extra proofs

In this Appendix we present some extra proofs of results in the work, which are more technical or are alternative formulations which are interesting on their own.

A.1 Contractibility results on some sub-loci of the link

In this section we prove Theorem 2.1.1, restated below. The proof uses techniques coming from [CGP22], and mildly generalizes some definitions used there.

Theorem A.1.1. *Let $g \geq 1$, $n \geq 1$ be integers, and let $\mathcal{A} \in \mathcal{D}_{g,n}$ be a weight datum. Then the following subcomplexes are either empty or contractible:*

1. *The subset $\Delta^{w}_{g,\mathcal{A}}$ of tropical curves with at least a strictly positive weighted vertex;*
2. *The subset $\Delta^{lw}_{g,\mathcal{A}}$ of tropical curves with at least a strictly positive weighted vertex and/or loops;*
3. *The closure of the subset of tropical curves with bridges $\Delta^{br}_{g,\mathcal{A}}$.*

Fix g, n and a weight datum $\mathcal{A} = (a_1, ..., a_n) \in \mathcal{D}_{g,n}$ as in the Theorem. Recall that p-simplicies in $\Delta_{g,\mathcal{A}}$ are indexed by pairs (G, τ) where $G \in Ob(\mathcal{G}_{g,\mathcal{A}})$ and $\tau : E(G) \to [p]$ is a bijection. Vertices of $\Delta_{g,\mathcal{A}}$ are (indexed by) 1-edge $(g, \mathcal{A})$-stable graphs. One of them is the graph with one vertex and a loop, the other vertices of the simplices are bridge graphs constructed as follows: let (g', S) be a pair where $0 \leq g' \leq g$ is an integer and $S \subset [n]$ such that

$$2g' - 1 + \textstyle\sum_{i \in S} a_i > 0 \quad \text{and} \quad 2(g - g') - 1 + \textstyle\sum_{i \notin S} a_i > 0.$$

Then, for any such pair (g', S) there is only a bridge graph with a g'-weighted vertex v with $L(v) = S$ (Figure A.1), we will call this point $B(g', S)$.

Now, for each pair of integers (g', n') such that $g \leq g$ and $n' \leq n$, we define the Property

$$P^{\mathcal{A}}_{g',n'} = \{B(g', S) : |S| = n'\} \subset \Delta_{g,\mathcal{A}}(0).$$

in the sense of Definition 1.2.27. We introduce some tools we need for the proof of Theorem 2.1.1.

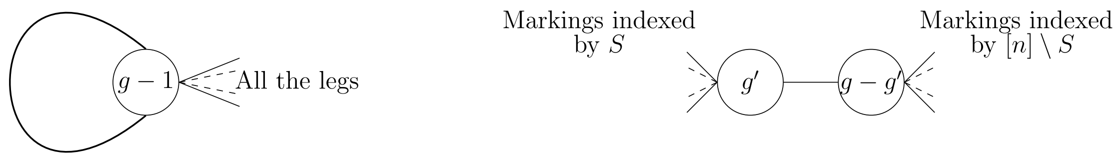

Figure A.1: The 1-edge $(g, \mathcal{A})$-stable graphs.

Definition A.1.2. Let G be a $(g, \mathcal{A})$-stable graph and let $v \in V(G)$. We say that v is an $\mathcal{A}$-articulation point if at least one of the following conditions is true:

1. v is a cut vertex of G;
2. $w(v) > 0$;
3. $|v|_{\mathcal{A}} \geq 2$.

Notice when $\mathcal{A} = 1^{(n)}$, this agrees with the definition of articulation point of [CGP22]. Let $\mathbf{V}_{\mathcal{A}}$ be the set of $\mathcal{A}$-articulation points of G, and let $\mathbf{B}$ be the set of blocks of G, i.e., the set of maximal connected subgraphs with at least one edge and no cut vertices of G. If $B \in \mathbf{B}$ is a block, denote by $V(B)$ the set of vertices $v \in V(G)$ which are contained in B.

Definition A.1.3. Let $G \in Ob(\mathcal{G}_{g,\mathcal{A}})$. The $\mathcal{A}$-block graph $\mathrm{Bl}_{\mathcal{A}}(G)$ of G is the bipartite graph defined as follows. The vertices of $\mathrm{Bl}_{\mathcal{A}}(G)$ are the $\mathbf{V}_{\mathcal{A}} \cup \mathbf{B}$, and there is an edge $e = (v, B)$ from $v \in \mathbf{V}_{\mathcal{A}}$ to $B \in \mathbf{B}$ if and only if $v \in V(B)$ as subgraph of G (and we denote the edge e by the notation (v, B)).

The $\mathcal{A}$-block graph is a weighted marked graph with decorations as follows: if $v \in \mathbf{V}_{\mathcal{A}}$, then we decorate the corresponding vertex in the $\mathcal{A}$-block graph with the weight and the markings it had as a vertex on G. If $B \in \mathbf{B}$ is a block in G, we depict it as the block itself, but keep track of the decorations only of vertices which are not in $\mathbf{V}_{\mathcal{A}}$. Again, $\mathrm{Bl}_{1^{(n)}}(G)$ =Bl(G) as defined in [CGP22]. Note that $\mathrm{Bl}_{\mathcal{A}}(G)$ does not verify necessarily any stability condition.

Example A.1.4. Fix $n = 5$ and $\mathcal{A} = (1, 1, \varepsilon, \varepsilon, \varepsilon)$ for $\varepsilon < \frac{1}{5}$. consider the graph in Figure A.2, which belongs to $Ob(\mathcal{G}_{5,\mathcal{A}})$. This graph has three $\mathcal{A}$-articulation points, which are u, v and z. Indeed, u is a cut vertex, v has positive weight and z is such that $|z|_{\mathcal{A}} \geq 2$. The $\mathcal{A}$-block graph of G depicted according to the conventions above is the one in Figure A.3. Notice that vertices a, b and c preserve their decorations in the block, as they are not $\mathcal{A}$-articulation points. In this way we can see each vertex of the block graph as a (eventually weighted and/or marked) graph on its own.

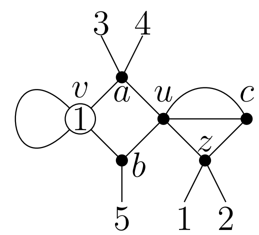

Figure A.2: A graph with five markings. We use letters to indicated the name of vertices. Here $\mathbf{V}_{\mathcal{A}} = \{u, v, z\}$, while a, b and c are not $\mathcal{A}$-articulation points.

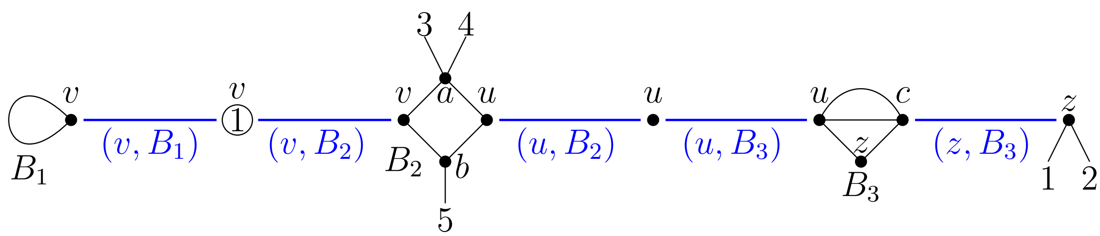

Figure A.3: The block graph of the graph in Figure A.2. We depict vertices in $\mathbf{B}$ as graphs, labeling them with B_i's. We depict edges between elements in $\mathbf{V}_{\mathcal{A}}$ and elements in $\mathbf{B}$ with thick blue lines, labeling them according to the conventions above.

