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The Prym-canonical Clifford index

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Abstract

The theme of the thesis is Brill–Noether theory for algebraic curves, with a special focus on Prym-canonical curves.

In Chapter 1, given a smooth irreducible Prym curve (C, η) of genus g, we define two new invariants. The first one is the Prym-canonical Clifford index, $\mathrm{Cliff}_\eta(C)$, which differs from the classical Clifford index in that it is computed with respect to the Prym-canonical bundle $\omega_C \otimes \eta$. The second one is the Prym-canonical Clifford dimension of (C, η). We prove the Prym-Clifford's Theorem, asserting that $\mathrm{Cliff}_\eta(C) \geq 0$, and equality holds if and only if $|\omega_C \otimes \eta|$ has base points. We conclude the chapter with some remarks about the relation between the Prym-canonical Clifford index of a curve and its étale double cover.

In Chapter 2 we classify curves such that $1 \leq \mathrm{Cliff}_\eta(C) \leq 2$. We prove that the first equality occurs if and only if the Prym-canonical bundle is base-point free but not very ample. On the other hand, Prym curves with $\mathrm{Cliff}_\eta(C) = 2$ are such that the Prym-canonical line bundle $\omega_C \otimes \eta$ embeds C in $\mathbb{P}^{g-2}$ with a trisecant line. Moreover, we compute the Prym-canonical Clifford dimension of bielliptic curves.

We analyze hyperelliptic and general Prym curves in Chapter 3. Both of them have Prym-canonical Clifford dimension $(0, 0)$. It turns out that the Prym-canonical Clifford index of a hyperelliptic Prym curve reflects the geometry of the curve, as it depends on the nontrivial 2-torsion line bundle η. For particular choices of η, it equals the maximal possible value $\lfloor \frac{g-1}{2} \rfloor$. By upper semicontinuity, we conclude that the Prym-canonical Clifford index of a general Prym curve (C, η) coincides with the classical Clifford index of C.

Finally, in Chapter 4 we define Prym-exceptional curves, and with the aim of constructing an example, we consider Prym-canonical curves on Nikulin surfaces. Our approach does not yield the desired result, but we provide a new example of exceptional curves with respect to the Clifford index.

Contents

Introduction

Let us begin by fixing some notation. By a *curve* C we mean a smooth, irreducible, projective curve over $\mathbb{C}$, that is, an algebraic variety of dimension 1, or alternatively, a compact Riemann surface. A fundamental invariant associated with a curve is its genus. The *arithmetic genus* of C is defined as $p_a(C) := 1 - \chi(\mathcal{O}_C)$, where $\chi(\mathcal{O}_C)$ is the Euler characteristic of the structure sheaf $\mathcal{O}_C$. The *geometric genus* is the arithmetic genus of the normalisation of C, thus since we assume C smooth unless otherwise indicated, they coincide, and we will denote the genus by g. A *divisor* $D = \sum_i a_i p_i$ is a formal sum of points $p_i \in C$, and it is *effective* if $a_i \geq 0$ for any i. We will frequently use Riemann-Roch formula, that is, $h^0(C, \mathcal{O}_C(D)) - h^1(C, \mathcal{O}_C(D)) = d - g + 1$, where $d = \deg(D)$. An effective divisor D is *special* if $h^1(C, \mathcal{O}_C(D) > d - g + 1$, and by the Serre duality, this is equivalent to $h^0(C, \omega_C \otimes \mathcal{O}_C(-D)) \geq 1$, which forces the degree d of D to satisfy $0 \leq d \leq 2g - 2$. Given an effective divisor D, the *complete linear series* $|D|$ is the set of all the effective divisors linearly equivalent to D; this is a projective space as one may identify $|D| = \mathbb{P}H^0(C, \mathcal{O}_C(D))$. By $r(D)$ we will mean the dimension of $|D|$. Any linear subspace of a complete linear series is a *linear series*, and with the notation g^r_d we refer to a linear series $\mathcal{D} = \mathbb{P}V$, where V is a vector subspace of $H^0(C, \mathcal{O}_C(D))$ of dimension $r + 1$ for some divisor D of degree d. When $r = 1$, a g^1_d is also called a *pencil*. If the sections $\{\sigma \in V\}$ have no common zeroes, then they globally generate the line bundle $\mathcal{O}_C(D)$, and the linear series g^r_d is *base-point free*. In this case, given a basis $\{s_0, \ldots, s_r\}$ of V, the g^r_d defines a well-defined morphism (as $\mathcal{O}_C(D)$ is globally generated by the sections in V)