Remark A.1.5. Observe that $\mathrm{Bl}_{\mathcal{A}}(G)$ is a tree. Suppose it is not, and let $\mathbf{V}_{\mathcal{A}} \cup \mathbf{B}$ be the set of vertices. Since $\mathrm{Bl}_{\mathcal{A}}(G)$ in not a tree, it contains a cycle or it is disconnected. If there is a cycle C in the $\mathcal{A}$-block graph, then there must be a cycle D in G, and all the vertices in D belong to the same block B. This implies that the cycle C in the $\mathcal{A}$-block graph can only pass through one vertex $B \in \mathbf{B}$. Since the block graph is bipartite, every cycle needs at least two vertices from each of the two sets of the partition, but this is a contradiction with what we already saw. Thus our block graph is acyclic. Now we need to show that it is connected, but this is a consequence of the fact that the original graph is connected.

Since $\mathrm{Bl}_{\mathcal{A}}(G)$ is a tree, in particular given an edge (v, B) the graph $\mathrm{Bl}_{\mathcal{A}}(G) \setminus \{(v, B)\}$ obtained deleting (v, B) has two connected components. In such situation, let T be the set of vertices of the component containing $B \in V(\mathrm{Bl}_{\mathcal{A}}(G))$; then we label the edge (v, B) with

$$(g(v, B), L(v, B)) := \left(\sum_{H \in T} g(H), \bigsqcup_{H \in T} L(H) \right),$$

where $g(H)$ is the genus of the sub-graph H of G. The set

$$\bigsqcup_{H \in T} L(H)$$

is the union of sets containing legs (seen as $[n]$).

Example A.1.6. Consider the $\mathcal{A}$-block graph of Example A.1.4, and choose the edge (u, B_3). Then the part of the $\mathcal{A}$-block graph containing B_3 is the one in Figure A.4, T is given by $\{B_3, z\}$ and the label becomes $(2, \{1, 2\})$.

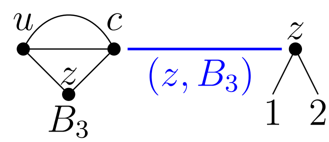

Figure A.4: The part of the block graph of the graph in Figure A.3 containing B_3.

Now we can generalize Lemma 4.21 of [CGP22]. Suppose $P \subset \Delta_{g,\mathcal{A}}([0])$ is a Property in the sense of Definition 1.2.27, G a $(g, \mathcal{A})$- stable graph. Recall that given a Property P and a symmetric Δ-complex X, we write $P(X)$ for the set of P-simplices, i.e., simplices in the symmetric Δ-complex X with vertices in P, and $P^*(X)$ for all the simplices being faces of a P-simplex. Recall that $\Delta_{g,\mathcal{A}}$ seen as a symmetric Δ-complex is a functor

$$\Delta_{g,\mathcal{A}} : I^{op} \to Sets,$$

with

$$X_p = \{\text{equivalence classes of pairs } (G, \tau)\},$$

with $G \in Ob(\mathcal{G}_{g,\mathcal{A}})$ and $\tau : E(G) \to [p]$ a bijection, where we consider $\tau = \tau'$ if they are in the same orbit under the evident action of $Aut(G)$. We adopt the following notation for the rest of the section:

- We say that $G \in P$ if $(G, \tau) \in P(\Delta_{g,\mathcal{A}})$ for every τ. Similarly we write $G \in P^*$ if $(G, \tau) \in P^*(\Delta_{g,\mathcal{A}})$ for every τ.
- We say that an edge $e \in E(G)$ is a (g', L')-bridge if $G/(E(G) \setminus \{e\})$ is isomorphic to $B(g', L')$.

Here by $G/(E(G)) \setminus \{e\}$ we intend the graph obtained contracting all the edges except e.

We also recall the equations 4.2.1 of [CGP22]:

$$\sum_{B \ni v} g(v, B) + w(v) = g \tag{A.1}$$

$$\sum_{B \ni v} n(v, B) + |v|_{1^{(n)}} = n \tag{A.2}$$

where $n(v, B) := |L(v, B)|$ and the sums are taken over all the possible blocks containing a fixed $v \in \mathbf{V}_{\mathcal{A}}$.

Lemma A.1.7. *Let $g \geq 1$, $n \geq 1$ be integers, and let $\mathcal{A} \in \mathcal{D}_{g,n}$ be a weight datum. Let $G \in Ob(\mathcal{G}_{g,\mathcal{A}})$.*

1. *If $e \in E(G)$ is a bridge then its image vertex in the graph obtained contracting e, $v \in V(G/e)$ is an $\mathcal{A}$-articulation point.*

2. *Let v be an $\mathcal{A}$-articulation point of G, $w(v) = u \geq 0$, $L(v) = M$, with edges of $Bl_{\mathcal{A}}(G)$ at v labeled $(g_1, L_1), ..., (g_s, L_s)$. Then v may be expanded into a bridge, with the result a $(g, \mathcal{A})$-stable marked, weighted graph, in any of the following ways. Choose a partition of M into two subsets M_1 and M_2, two integers w_1, w_2 such that $w_1 + w_2 = u$ for $j = 1, 2$, and a partition of $E(Bl_{\mathcal{A}}(G))$ into P_1 and P_2 such that*

$$\sum_{(v,B)\in P_j} |v|_{E,B} + \sum_{x_k \in M_j} a_k + 2w_j \geq 2,$$

where with $|v|_{E,B}$ we denote the number of half-edges which are not legs incident to v lying in B. Then v may be expanded into a bridge of type

$$\left(\sum_{(v,B)\in P_1} g((v,B)) + w_1, \bigsqcup_{(v,B)\in P_1} L((v,B)) \sqcup M_1 \right)$$

such that the result is $(g, \mathcal{A})$-stable, and no other $(g, \mathcal{A})$-stable expansions of v into bridges are possible.

3. *If $Bl_{\mathcal{A}}(G)$ has an edge labelled (g', L), $|L| = n'$, then $G \in (P^{\mathcal{A}}_{g',n'})^*$.*

4. *Suppose $g' \geq 1$ and $w(v) = 0$ for all $v \in V(G)$, and suppose every label (g'', L) on $E(Bl_{\mathcal{A}}(G))$ satisfies either $g'' > g'$, or $g'' = g'$ and $|L| > n'$. Then $G \notin (P^{\mathcal{A}}_{g',n'})^*$.*

Proof. The proof of (1) is an easy check since every vertex obtained contracting a bridge is a cut vertex, so an $\mathcal{A}-$articulation point. As for (2), it is enough to observe that the described expansion keeps the $\mathcal{A}$-stability, since the stability condition on the two new vertices is satisfied by construction, while on the other vertices nothing changes. To show statement 4, suppose by contradiction $G \in (P^{\mathcal{A}}_{g',n'})^*$, so it admits an uncontraction G' which contracts to a bridge $B(g', L')$, $g' < g$ and $|S| = n'$. So by 2, on G there must be an $\mathcal{A}$-articulation point v which can be expanded into a bridge of type (g', L'), with $|L'| = n'$ and

$$(g', L') = \left(\sum_{(v,B)\in P_1} g((v,B)) + w_1, \bigsqcup_{(v,B)\in P_1} L((v,B)) \sqcup M_1 \right),$$

for some choice of partition $P_1 \sqcup P_2$ of the blocks at v, which is impossible since this label is not allowed by hypothesis. For (3), suppose $\epsilon = (v, B)$ is labeled as (g', L'). If B is already a block made by a bridge there is nothing to show,since we can contract everything but the bridge to obtain $B(g', L')$, ans so $G \in (P^{\mathcal{A}}_{g',n'})^*$. Suppose $|v|_{E,B} \geq 2$, and let $B_1, ..., B_s$ be the remaining blocks on v.