$$\begin{aligned} f \colon C &\to \mathbb{P}^r \\ p &\mapsto [s_0(p) : \ldots : s_r(p)] \end{aligned}$$

and $d = \deg(D) = \deg(f) \cdot \deg(f(C))$. When the g^r_d is very ample, then the morphism f is an embedding. We also notice that the image of f is a *non-degenerate* curve, i.e., it is not contained in any hyperplane of $\mathbb{P}^r$. Finally, we recall that there exists a bijective correspondence between divisors and line bundles on a curve. Indeed, to any divisor D on C, one can attach the sheaf $\mathcal{O}_C(D)$. Viceversa, given a line bundle $L \in \mathrm{Pic}(C)$ and a non-zero meromorphic section s, one may associate to L the divisor $D = (s)$ of s, and dividing by s, one obtains the isomorphism $L \simeq \mathcal{O}_C(D)$.

0.1 Brill-Noether theory

Brill-Noether theory was introduced in 1874 by the German mathematicians Brill and Noether ([BN74]), and it is one of the principal approaches to studying algebraic curves. It aims to describe all the ways in which a curve can be mapped into some projective space. Since the set of all the non-degenerate morphsims $C \to \mathbb{P}^n$ for some n has the structure of a variety, it is possible to ask questions about its geometry, such as when it is empty, what its dimension is, whether it is connected, when it is irreducible, and so on. Brill-Noether theory replies to these kinds of questions.

0.1.1 Brill-Noether varieties

We attempt to briefly describe the variety mentioned above (for a detailed discussion, we refer to [ACGH85, Ch. IV]), and we state the main theorems. Actually, there exist three main kinds of varieties, that set-theoretically are:

- $C_d^r = \{D \in C_d \mid \dim|D| \geq r\}$ parametrizes effective divisors of degree d on C moving in dimension at least r, where C_d is the d-fold symmetric product of C;
- $W_d^r(C) = \{L \in \operatorname{Pic}(C) \mid \deg(L) = d,\ h^0(L) \geq r+1\} \subseteq \operatorname{Pic}^d(C)$ introduced in 1971 by Kempf ([Kem71]), it parametrizes complete linear series of degree d and dimension at least r;
- $G_d^r(C) = \{g_d^r\text{'s on } C\}$ introduced in 1981 by Arbarello and Cornalba ([AC81]), it parametrizes linear series of degree d and dimension exactly r.

The abelian sum mapping $u\colon C_d \to \operatorname{Pic}^d(C)$ relates the varieties C_d^r and $W_d^r(C)$, as $u(C_d^r) = W_d^r(C)$. Moreover, C_d^r and $W_d^r(C)$ carry a determinantal structure, while $G_d^r(C)$ can be constructed as the canonical blow-up of $W_d^r(C)$ along $W_d^{r+1}(C)$ (cf. [ACGH85, Ch. II, Sec. 4]). To sketch the construction of C_d^r, we introduce the Brill-Noether matrix. Let $D = \sum_i p_i$ be an effective degree d divisor on C. Given $\omega_1, \dots \omega_g$ a basis for the vector space of holomorphic differentials on C, we can write $\omega_j = f_j dz$ for $j = 1, \dots, g$, where f_j's are holomorphic functions in a neighborhood of $p_1, \dots, p_d$ and z is a local coordinate. The *Brill-Noether matrix* is

$$A = \begin{pmatrix} \omega_1(p_1) & \dots & \omega_1(p_d) \\ \vdots & \dots & \vdots \\ \omega_g(p_1) & \dots & \omega_g(p_d) \end{pmatrix},$$

where by $\omega_j(p_j)$ we mean $f_j(z(p_j))$. If the points $p_1, \dots p_d$ are distinct, Riemann-Roch yields that $r(D) \leq r$ if and only if $\operatorname{rank}(A) \leq d-r$. It follows that, in a neighborhood of D in C_d, the set of divisors in C_d moving in dimension at least r is the common zero locus of the $(d-r+1) \times (d-r+1)$ minors of A. Given the variety of $m \times n$ complex matrices, the locus of matrices of rank at most k, with $0 \leq k \leq \min(m,n)$, has codimension $(m-k)(n-k)$ ([ACGH85, p. 67]). This yields that

$$\operatorname{codim}(C_d^r) \leq [d-(d-r)][g-(d-r)] = r(g-d+r),$$

and every component of $C_d^r \subset C_d$ has dimension at least

$$d - r(g-d+r) = r + g - (r+1)(g-d+r).$$

The *Brill-Noether number* is defined as

$$\rho(g,r,d) := g - (r+1)(g-d+r),$$

and we will see that it plays a fundamental role in the study of special divisors. Note that the variety C_d^r can be also determinantally described regardless of whether the points $p_1, \dots, p_d$ are distinct or not. Indeed, given the natural polarization Θ_{C_d} on C_d and the principal polarization $\Theta_{J(C)}$ on $J(C)$, C_d^r can be defined as the $(d-r)$th determinantal variety attached to u_*, where

$$u_* \colon \Theta_{C_d} \to u^*\Theta_{J(C)}$$

and Lemma [ACGH85, Lem. 1.1, Ch. IV, Sec. 1] ensures that the condition $r(D) \geq r$ is equivalent to requiring that the rank of u_* at D is at most $d-r$, and in particular

$$\operatorname{Supp}(C_d^r) = \{D \in C_d \mid r(D) \geq r\}.$$

Moreover,

Lemma 0.1. *[ACGH85, Lemma 1.6, Ch. IV, Sec. 1] The variety C_d^r is smooth and has the "expected" dimension $\rho(g,r,d)+r$ at $D \in C_d^r - C_d^{r+1}$ if and only if the cup-product homomorphism*

$$\mu_0 \colon H^0(C, \mathcal{O}_C(D)) \otimes H^0(C, \omega_C \otimes \mathcal{O}_C(-D)) \to H^0(C, \omega_C)$$

is injective.

In order to describe the variety $W_d^r(C)$, we recall the definitions of universal divisor and Poincaré line bundle. Given a fixed smooth curve C of genus g, the *universal effective divisor* of degree d on C is the divisor $\Delta \subset C \times C_d$, which, for any $D \in C_d$, cuts on $C \cong C \times \{D\}$ exactly the divisor D. The adjective "universal" refers to the universal property of Δ ([ACGH85, Lemma 2.1, Ch. IV, Sec. 1]). The *Poincaré line bundle* of degree d for C can be thought of as analogous to the divisor Δ, and it is defined as the line bundle $\mathcal{L}$ on $C \times \mathrm{Pic}^d(C)$ which, for each $L \in \mathrm{Pic}^d(C)$, restricts exactly to L on $C \cong C \times \{L\}$. Also, the Poincaré line bundles satisfy a universal property ([ACGH85, Lemma 2.2, Ch. IV, Sec. 2]), and $\mathrm{Supp}(W_d^r(C)) = \{L \in \mathrm{Pic}^d(C) : h^0(C, L) \geq r+1\}$ (see [ACGH85, p. 178]). Given $\mathcal{L}$ a Poincaré line bundle of degree d on $C \times \mathrm{Pic}^d(C)$ and the projection $\nu \colon C \times \mathrm{Pic}^d(C) \to \mathrm{Pic}^d(C)$, we fix an effective divisor E on C of degree $m \geq 2g - d - 1$, and take $\Gamma = E \times \mathrm{Pic}^d(C)$ the product divisor in $C \times \mathrm{Pic}^d(C)$. Then, $\nu_*(\mathcal{L}(\Gamma))$ is a locally free sheaf of rank $n = d + m - g + 1$, and the short exact sequence

$$0 \to \mathcal{L} \to \mathcal{L}(\Gamma) \to \mathcal{L}(\Gamma)/\mathcal{L} \to 0$$

yields the following

$$0 \to \nu_*\mathcal{L} \to \nu_*\mathcal{L}(\Gamma) \xrightarrow{\gamma} \nu_*(\mathcal{L}(\Gamma)/\mathcal{L}) \to R^1\nu_*\mathcal{L} \to 0,$$

where $R^1\nu_*\mathcal{L}(D) = 0$ by [ACGH85, Thm 2.6, Ch. IV, Sec.2]. Moreover, the two middle terms are locally free of ranks n and m, so that the variety $W_d^r(C)$ can be defined as the locus where the fiber of $R^1\nu_*\mathcal{L}$ has dimension at least $g-d+r$, or as the $(m+d-g-r)$th determinantal variety attached to γ. Its support is given by

$$\mathrm{Supp}(W_d^r(C)) = \{L \in \mathrm{Pic}^d(C) \mid h^0(C, L) \geq r+1\}$$