If we have

$$\sum_{j=1}^{s} |v|_{E,B_j} + |v|_{\mathcal{A}} + 2w(v) \geq 2$$

then v can be expanded into a (g', L') bridge as in (2), with the partition given by leaving the blok B on a side of the bridge and the rest of the blocks $B_1, \dots, B_s$ with all the decorations on the other side. Assume

$$\sum_{j=1}^{s} |v|_{E,B_j} + \sum_{i \in L(v)} a_i + 2w(v) \leq 1,$$

so $w(v) = 0$, and $|v|_{E,B_1} = 1$ since it can not be zero, implying also $|v|_{\mathcal{A}} = 0$. So B_1 is a bridge, and by the identities A.1 and A.2 we can give it the label $(g - g', [n] \setminus L')$, which is equivalent to have the label (g', L') since the contraction to the bridge graph gives $B(g - g', [n] \setminus L') = B(g', L')$. $\square$

Proof of Theorem 2.1.1. The rest of this section is devoted to show Theorem 2.1.1. Recall that we have $g \geq 1$, $n \geq 1$ integers and $\mathcal{A} \in \mathcal{D}_{g,n}$ an input datum. We divide the proof in three cases, and show each of them separately.

Case 1: Assume $n = 1$. When $n = 1$, we have $\mathcal{A} = a \in (0, 1] \cap \mathbb{Q}$ and $\Delta_{g,a}$ is equal to $\Delta_{g,1}$, so the proof is the same as Theorem 1.1 of [CGP22] for $(g, n) = (g, 1)$. In this case, $\Delta^{w}_{1,1}$ and $\Delta^{br}_{1,1}$ are empty, and $\Delta^{lw}_{1,1} = \Delta_{1,1}$ is a point corresponding to the loop graph with a single leg attached to the base-point, so it is trivially contractible.

Case 2: Assume $g = 1$, $n \geq 2$ and $\mathcal{A} = (a_1, \dots, a_n)$ is such that $\sum_{i=0}^{n} a_i \leq 1$. We have the maximum number of edges in our graphs to be n, i.e., it is equal to the number of legs. The only combinatorial type with 1 edge is is the loop graph with a single vertex of weight zero and all the legs attached to it, and so so $\Delta^{lw}_{1,\mathcal{A}}$ is a point, which is trivially contractible. Since the weight of a vertex never grows through uncontractions, every $(1, \mathcal{A})$-stable graph G will be pure, so $\Delta^{w}_{1,\mathcal{A}} = \varnothing$. Moreover, by the fact that the genus is one and that the graphs are pure we can deduce that the number of edges equals the number of vertices, and this shows that all of our graphs are cycles, so the subcomplex $\Delta^{br}_{1,\mathcal{A}}$ is empty since cyclic graphs have no bridges.

Case 3. Here we take all the remaining cases: the first case is when $g = 1$, $n \geq 2$ and $\mathcal{A} = (a_1, ..., a_n)$, is such that $\sum_{i=0}^{n} a_i > 1$. The last case is when $g \geq 2$, $n \geq 2$ and $\mathcal{A}$ is any weight datum. We start observing that $P^{\mathcal{A}}_{1,0}$ is always non-empty with this assumptions. Moreover, whenever $\sum_{i=0}^{n} a_i > 1$ the Properties $P^{\mathcal{A}}_{0,i}$ are non-empty for every $i = 1, ..., n$. Let us consider the following sequence of Properties:

$$P^{\mathcal{A}}_{0,n}, ..., P^{\mathcal{A}}_{0,2}, P^{\mathcal{A}}_{1,0}, ..., P^{\mathcal{A}}_{1,n}, P^{\mathcal{A}}_{2,0}, ..., P^{\mathcal{A}}_{2,n} \cdots$$

If g is even, the last term of the sequence is chosen to be $P^{\mathcal{A}}_{g/2,\lfloor n/2 \rfloor}$, while if g is odd the last term is $P^{\mathcal{A}}_{(g-1)/2,n}$. With this choice each possible bridge graph appears in exactly one Property. We denote by $\mathbf{S}_{\mathcal{A}}$ the sequence above. When $\sum_{i=0}^{n} a_i \leq 1$, the terms $P^{\mathcal{A}}_{0,i}$ of the sequence are empty.

The next Theorem will serve us as a tool to complete the proof of the Case 3. It is a generalized version of Theorem 4.9 of [CGP22].

Theorem A.1.8. *Let $X = \Delta_{g,\mathcal{A}}$, and let $g \geq 1$ and $n \geq 2$. Let $\mathcal{A} \in \mathcal{D}_{g,n}$ be a weight datum verifying the hypothesis of Case 3. Then the sequence of properties $\mathbf{S}_{\mathcal{A}}$ satisfies both conditions of Corollary 1.2.32.*

Proof. Let us write the sequence above as $P_1, ..., P_N$. Then we have to check the following two conditions:

1. for every $i = 2, ..., N-1$ the properties $P = P_1 \cup ... \cup P_{i-1}$ and $Q = P_i$ verify both the conditions of Proposition 1.2.31, and every strictly Q-simplex is in P^*.
2. The symmetric orbits of X admit canonical co-P_1 maximal faces: to verify this, it is enough to check that the properties $P = \varnothing$ and $Q = P_1$ satisfy the second condition of Proposition 1.2.31.

Condition 1 of Proposition 1.2.31: Let $P = P_1 \cup ... \cup P_i$, $Q = P_i = P^{\mathcal{A}}_{g',n'}$, $i = 2, ..., N-1$. We have to show that the set of simplices $P^*(X)$ is co-Q saturated, i.e., we want to show that given $G \in Ob(\mathcal{G}_{g,\mathcal{A}})$ which is not in P^*, and G' is obtained contracting (g', L')-bridges, $|L'| = n'$, then $G' \notin P^*$. We distinguish three cases:

- $Q = P^{\mathcal{A}}_{g',n'}$, $(g', n') \neq (1, 0)$, $g' \geq 1$;
- $Q = P^{\mathcal{A}}_{1,0}$;
- $Q = P^{\mathcal{A}}_{0,n'}$, for $n' = 2, ..., n$ (this case occurs only when $\sum_{i=0}^{n} a_i > 1$).