(cf. [ACGH85, p.178]), and by [ACGH85, Lemma 3.3, Ch. IV, Sec. 3], if $r \geq d - g$ then every component of $W_d^r(C)$ has dimension greater than or equal to the Brill-Noether number $\rho(g, r, d)$. Observe that, even when we take into account the scheme structure of C_d^r and $W_d^r(C)$, it is still true that $u^{-1}(W_d^r(C)) = C_d^r$ ([ACGH85, Prop. 3.4, Ch. IV, Sec. 3]).
We mentioned that the variety $G_d^r(C)$ can be constructed as the canonical blow-up of $W_d^r(C)$ along $W_d^{r+1}(C)$ (cf. [ACGH85, Ch. II, Sec. 4]), but to see that the points of $G_d^r(C)$ are the g_d^r's on C, consider $x \in \mathrm{Pic}^d(C)$ and V a $(r+1)$-dimensional vector subspace of the kernel of

$$\gamma_x \colon \nu_*\mathcal{L}(\Gamma) \otimes k(x) \to \nu_*(\mathcal{L}(\Gamma)/\mathcal{L}) \otimes k(x).$$

By Lemma [ACGH85, Lemma 3.1, Ch. IV, Sec. 3], the kernel of γ_x is canonically isomorphic to $H^0(C, L)$, so that in symbols the support of $G_d^r(C)$ is

$$\mathrm{Supp}(G_d^r(C)) = \{(L, V) \mid L \in \mathrm{Pic}^d(C), V \in G(r+1, H^0(C, L))\}.$$

Furthermore, every component of $G^r_d(C)$ has dimension at least equal to $\rho(g,r,d)$ ([ACGH85, Prop. 4.1, Ch. IV, Sec. 4]).

0.1.2 Basic results

The moduli space M_g of curves of genus g is the set of isomorphism classes of smooth, genus g curves. It has a natural structure of quasi-projective normal variety of dimension $3g-3$ ([Bai62]) and it is irreducible ([Kle82, no. 19]). A genus g curve C is general if it corresponds to a general point of M_g, or equivalently, if it is contained in an open and dense subset of M_g. In the previous section, we recalled that the varieties $W^r_d(C)$ and $G^r_d(C)$, if non-empty, have dimension at least equal to $\rho(g,r,d)$. This naturally leads one to ask when, given a general curve in M_g, these varieties are non-empty. The Existence Theorem replies to this question. It was proved by Meis for g^1_d's ([Mei60]), and later by Kempf ([Kem71]) and Kleiman-Laksov ([KL72]) independently.

Theorem 0.2. *[ACGH85, Thm 1.1, Ch.V] Let C be a smooth curve of genus g. Let d, r be integers such that $d \geq 1$, $r \geq 0$. Then if*

$$\rho(g,r,d) = g-(r+1)(g-d+r) \geq 0,$$

$G^r_d(C)$, and hence $W^r_d(C)$, are non-empty.

The reverse is also true, and it is known as the Brill–Noether Theorem or the Dimension Theorem.

Theorem 0.3. *[ACGH85, Thm 1.5, Ch. V] Let C be a general curvee of genus g. Let d and r be integers such that $d \geq 1$, $r \geq 0$. Then if*

$$\rho(g,r,d) = g-(r+1)(g-d+r) < 0,$$

$G^r_d(C)$ is empty. If $\rho(g,r,d) \geq 0$, then $G^r_d(C)$ is reduced and of pure dimension $\rho(g,r,d)$.

The first proof of the Brill-Noether Theorem is due to Griffiths and Harris ([GH80]). Moreover, Fulton and Lazarsfeld proved in [FL81] that given a smooth curve C of genus g, if the Brill-Noether number $\rho(g,r,d)$ is at least one, then the variety $G^r_d(C)$, and hence $W^r_d(C)$, is connected. A first (and indirect) statement about smoothness was given by Petri ([Pet25]), but it was Gieseker who proved in [Gie82] that given a general curve $C \in M_g$, then G^r_d is smooth of dimension $\rho(g,r,d)$. As a consequence of the theorems about the connectedness and the smoothness, Fulton and Lazarsfeld obtained that if $\rho(g,r,d) \geq 1$, then $G^r_d(C)$, and hence $W^r_d(C)$, are irreducible.

0.2 Remarks on secant varieties

Brill–Noether theory investigates the ways in which an algebraic curve can be mapped into some projective space. Given a smooth projective curve C of genus g, one of the main subject of study is the variety $G^r_d(C) = \{(L, V) \,:\, L \in \mathrm{Pic}^d(C), V \in G(r+1, H^0(L))\}$ of linear series of type g^r_d on C. A considerable number of questions arise spontaneously, such as to determine the number of singularities that occur under a morphism defined by a g^r_d on the curve. For instance, given a pair of pencils $l_1 = (L_1, V_1) \in G^1_{d_1}(C)$ and $l_2 = (L_2, V_2) \in G^1_{d_2}(C)$, they define a morphism $f\colon C \to \mathbb{P}^1 \times \mathbb{P}^1$, and the double points of $f(C)$ correspond to pairs of points (p_1, p_2) such that

$$\dim(l_1(-p_1 - p_2)) \geq 0 \quad \text{and} \quad \dim(l_2(-p_1 - p_2)) \geq 0, \tag{1}$$

where for an effective divisor $D \in C_e$ and for $i = 1, 2$, write

$$l_i(-D) := (L_i \otimes \mathcal{O}_C(D), V_i \cap H^0(L_i \otimes \mathcal{O}_C(D))).$$

Generalizing (1), the problem of computing the type of singularities can be reformulated in terms of incidence varieties, that is, for a linear system l

$$\Gamma_e(l) := \{D \in C_e \mid \dim(l(-D)) \geq 0\} \subset C_e$$

for $e \leq \deg(l) = d$, and where by C_e we denote the e-th symmetric product of the curve. As $\Gamma_e(l)$ has the structure of a degeneracy locus, it is indeed a variety, and it has dimension r in the case where linear series l has dimension $r \leq e \leq \deg(l) = d$ ([ACGH85, Lemma 3.2, ch. VIII]. The idea of incidence varieties can be generalised with the definition of secant varieties, that is, determinantal cycle of effective divisors of degree e which impose at most $e - f$ independent conditions on a linear series l of type g^r_d, where e, f are fixed integers $0 \leq f < e$. More precisely,

$$V^{e-f}_e(l) := \{D \in C_e \mid \dim(l(-D)) \geq r - e + f\}.$$

When l is very ample and we consider C as embedded in $\mathbb{P}^r$, then $V^{e-f}_e(l)$ parametrises e-secant $(e - f - 1)$-planes to C. As previously stated, incidence varieties are special cases of secant varieties, in fact if $f = e - r$, then $\Gamma_e(l) = V^r_e(l)$, and $V^{e-f}_e(l) \subset \Gamma_e(l)$ for $r - e + f \geq 0$. Moreover, for any line bundle L on C with $h^0(C, L) = r + 1$, one may define a vector bundle E_L on C_e by

$$E_L = (\pi_2)_*(\mathcal{O}_\Delta \otimes \pi_1^* L),$$

where $\pi_1\colon C_e \times C \to C$, $\pi_2\colon C_e \times C \to C_e$ are the two projections, and

$$\Delta := \{(Z, x) \in C_e \times C \mid x \in \mathrm{supp}(Z)\}$$

is the universal degree e effective divisor. The idea is to construct a bundle E_L on C_e whose fiber $(E_L)_D$ over a point $D \in C_e$ is the space of sections of L over D. Since Δ is flat over C_e and $\pi_1^* L$ is locally free, the sheaf $\mathcal{O}_\Delta \otimes \pi_1^* L$ is flat over C_e. Thus, E_L is locally free, and given $D \in C_e$, the identification

$$(E_L)_D \cong H^0(C, L/L(-D))$$

holds (see [ACGH85, p. 339]). Given a subspace $V \subset H^0(C, L)$, one may consider the map

$$\alpha_V : V \otimes \mathcal{O}_{C_e} \to E_L$$

obtained by restricting the evaluation map

$$\alpha_L : H^0(C, L) \otimes \mathcal{O}_{C_e} \to E_L$$

to V, where $H^0(C, L) \otimes \mathcal{O}_{C_e}$ is the trivial bundle. Then, the locus $V_e^{e-f}(L)$ is set-theoretically the $(e-f)$-th degeneracy locus of the map α_V, and so its expected dimension is equal to $e - f(r + 1 - e + f)$. In [Far08], Farkas proves the following theorem:

Theorem 0.4. *Let $[C] \in \mathcal{M}_g$ be a general curve and we fix non-negative integers $0 \le f < e$, r and d such that $r - e + f \ge 0$. Then we have that*

$$\dim\{l \in G_d^r(C) \mid V_e^{e-f}(l) \neq \emptyset\} \le \rho(g, r, d) - f(r + 1 - e + f) + e.$$

In particular, if $\rho(g, r, d) - f(r + 1 - e + f) + e < 0$, then $V_e^{e-f}(l) \neq \emptyset$, for every $l \in G_d^r(C)$.