In the first case $Q = P^{\mathcal{A}}_{g',n'}$, $(g', n') \neq (1, 0)$, $g \geq 1$. Let $e \in E(G)$ be a (g', L')-bridge with $|L'| = n'$, our goal is to show that $G/e \notin P^*$. To do this, we want to show that $G/e \notin (P^{\mathcal{A}}_{0,n''})^*$ for every n'', i.e., G has no repeated markings. It suffice to show that not both ends v_1 and v_2 of e are marked. Assume the edge $(v_1, e) \in E(\mathrm{Bl}_{\mathcal{A}}(G))$ is labeled $(g - g', [n] \setminus L')$ for some L' with n' elements. We show v_1 is unmarked. Since $G \notin (P^{\mathcal{A}}_{0,n''})^*$ for any n'' and $G \notin (P^{\mathcal{A}}_{1,0})^*$, we have $w(v_1) = 0$ and $|L(v_1)| \leq 1$. So there must be another block $B \neq e$ at v_1 and $(v_1, B) \in E(\mathrm{Bl}_{\mathcal{A}}(G))$ is labeled (g'', L) for $g'' > g'$ or $g' = g$ and $|L| \geq n'$. By equations A.1 and by the fact that the other label is $(g - g', [n] \setminus L')$, (v_1, B) has to be labeled (g', L'), and v_1 is unmarked. Next, by Lemma A.1.7 point 3, every label (g'', L'') with $|L''| = n''$ on $E(\mathrm{Bl}_{\mathcal{A}}(G))$ satisfies either $g'' > g'$, or $g'' = g'$ and $n'' \geq n'$. Furthermore, the labels on $E(\mathrm{Bl}_{\mathcal{A}}(G/e))$ are a subset of those on $E(\mathrm{Bl}_{\mathcal{A}}(G))$. Therefore by Lemma A.1.7 point 4, $G/e \notin (P^{\mathcal{A}}_{g'',n''})^*$ for any $g'' < g'$ or $g'' = g'$ and $n'' < n'$, as long as $g'' \geq 1$.

In the second case, assume $Q = P^{\mathcal{A}}_{1,0}$. If $\sum_{i=0}^{n} a_i \leq 1$ then $P = \varnothing$ and we are done. Otherwise, $P = P^{\mathcal{A}}_{0,n} \cup ... \cup P^{\mathcal{A}}_{0,2}$, and a graph G is in P^* if and only if G has repeated markings. The Property P is evidently preserved by uncontracting $(1, \varnothing)$-bridges, so we are done.

Third, assume $g' = 0$, that is $Q = P^{\mathcal{A}}_{0,n'}$, so $\sum_{i=0}^{n} a_i > 1$. The assumption $G \notin P^*$ means that $G \notin (P^{\mathcal{A}}_{0,n''})^*$ for any $n'' > n'$. Let C denote the core of G, i.e., the smallest connected subgraph of G containing all of its cycles and vertices of positive weight. Then $G \setminus E(C)$ is a disjoint union of trees Y_v, for $v \in V(C)$. We say that a core vertex $v \in V(C)$ supports a leg $x \in [n]$ if it belongs to Y_v. Then observe that for any $G \in Ob(\mathcal{G}_{g,\mathcal{A}})$, the following are two conditions are equivalent:

1. $G \notin (P^{\mathcal{A}}_{0,n''})^*$ for any $n'' > n'$;

2. every core vertex of G supports at most n' markings.

Now, since we are assuming that G satisfies (1), it does with (2), which is evidently preserved by contracting $(0, L')$-bridges, $|L'| = n'$, since those operations never increase the number of legs supported by a core vertex. So (1) is also preserved by contracting $(0, L')$-bridges, which is what we wanted to show.

CONDITION 2: Let $P = P_1 \cup ... \cup P_{i-1}$, $Q = P_i = P^{\mathcal{A}}_{g',n'}$, $i = 2, ..., N-1$. We have to show that the set of symmetric orbits of $X \setminus P^*(X)$ admits canonical co-Q maximal faces, i.e., given a graph $G \in Ob(\mathcal{G}_{g,\mathcal{A}})$ that is not in P^*, we need to show that there is a maximal uncontraction $\alpha : \tilde{G} \to G$ by (g', L')-bridges, $|L'| = n'$ such that for every $\alpha' : \tilde{G} \to G$ there is $\theta \in Aut(\tilde{G})$ such that $\alpha'\theta = \alpha$. Again we distinguish the three possible cases for Q as before.

Let $Q = P^{\mathcal{A}}_{g',n'}$, $(g', n') \neq (1, 0)$, $g \geq 1$;. Let v be an $\mathcal{A}$-articulation point of G. Again G is not a $(0, L)$-contraction, since $P^{\mathcal{A}}_{0,n'} \subset P$ when $\sum_{i=0}^n a_i > 1$, or it is not in the sequence when $\sum_{i=0}^n a_i \leq 1$, for any $n' = 2, ..., n$. Then it has not vertices with $|L(v)| \geq 2$, i.e., it has no repeated markings. Moreover, since $P^{\mathcal{A}}_{1,0} \subset P$ and $G \notin P^*$, every vertex of G has weight 0. Let $B \in Bl_{\mathcal{A}}(v)$, (g'', L'') be the label of $\epsilon = (v, B) \in E(\mathrm{Bl}_{\mathcal{A}}(G))$. By Lemma A.1.7, point 3 $G \in P^{\mathcal{A}}_{g'',n''}$, with $n'' = |L''|$. It follows that $g'' \geq g'$ and if $g'' = g'$, then $n'' > n'$. Lemma A.1.7, point 2 implies that there is a unique maximal uncontraction which is canonical in the sense we required above.

Assume now $Q = P^{\mathcal{A}}_{1,0}$. Then by Lemma A.1.7, point 2, the maximal $(1, \varnothing)$-bridge expansion of G is obtained by replacing for any vertex v with $|v|_E + 2w(v) > 3$, every loop based at v with a bridge from v to a loop; adding $w(v)$ bridges to vertices of weight 1, and setting $w(v) = 0$, leaving the legs on v. Moreover this expansion is canonical.

Lastly, suppose $Q = P^{\mathcal{A}}_{0,n'}$, for $n' = 2, ..., n$ and $\sum_{i=0}^n a_i > 1$. Let v be an $\mathcal{A}$-articulation point, B_1,...,B_k the blocks at v labelled $(0, L_1)$,...,$(0, L_k)$ for some $L_i \subset [n]$. By how we ordered the properties in $\mathbf{S}_{\mathcal{A}}$ we have that $P^{\mathcal{A}}_{0,n''} \subset P$ for every $n'' \geq n'$. By the Lemma A.1.7 point 2, since we are assuming $G \notin P^*$ we have $|(\bigsqcup_{i=1}^k L_k) \sqcup L(v)| \leq n'$, and if the equality holds, v can be expanded into a $(0, L')$-bridge with $|L'| = n'$. So this concludes the proof of Condition 2 Observe that this shows also that the symmetric orbits of $\Delta_{g,\mathcal{A}}$ admit canonical co-P_1 maximal faces, as our proves work in the cases $P = \varnothing$ and $Q = P_1$, where P_1 can be both $P^{\mathcal{A}}_{0,n}$ or $P^{\mathcal{A}}_{1,0}$ depending on $\sum_{i=0}^n a_i$.