As a consequence ([Far08, Cor. 0.3]), the author obtains that, when the curve is general and $V_e^{e-f}(l)$ is non-empty with l a general g_d^r, then its dimension is the expected one. Nevertheless, it took many years before a theorem comparable to the Brill–Noether Theorem for secant varieties was proved. Before the result mentioned above, a first result can be found in [ACGH85, p.356], where it is proved that if $g \ge (e-f+1)(r-d+e-f+1)$, then $V_e^{e-f}(l) \neq \emptyset$; while if $g < (e - f + 1)(r - d + e - f)$, then $V_e^{e-f}(l)$ is either empty or of the wrong dimension. Later, in [CM91, Thm 1.2] Coppens and Martens showed a similar result but for any genus. Precisely, given a complete $l = g_d^r$, if $d \ge 2e - 1$ and $f(r - e + f + 1) \le r + 2e - f$, then $V_e^{e-f}(l)$ is non-empty. Very recently, Farkas and Lelli-Chiesa established in [FLC25, Thm 1.1] the dimension of the secant variety for a general Prym-canonical curve C of genus g using degeneration, and they give a sufficient condition for its emptiness. The result is the following (we recall that $\mathcal{R}_g$ is the moduli space parametrizing (C, η), where C is a smooth curve of genus g and η is a non trivial 2-torsion line bundle on C; see Section 0.6):

Theorem 0.5. *We fix integers $0 \le f < e < g$. Then for a general Prym curve $(C, \eta) \in \mathcal{R}_g$ the secant locus $V_e^{e-f}(\omega_C \otimes \eta)$ is equidimensional of dimension*

$$\dim V_e^{e-f}(\omega_C \otimes \eta) = e - f(g - 1 - e + f).$$

In particular, if $e - f(g-1-e+f) < 0$, then $V_e^{e-f}(\omega_C \otimes \eta) = \emptyset$.

0.3 Clifford index

In order to study algebraic curves, and in particular to classify them, it is useful to fix some invariants. A classical one is the gonality of a curve, that is, the minimum degree of a morphism from the curve into $\mathbb{P}^1$. For instance, if the gonality is one, then the curve is said rational, while hyperelliptic curves are those with gonality two. In terms of the variety $G_d^r(C)$, the gonality of C is the minimum integer d such that $G_d^1(C) \neq \emptyset$, and the Brill–Noether Theorem implies that the gonality of a general curve is $\lfloor \frac{g+3}{2} \rfloor$. There exists another classic invariant for curves, strictly related to the gonality. Given a line bundle $L \in \mathrm{Pic}(C)$ such that both $h^0(C, L) \ge 2$ and $h^1(C, L) \ge 2$, the Clifford index of L is

$$\mathrm{Cliff}(L) := \deg(L) - 2(h^0(C, L) - 1),$$

and it measures how many independent sections a line bundle has for its degree. The Clifford index of the curve C is defined as

$$\mathrm{Cliff}(C) := \min\{\mathrm{Cliff}(L) \mid L \in \mathrm{Pic}(C),\ h^i(C, L) \ge 2 \text{ for } i = 1, 2\}.$$

If $L \in \mathrm{Pic}(C)$ satisfies $h^i(C, L) \ge 2$, then L contributes to $\mathrm{Cliff}(C)$, and if $\mathrm{Cliff}(L) = \mathrm{Cliff}(C)$, it computes the Clifford index of the curve. In [Cli65] Clifford proved the following:

Theorem 0.6. *(Clifford's Theorem) If D is an effective divisor of degree d on C with $d \le 2g - 1$, then*

$$r(D) \le \frac{d}{2};$$

if the equality holds then either D is zero, or D is a canonical divisor, or C is hyperelliptic and D is linearly equivalent to a multiple of the hyperelliptic divisor.

For the proof we refer to [ACGH85, Ch. III, Sec. 1] and [Har77, p. 108].
Clifford's Theorem implies that $\mathrm{Cliff}(L) \ge 0$, and if $\mathrm{Cliff}(L) = \mathrm{Cliff}(C) = 0$, then C is hyperelliptic and L is a multiple of the g_2^1. Hence, another way to interpret the Clifford index of a curve is as a measure of how far the curve is from being hyperelliptic.