STRICTLY CO-Q FACES BEING IN P^*: We can assume all edges of $G \in Ob(\mathcal{G}_{g,\mathcal{A}})$ are (g', L')-bridges, $|L'| = n'$, eventually after contracting blocks of G to some vertices. Then G must be a tree with a single non-leaf vertex v, while every other vertex v' has $w(v') = g'$ and $|L(v')| = n'$. We treat the following cases. Suppose $Q = P^{\mathcal{A}}_{g',n'}$ with $g' \geq 1$ and $(g', n') \neq (1, 0)$. If G has only (g', L')-edges, $|L'| = n'$ then $G \in (P^{\mathcal{A}}_{1,0})^*$, since G has positive weights. Therefore $G \in P^*$. Next, suppose $Q = P^{\mathcal{A}}_{1,0}$. If G has only $(1, \varnothing)$-edges, then either $\sum_{i=0}^n a_i \leq 1$ and there is nothing to check, or $\sum_{i=0}^n a_i > 1$ and so v supports $n \geq 2$ markings. Then $G \in (P^{\mathcal{A}}_{0,2})^*$, so $G \in P^*$. Finally, suppose $\sum_{i=0}^n a_i > 1$ and $Q = P^{\mathcal{A}}_{0,n'}$ for $n' < n$. If G has only $(0, L')$-edges, $|L'| = n'$ for some $n' < n$, note that $w(v) = g$ and so v may be expanded into a $(g, \varnothing)$-bridge, which is equivalent to a $(0, [n])$-bridge. So $G \in (P^{\mathcal{A}}_{0,n})^*$, and hence $G \in P^*$, as required. □

With this we can conclude the proof of Theorem 2.1.1, Case 3. To show 3, note that for every $(g, \mathcal{A})$ we have in Case 3, $\cup P_i$ is the Property of being a $(g, \mathcal{A})$-stable bridge graph, so $(\Delta_{g,\mathcal{A}})_{\cup P_i} = \Delta^{br}_{g,\mathcal{A}}$. If $\sum_{i=0}^{n} a_i > 1$, $P_1 = P^{\mathcal{A}}_{0,n}$ is the Property of being a $(0, [n])$-bridge graph and $(\Delta_{g,\mathcal{A}})_{P^{\mathcal{A}}_{0,n},0}$ is a point. If $\sum_{i=0}^{n} a_i \leq 1$ and $g \geq 2$, $P = P^{\mathcal{A}}_{1,0}$ is the Property of being a $(1, \varnothing)$-bridge graph, with $(\Delta_{g,\mathcal{A}})_{P^{\mathcal{A}}_{1,0},0}$ being a $(g-1)$-simplex parametrizing non-negative edge lengths on a tree with g leaves of weight 1, and a central vertex supporting all the markings: here the Property of being a bridge is given by the union of Properties of the sequence $\mathbf{S}_{\mathcal{A}}$ starting from $P^{\mathcal{A}}_{1,0}$. In either cases, by Theorem A.1.8, we can apply Corollary 1.2.32 to produce a deformation retract of $(\Delta_{g,\mathcal{A}})_{\cup P_i} = \Delta^{br}_{g,\mathcal{A}}$ to a contractible space.

For 2, we have to verify that the properties $P = \varnothing$ and $Q = P_{1,0}$ satisfy the conditions (1) and (2) of Proposition 1.2.31. If G has no loops or weights the expansion is trivial. Otherwise, we can expand it without changing stability type as follows: for any $v \in V(G)$ such that $|v|_E + 2w(v) > 3$ replace loops on v with bridges from v to a loop, add $w(v)$ bridges to vertices of weight 1 and set $w(v) = 0$. Then the contractibility of $\Delta^{lw}_{g,\mathcal{A}} = (\Delta_{g,\mathcal{A}})_{1,0}$ follows by Proposition 1.2.31.

For 1, consider the same properties and the expansion already described but on $\Delta^{w}_{g,\mathcal{A}}$. Then $\Delta^{w}_{g,\mathcal{A}} = (\Delta_{g,\mathcal{A}})_{P_{1,0}}$ deformation retracts to the subcomplex of graphs with only $(1, \varnothing)$-edges, which is contractible. This concludes the proof of the Theorem 2.1.1.

□

A.2 Alternative proof of Theorem 2.1.6

Here we present an alternative proof of Theorem 2.1.6. It relies on the fact that $\pi_1(\Delta_{g,\mathcal{A}})$ is trivial when $\mathcal{A} = 1^{(n)}$, shown by Allcock, Corey, and Payne in [ACP22], Theorem 1.4, were they also show that $\Delta_{g,n}$ are simply connected for $(g, n) \neq (0, 4), (0, 5)$. The purposes to present this proof are two: to start, this is the very first proof which we found about the simply connectedness of $\Delta_{g,\mathcal{A}}$ while working on [KLSY22], and the second reason is that this proof involves different ideas which really show how deeply the combinatorics of this space influence its topology.

Let $g \geq 1$, $n \geq 1$, and fix a weight datum $\mathcal{A} \in \mathcal{D}_{g,n}$. For each subset $S \subseteq [n]$, we define a subcomplex $\Delta_{g,\mathcal{A}}(S) \subseteq \Delta_{g,\mathcal{A}}$ as follows. Given a $(g, \mathcal{A})$-stable graph G and a subset $S \subseteq [n]$, we call $v \in V(G)$ an S-vertex if $S \subseteq m^{-1}(v)$. Given a $S \subseteq [n]$, we define $\mathcal{G}_{g,\mathcal{A}}(S)$ to be the set of objects

$$\{G \in \mathcal{G}_{g,\mathcal{A}} : G \text{ has an } S\text{-vertex}\}.$$

In the same way, we define a subcomplex $\Delta_{g,\mathcal{A}}(S)$ of $\Delta_{g,\mathcal{A}}$ by defining $\Delta_{g,\mathcal{A}}(S)(p)$ as

$$\{[G, \tau] \in \Delta_{g,\mathcal{A}}(p) : G \text{ has an } S\text{-vertex}\},$$

for each $p \geq -1$. As defined, $\Delta_{g,\mathcal{A}}(S)$ is a subcomplex of $\Delta_{g,\mathcal{A}}$ because the property of having an S-vertex is closed under edge contractions. The following lemma follows from the definition of an S-vertex.

Lemma A.2.1. *If a collection of subsets $\{S_\alpha\}_{\alpha \in \mathbf{A}}$ of $[n]$ satisfies $\bigcap_{\alpha \in \mathbf{A}} S_\alpha \neq \varnothing$, then*

$$\bigcap_{\alpha \in \mathbf{A}} \Delta_{g,\mathcal{A}}(S_\alpha) = \Delta_{g,\mathcal{A}}\left(\bigcup_{\alpha \in \mathbf{A}} S_\alpha\right).$$

Define a weight vector $\mathcal{A}^S \in ((0,1]\cap\mathbb{Q})^{n-|S|+1}$ by removing from $\mathcal{A}$ in order those entries indexed by S, and then appending an entry of weight $min\left(\sum_{i\in S} a_i, 1\right)$.

Example A.2.2. As an example, if we take $\mathcal{A} = (1/4, 2/3, 1/2, 1)$, $S = \{1,3\}$, and $T = \{2,3\}$, then $\mathcal{A}^S = (2/3, 1, 3/4)$, while $\mathcal{A}^T = (1/4, 1, 1)$.

For a topological space T, denote by $Cone(T)$ the space $T\times[0,1]/(T\times\{1\})$.

Proposition A.2.3. *Let $S \subseteq [n]$. If $\sum_{i\in S} a_i \leq 1$, then we have an isomorphism of symmetric Δ-complexes $\Delta_{g,\mathcal{A}}(S) \cong \Delta_{g,\mathcal{A}^S}$. Otherwise, there is a homeomorphism of topological spaces $|\Delta_{g,\mathcal{A}}(S)| \cong Cone(|\Delta_{g,\mathcal{A}^S}|)$.*

Proof. For the first part, suppose $\sum_{i\in S} a_i \leq 1$. We define a natural transformation of functors $\eta : \Delta_{g,\mathcal{A}^S} \longrightarrow \Delta_{g,\mathcal{A}}$, by defining its component at $\overline{[p]} \in I^{op}$, $\eta_p : \Delta_{g,\mathcal{A}^S}(p) \to \Delta_{g,\mathcal{A}}(p)$, as follows. For every $[G,\tau] \in \Delta_{g,\mathcal{A}^S}(p)$, the image $\eta_p([G,\tau])$ is the graph obtained replacing the last marking by the set of markings indexed by S (nothing changes on τ). This is a natural transformation since for every $i : \overline{[p]} \to \overline{[q]}$, the following diagram is commutative:

$$\begin{array}{ccc}
\Delta_{g,\mathcal{A}^S}(q) & \xrightarrow{\eta_q} & \Delta_{g,\mathcal{A}}(p) \\
\downarrow{\scriptstyle \Delta_{g,\mathcal{A}^S}(i)} & & \downarrow{\scriptstyle \Delta_{g,\mathcal{A}}(i)} \\
\Delta_{g,\mathcal{A}^S}(p) & \xrightarrow{\eta_p} & \Delta_{g,\mathcal{A}}(p).
\end{array}$$

Indeed, the relative positions of the markings with respect to the last marking (or on the right hand side of the diagram, the S-indexed markings) in each graph involved undergo the same changes during edge contractions and relabellings of edges involved in $\Delta_{g,\mathcal{A}}(i)$ and $\Delta_{g,\mathcal{A}^S}(i)$. Moreover, the stability at each vertex persists in all the involved graphs under η_p and η_q by the construction of $\mathcal{A}^S$. Therefore, $\eta_p \circ \Delta_{g,\mathcal{A}^S}(i) = \Delta_{g,\mathcal{A}}(i)\circ\eta_q$. Furthermore, η is an isomorphism of sub-complexes between $\Delta_{g,\mathcal{A}^S}$ and $\Delta_{g,\mathcal{A}}(S)$. This amounts to checking that all the components $\eta_p : \Delta_{g,\mathcal{A}^S}(p) \to \Delta_{g,\mathcal{A}}(S)(p)$ for all $p \geq -1$ are isomorphisms. Indeed, given $[\mathbf{H},\pi] \in \Delta_{g,\mathcal{A}}(S)(p)$, the morphism that replaces the set of markings indexed by S by a marking of weight $\sum_{i\in S} a_i$ is the inverse of η_p (again, nothing changes in the edge label function).

For the second part, we construct a homeomorphism of topological spaces from the cone over $\Delta_{g,\mathcal{A}^S}$ to $\Delta_{g,\mathcal{A}}(S)$ $f : \left(\Delta_{g,\mathcal{A}^S}\times[0,1]\right)/\left(\Delta_{g,\mathcal{A}^S}\times\{1\}\right) \cong \Delta_{g,\mathcal{A}}(S)$, where both $\Delta_{g,\mathcal{A}^S}$ and $\Delta_{g,\mathcal{A}}(S)$ are equipped with the final topology. Recall that each point in $\Delta_{g,\mathcal{A}^S}$ is represented by a pair (G,ℓ) where $G \in \mathcal{G}_{g,\mathcal{A}^S}$ and $\ell : E(G) \to \mathbb{R}_{\geq 0}$. Similarly for each point in $\Delta_{g,\mathcal{A}}(S)$. We let $\mathcal{A}^S$ be indexed by $([n]\setminus S)\cup\{n+1\}$. Now to construct the homeomorphism f, for each $G = (G,m,h) \in \mathcal{G}_{g,\mathcal{A}^S}$, we set $f(G)$ to be the graph $G' = (G',m',h') \in \mathcal{G}_{g,\mathcal{A}}(S)$ that is the new graph obtained by adding a new vertex with zero weight, connecting it to the vertex that houses the marking $n+1$, and replacing the marking $n+1$ with S-indexed markings; see Figure A.5. Formally, G' is defined as follows. Let v be the vertex where the marking $n+1$ is attached; that is, $v = m(n+1) \in V(G)$. Let v_0 be a new vertex and set

$$V(G') = V(G)\cup\{v_0\},$$
$$E(G') = E(G)\cup\{v,v_0\}.$$

Set the new marking function $m' : [n] \to V(G')$ to be $m'(i) = \begin{cases} m(i) & i \in [n] \setminus S, \\ v_0 & i \in S, \end{cases}$ and finally, set the new weight function $h' : V(G') \to \mathbb{Z}_{\geq 0}$ to be $h'(u) = \begin{cases} h(u) & u \in V(G), \\ 0 & u = v_0. \end{cases}$

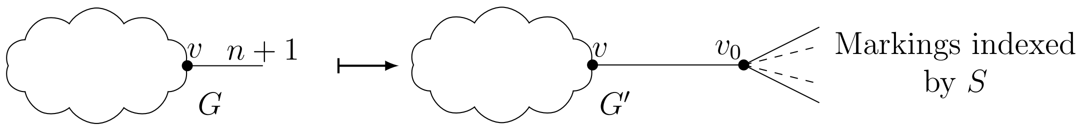

Figure A.5: A graph $G \in \mathcal{G}_{g,\mathcal{A}^S}$ and $f(G) = G' \in \mathcal{G}_{g,\mathcal{A}}(S)$. The cloud shapes represent the parts of the graphs that are unchanged under f.

It remains to check that G' is an object in $\mathcal{G}_{g,\mathcal{A}}(S)$. First, notice that v_0 is an S-vertex and that $g(G') = g(G) = g$. It now suffices to check that v and v_0 satisfy the stability condition. For v we have

$$\begin{aligned} 2h'(v) - 2 + \mathrm{val}_{G'}(v) + \sum_{i \in (m')^{-1}(v)} a_i &= 2h(v) - 2 + \mathrm{val}_G(v) + 1 + \left(\sum_{i \in m^{-1}(v)} a_i \right) - 1 \\ &= 2h(v) - 2 + \mathrm{val}_G(v) + \sum_{i \in m^{-1}(v)} a_i \\ &> 0, \end{aligned}$$

by stability of v in $\mathbf{G}$.

For v_0, we use the assumption that $\sum_{i \in S} a_i > 1$ to obtain

$$2h'(v_0) - 2 + \mathrm{val}_{G'}(v_0) + \sum_{i \in} m'^{-1}(v_0) = 0 - 2 + 1 + \sum_{i \in S} a_i > 0.$$

Now we define the morphism $f : \left(\Delta_{g,\mathcal{A}^S} \times [0,1]\right) / \left(\Delta_{g,\mathcal{A}^S} \times \{1\}\right) \to \Delta_{g,\mathcal{A}}(S)$ by sending $((G,\ell),t) \mapsto (G',\ell')$, where

$$\ell'(e) = \begin{cases} t & e = \{v, v_0\}, \\ (1-t)\ell(e) & e \in E(G). \end{cases}$$

When $t = 0$, $f(-,0)$ coincides with the map η_p for all $p \geq -1$. When $t = 1$, $((G,\ell),1)$ is sent to the cone point in $\Delta_{g,\mathcal{A}}(S)$ represented by the graph consisting of a weight-g vertex marked by $[n] \setminus S$ and a weight-0 vertex v_0 marked by S (Figure A.6). Since both $\Delta_{g,\mathcal{A}^S}$ and $\Delta_{g,\mathcal{A}}(S)$ can be given the structure of finite CW-complexes, they are compact and Hausdorff. Since the map f is a continuous bijection, it is a homeomorphism.

□

Before we prove that $\Delta_{g,\mathcal{A}}$ is simply connected, i.e., it is nonempty, path connected and has trivial fundamental group, we need some auxiliary definitions.

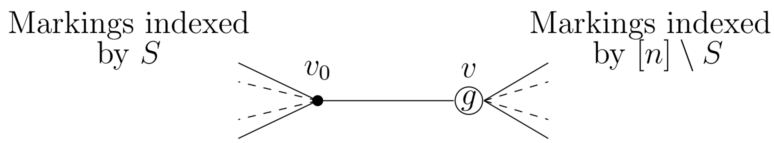

Figure A.6: The graph representing the cone point in $\Delta_{g,\mathcal{A}}(S)$.

Recall that a collection K of non-empty finite subsets of a set X is called a set-family. A set-family K is called an abstract simplicial complex if, for every set S in K, and every non-empty subset $T \subset S$, the set T also belongs to K. The finite sets that belong to K are called faces of the complex, and a face T is said to belong to another face S if $T \subset S$, so the definition of an abstract simplicial complex can be restated as saying that every face of a face of a complex K is itself a face of K. The vertex set of K is defined as the union of all faces. For every vertex v the set $\{v\}$ is a face of the complex, and every face of the complex is a finite subset of the vertex set.

Definition A.2.4. Given an integer $n \geq 1$, and $\mathcal{A} \in \mathcal{D}_{g,n}$, we associate an abstract simplicial complex $K(\mathcal{A})$ with vertex set equal to $[n]$ by declaring that $S \subseteq [n]$ belongs to $K(\mathcal{A})$ if and only if $\sum_{i \in S} a_i \leq 1$.

Observe that the association of $K(\mathcal{A})$ to the vector $\mathcal{A}$ is order-reversing: if $\mathcal{A}, \mathcal{A}' \in \mathcal{D}_{g,n}$ with $\mathcal{A} \leq \mathcal{A}'$, then $K(\mathcal{A}')$ is a subcomplex of $K(\mathcal{A})$.

Definition A.2.5. Let $S \subseteq [n]$ and let $G \in \mathcal{G}_{g,\mathcal{A}}$. We call a vertex $v \in V(\mathbf{G})$ an S-antenna if $w(v) = 0$, $\mathrm{val}(v) = 1$, and $m^{-1}(v) = S$.

By the definition of $\mathcal{A}$-stability, there exist graphs with S-antennas in $\mathcal{G}_{g,\mathcal{A}}$ if and only if $S \notin K(\mathcal{A})$. We will also require the following basic topological lemma obtained via the Seifert-van Kampen theorem (see [Hat02]) and induction.

Lemma A.2.6. *Let X be a path-connected CW-complex, and suppose that $X = \cup_{i=1}^{N} U_i$ where each U_i is a simply connected CW-subcomplex. Suppose further that for any $1 \leq i_1, \ldots, i_k \leq N$, the intersection $\cap_{j=1}^{k} U_{i_j}$ is simply connected. Then X is simply connected.*

Lemma A.2.7. *Let $g, n \geq 1$ and $\mathcal{A} \in (\mathbb{Q} \cap (0, 1])^n$. Then $\Delta_{g,\mathcal{A}}$ is path-connected.*

Proof. It is enough to show that points corresponding to $\Delta_{g,\mathcal{A}}(0)$ are path-connected to each other, because each point $(G, \ell) \in \Delta_{g,\mathcal{A}}$ is path-connected to $(G/T, \ell')$ for any edge set $T \subset E(G)$ and length function ℓ', and every graph can be contracted to a graph with only one edge. If a $\mathcal{A}$-stable graph G with genus g has only one edge, then G is either a loop at a vertex with weight $(g-1)$ supporting all the markings, or a bridge connecting two vertices with weights summing up to g.

Suppose G is the former and G' is the latter. Further suppose that G' has one vertex of weight $k \geq 1$ and supports markings indexed by some subset $S \subseteq [n]$. Note that such vertex always exists because G' has genus greater than 0. Since G' is stable, we have $2(g-k) + \sum_{i \notin S} w_i > 1$. Then G and G' are vertices of the 1-simplex corresponding to the w-stable genus g graph H consisting of two vertices, with weights $k-1$ and $g-k$, respectively, an edge that joins them and a loop at the vertex with weight $k-1$. Therefore G and G' are path-connected to each other. □

We now have the necessary framework to prove that $\pi_1(\Delta_{g,\mathcal{A}})$ is trivial. Let us denote by $\ell(\mathcal{A})$ the length of $\mathcal{A}$, and by $j(\mathcal{A})$ the number of entries of $\mathcal{A}$ which are strictly less than 1, i.e., $j(\mathcal{A}) := |\{i \in [n] \mid a_i < 1\}|$.

Theorem A.2.8. *Let $g, n \geq 1$ and $\mathcal{A} \in \mathcal{D}_{g,n}$. Then $\Delta_{g,\mathcal{A}}$ is simply connected.*

Proof. Fix $g \geq 1$. We will proceed by induction on the pair $(\ell(\mathcal{A}), j(\mathcal{A}))$, where $\mathbb{Z}^2$ is given the lexicographic order. In the base case when $(\ell(\mathcal{A}), j(\mathcal{A})) = (1, 0)$ and $(\ell(\mathcal{A}), j(\mathcal{A})) = (1, 1)$, we have $\Delta_{g,\mathcal{A}} \cong \Delta_{g,1}$, and $\pi_1(\Delta_{g,1})$ is trivial by [ACP22]. Suppose $\ell(\mathcal{A}) \geq 2$, and the statement is true for all $\mathcal{A}'$ such that $(\ell(\mathcal{A}'), j(\mathcal{A}'))$ is strictly less than $(\ell(\mathcal{A}), j(\mathcal{A}))$ with respect to the lexicographic order. Since $\Delta_{g,n}$ is simply connected for all n as $g \geq 1$ again by [ACP22], reordering the entries of $\mathcal{A}$ if necessary, we can assume that $a_1 < 1$ (we can reorder without changing the homotopy type by 4.2.10, number (3)). Now denote by $\overline{\mathcal{A}}$ the weight vector obtained from $\mathcal{A}$ by changing a_1 to 1. So in the lexicographic order, $(\ell(\overline{\mathcal{A}}), j(\overline{\mathcal{A}})) < (\ell(\mathcal{A}), j(\mathcal{A}))$. We also have an embedding $\Delta_{g,\mathcal{A}} \hookrightarrow \Delta_{g,\overline{\mathcal{A}}}$ of topological spaces, as $a_i \leq \overline{a}_i$ for all i. Now we analyze the locus $\Sigma_{g,\mathcal{A}} := \overline{\Delta_{g,\overline{\mathcal{A}}} \setminus \Delta_{g,\mathcal{A}}} \subseteq \Delta_{g,\overline{\mathcal{A}}}$. We suppose that $\Sigma_{g,\mathcal{A}}$ is nonempty; otherwise $\Delta_{g,\mathcal{A}} \cong \Delta_{g,\overline{\mathcal{A}}}$, and we have that $\pi_1(\Delta_{g,\overline{\mathcal{A}}})$ is trivial by assumption.

Now consider the decomposition $\Delta_{g,\overline{\mathcal{A}}} = \Delta_{g,\mathcal{A}} \cup \Sigma_{g,\mathcal{A}}$. Recall that the Seifert-van Kampen theorem for CW-complexes expresses $\pi_1(X)$ for a path-connected CW-complex X as the amalgamated free product $\pi_1(X) = \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V)$, where U and V are path-connected CW-subcomplexes that cover X, such that $U \cap V$ is path-connected. We will show that $\Sigma_{g,\mathcal{A}}$ is a subcomplex of $\Delta_{g,\overline{\mathcal{A}}}$ and that both $\Sigma_{g,\mathcal{A}}$ and $\Delta_{g,\mathcal{A}} \cap \Sigma_{g,\mathcal{A}}$ are simply connected, so that Seifert-van Kampen applies and we have the following pushout diagram of groups:

$$\begin{array}{ccc} \pi_1(\Delta_{g,\mathcal{A}} \cap \Sigma_{g,\mathcal{A}}) & \longrightarrow & \pi_1(\Sigma_{g,\mathcal{A}}) \\ \downarrow & & \downarrow \\ \pi_1(\Delta_{g,\mathcal{A}}) & \longrightarrow & \pi_1(\Delta_{g,\overline{\mathcal{A}}}) \end{array}, \tag{A.3}$$

where all arrows are induced by inclusions. In particular, the two groups in the top row in Diagram A.3 will be trivial. Since $\pi_1(\Delta_{g,\overline{\mathcal{A}}})$ is trivial by induction, it will follow that $\pi_1(\Delta_{g,\mathcal{A}})$ must be trivial.

To show that $\Sigma_{g,\mathcal{A}}$ is a simply connected subcomplex of $\Delta_{g,\overline{\mathcal{A}}}$, we first establish the equality

$$\Sigma_{g,\mathcal{A}} = \bigcup_{S \in K(\mathcal{A}) \setminus K(\overline{\mathcal{A}})} \Delta_{g,\overline{\mathcal{A}}}(S). \tag{A.4}$$

Indeed, the graphs parameterized by the subspace $\Delta_{g,\overline{\mathcal{A}}} \setminus \Delta_{g,\mathcal{A}}$ are precisely those that are $\overline{\mathcal{A}}$-stable but not $\mathcal{A}$-stable. Any such graph must have an S-antenna for some $S \in K(\mathcal{A}) \setminus K(\overline{\mathcal{A}})$; since S-antennas are also S-vertices, we have the containment

$$\Delta_{g,\overline{\mathcal{A}}} \setminus \Delta_{g,\mathcal{A}} \subseteq \bigcup_{S \in K(\mathcal{A}) \setminus K(\overline{\mathcal{A}})} \Delta_{g,\overline{\mathcal{A}}}(S).$$

The object on the right is a closed subcomplex of $\Delta_{g,\overline{\mathcal{A}}}$, so we must have

$$\overline{\Delta_{g,\overline{\mathcal{A}}} \setminus \Delta_{g,\mathcal{A}}} = \Sigma_{g,\mathcal{A}} \subseteq \bigcup_{S \in K(\mathcal{A}) \setminus K(\overline{\mathcal{A}})} \Delta_{g,\overline{\mathcal{A}}}(S).$$

For the reverse containment, note that any metric graph in $\Delta_{g,\overline{\mathcal{A}}}(S)$ for $S \in K(\mathcal{A}) \setminus K(\overline{\mathcal{A}})$ either contains an S-antenna, or has a $\overline{\mathcal{A}}$-stable uncontraction with an S-antenna. As such, $\Delta_{g,\overline{\mathcal{A}}}(S) \subseteq \Sigma_{g,\mathcal{A}}$ and the equality is proven; in particular, $\Sigma_{g,\mathcal{A}}$ is a subcomplex of $\Delta_{g,\overline{\mathcal{A}}}$. We now argue that $\Sigma_{g,\mathcal{A}}$ is simply connected using the criterion of Lemma A.2.6 and Equation A.4. Indeed, for all $S \in K(\mathcal{A}) \setminus K(\overline{\mathcal{A}})$, we have $\sum_{i \in S} \overline{\mathcal{A}}_i > 1$ by the definition of $K(\overline{\mathcal{A}})$. Therefore for each such S, we have $\Delta_{g,\overline{\mathcal{A}}}(S) \cong Cone(\Delta_{g,\overline{\mathcal{A}}^S})$ by Lemma A.2.3, so each $\Delta_{g,\overline{\mathcal{A}}}(S)$ appearing in the union of Equation A.4 is contractible. Moreover, given any $S_1, \ldots, S_N \in K(\mathcal{A}) \setminus K(\overline{\mathcal{A}})$, we have $1 \in \cap_{i=1}^N S_i$, so Lemma A.2.1 implies that $\bigcap_{i=1}^N \Delta_{g,\overline{\mathcal{A}}}(S_i) = \Delta_{g,\overline{\mathcal{A}}}\left(\bigcup_{i=1}^N S_i\right)$, which is again contractible by Lemma A.2.3 and hence is simply connected. Thus Lemma A.2.6 gives that the subcomplex $\Sigma_{g,\mathcal{A}}$ is simply connected.

We will now prove that $\Delta_{g,\mathcal{A}} \cap \Sigma_{g,\mathcal{A}}$ is simply connected, again using the criterion of Lemma A.2.6. Identifying $\Delta_{g,\mathcal{A}}(S)$ with its image under the embedding $\Delta_{g,\mathcal{A}} \hookrightarrow \Delta_{g,\overline{\mathcal{A}}}$, we have $\Delta_{g,\mathcal{A}} \cap \Sigma_{g,\mathcal{A}} = \bigcup_{S \in K(\mathcal{A}) \setminus K(\overline{\mathcal{A}})} \Delta_{g,\mathcal{A}}(S)$. For each $S \in K(\mathcal{A}) \setminus K(\overline{\mathcal{A}})$, we have $\sum_{i \in S} w_i \leq 1$, so $\Delta_{g,\mathcal{A}}(S) \cong \Delta_{g,\mathcal{A}^S}$ by Lemma A.2.3. Since $|S| \geq 2$ for any such S, we have $\ell(\mathcal{A}^S) = \ell(\mathcal{A}) - |S| + 1 < \ell(\mathcal{A})$, so by induction, each $\Delta_{g,\mathcal{A}}(S)$ appearing in the union above is simply connected. Given $S_1, \ldots, S_N \in K(\mathcal{A}) \setminus K(\overline{\mathcal{A}})$, we have $\bigcap_{i=1}^N \Delta_{g,\mathcal{A}}(S_i) = \Delta_{g,\mathcal{A}}\left(\bigcup_{i=1}^N S_i\right)$, again by Lemma A.2.1. Setting $\mathcal{S} = \cup_{i=1}^N S_i$, Lemma A.2.3 gives that $\Delta_{g,w}(\mathcal{S})$ is either isomorphic to $\Delta_{g,\mathcal{A}^\mathcal{S}}$, or the cone over it. In the first case $\Delta_{g,\mathcal{A}}(\mathcal{S})$ is simply connected by induction, and in the second it is simply connected because it is contractible. Hence Lemma A.2.6 gives that $\Delta_{g,\mathcal{A}} \cap \Sigma_{g,\mathcal{A}}$ is simply connected. As already discussed, that $\pi_1(\Delta_{g,\mathcal{A}})$ is trivial now follows from the fact that Diagram A.3 is a pushout square and the inductive assumption that $\pi_1(\Delta_{g,\overline{\mathcal{A}}})$ is trivial. □

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