We recall also the definition of the Clifford dimension of a curve, that is, the minimum $r \geq 1$ such that there exists a line bundle L that computes $\mathrm{Cliff}(C)$ and such that $h^0(C, L) = r + 1$. A general curve C has Clifford dimension 1. Indeed, a general C has a g^r_d if and only if $\rho(g, r, d) \geq 0$, that can be rewritten as

$$d - 2r \geq \frac{r}{r+1} g - r. \tag{2}$$

The right handside of (2) attains its minimum for $r = 1$, and since $\mathrm{gon}(C) = \lfloor \frac{g+3}{2} \rfloor$, it follows that the Clifford index of a general curve C is $\mathrm{Cliff}(C) = \lfloor \frac{g+3}{2} \rfloor - 4 + 2 = \lfloor \frac{g-1}{2} \rfloor$, namely, $\mathrm{gon}(C) - 2$. More generally, every curve C with Clifford dimension 1 has Clifford index equals to $\mathrm{gon}(C) - 2$. However, there exist curves with Clifford dimension greater than one, that are called *exceptional*. We discuss them more in detail in the next section.

0.4 Clifford dimension and exceptional curves

Curves with Clifford dimension 2 are smooth plane curves of degree greater or equal than 5 and genus at least 2. Indeed, in [ELMS89, Lemma 1.1] it is proved that if L is a line bundle computing the Clifford dimension of the curve with $h^0(C, L) \geq 3$, then L is very ample and thus defines an embedding $C \hookrightarrow \mathbb{P}^r$. In particular, for $r = 2$, it realizes C as a smooth planar curve. Moreover, by unicity of the g^2_d on a smooth planar curve of degree d ([ACGH85, p.56]), the embedding defined by L is unique. In [CM91] Coppens and Martens proved that if C is an exceptional curve of gonality k, then $\mathrm{Cliff}(C) = k-3$, and C has a 1-dimensional family of g^1_k. By a result of Max Noether, when $r = 2$, every g^1_k is obtained by projecting from a point of C. Curves with Clifford dimension ≥ 3 are investigated by Eisenbud, Lange, Martens and Schreyer in [ELMS89]. Indeed, they conjectured the following:

Conjecture 1. *If C has Clifford dimension $r \geq 3$ then:*

1. *C has genus $g = 4r - 2$ and Clifford index $2r - 3$ (and thus degree $g - 1$).*
2. *C has a unique line bundle L computing the Clifford index.*
3. *$L^{\otimes 2}$ is the canonical bundle of C, and L embeds C as an arithmetically Cohen-Macaulay curve in $\mathbb{P}^r$.*
4. *C is $2r$-gonal, and there is a 1-dimensional family of pencils of degree $2r$, all of the form $|L(-D)|$ where D is the divisor of $2r - 3$ points of C.*

In particular, curves with Clifford dimension 3 were previously studied by Martens in [Mar82]. They are precisely the complete intersections of pairs of cubics in $\mathbb{P}^3$, and they have genus 10 and their Clifford index is 3. For numerical reasons, they are extremely rare. Indeed, by [ELMS89, Lemma 1.1] and a theorem due to Castelnuovo ([ELMS89, Thm 1.2]) establishing the number $C(d,g,r)$ of $(2r-2)$-secant $(r-2)$-planes with $r \geq 2$, an exceptional curves has no $(2r-2)$-secant $(r-2)$-planes. When $r \geq 3$, the diophantine equation $C(d,g,r)=0$ has very few solutions. As concerns Conjecture 1, the authors proved that if (1) holds, then the statements $(2)-(3)-(4)$ follow. In particular, under the assumptions that C has Clifford dimension $r \geq 2$ and Clifford index $2r-3$, they firstly showed that $\mathrm{gon}(C) = 2r$ ([ELMS89, Prop. 3.2]), and then that C is not contained in any quadric of rank ≤ 4 ([ELMS89, Thm 3.3]). Moreover, with the so-called "Recognition Theorem" ([ELMS89, Thm 3.6]) they established that the following conditions are equivalent:

- C has Clifford dimension r;
- $C \subseteq \mathbb{P}^r$ is not contained in any quadrics of rank ≤ 4;
- C is $2r$-gonal,

and that if one of the above is satisfied, then $L = \mathcal{O}_C(1)$ computes the Clifford dimension and it is semi-canonical. The unicity of L follows from [ELMS89, Thm 3.7]. Finally, they proved that for $r \leq 9$ the statement (1) is satisfied ([ELMS89, p. 203]), and so for $r \leq 9$, the conjecture is completely proved.
In Section 4, they also contructed examples of exceptional curves of Clifford dimension $r \geq 3$ lying on certain $K3$ surfaces. In particular, they considered a $K3$ surface whose Picard group is generated by a very ample divisor D and an irreducible curve Γ with intersection matrix

$$\begin{pmatrix} D^2 & \Gamma.D \\ \Gamma.E & \Gamma^2 \end{pmatrix} = \begin{pmatrix} 2r-2 & 1 \\ 1 & -2 \end{pmatrix},$$

and they proved that curves $C \sim 2D + \Gamma$ are smooth, irreducible, of Clifford dimension r and Clifford index $2r-3$ ([ELMS89, Thm 4.3]).

0.5 Syzygies of a canonical curve

Syzygies were originally used in astronomy, and they first appeared in Mathematics in a XIX-century paper by Sylvester, where they expressed the idea of linear relation between certain functions with arbitrary functional coefficients. In the following century, they became widely known and extensively studied in algebraic geometry after Green's

introduction of Koszul cohomology in [Gre84]. He showed how one can read all information about the minimal free resolution of the coordinate ring of an algebraic variety from Koszul cohomology, which turned out to be the most effective tool for computing the syzygies of a curve. In particular, the syzygies of a canonical curve C are related to $\mathrm{Cliff}(C)$ via Green's conjecture ([Gre84, Conj. 5.1]).
Let C be a smooth curve of genus g, and consider $S := \mathrm{Sym}H^0(C, \omega_C)$ the homogeneous coordinate ring of the projective space $\mathbb{P}(H^0(C, \omega_C)^\vee)$. We set $R(C) := \bigoplus_n H^0(C, \omega_C^{\otimes n})$; since this is a finitely generated S-module, it admits a minimal graded free resolution

$$0 \to E_{g-2} \to \ldots \to E_1 \to E_0 \to R(C),$$

where $E_k := \sum_{i \geq k} S(-i-1)^{\beta_{k,i}}$, and the integers $\beta_{k,i}$ are called *graded Betti numbers* of $R(C)$. The *syzygies* of C of order k are by definition the graded components of the S-module E_k. The *Koszul cohomology group* $K_{p,q}(C, \omega_C)$ is defined as the cohomology group of the complex

$$\bigwedge^{p+1} H^0(C, \omega_C) \otimes H^0(C, \omega_C^{q-1}) \to \bigwedge^{p} H^0(C, \omega_C) \otimes H^0(C, \omega_C^{q}) \to \bigwedge^{p-1} H^0(C, \omega_C) \otimes H^0(C, \omega_C^{q+1}),$$

and with this notation, it turns out that $\beta_{i,j} = \dim K_{i,j}(C, \omega_C)$. Green's Conjecture ([Gre84]) states the following:

Conjecture 2. $K_{p,2}(C, \omega_C) = 0$ *if* $p < \mathrm{Cliff}(C)$.

In terms of Betti numbers, it can reformulated as $\beta_{j,2} = 0$ for $j < \mathrm{Cliff}(C)$. In [Gre84] Green introduced also the property (N_p), which for (C, ω_C) is satisfied if and only if

$$K_{i,q}(C, \omega_C) = 0 \quad \text{for all} \quad i \leq p \text{ and } q \geq 2,$$

so that the conjecture can be rephrased by asserting that (C, ω_C) satisfies property (N_p) whenever $p < \mathrm{Cliff}(C)$.
When $p = 0$, Conjecture 2 concerns the vanishing $K_{0,q}(C, \omega_C) = 0$ for all $q \geq 2$, that is, the projective normality of a canonical curve $C \subset \mathbb{P}^{g-1}$ (we recall that a curve $C \subset \mathbb{P}^r$ is projectively normal for every k the hypersurfaces of degree k cut out the complete linear series $|\mathcal{O}_C(k)|$). Max Noether's Theorem (see [ACGH85, p. 117]) ensures that a canonical curve is projectively normal whenever it is not hyperelliptic. Analogously, when $p = 1$, the conjecture predicts that as long as $\mathrm{Cliff}(C) > 1$, $K_{1,q}(C, \omega_C) = 0$ for all $q \geq 2$, or equivalently, the ideal $\mathcal{I}_{\omega_C}$ is generated by quadrics. This statement is true and is the content of the Enriques–Babbage Theorem ([ACGH85, p. 124]). Note that curves with $\mathrm{Cliff}(C) = 1$ were classified by Marten ([ACGH85, p. 191]) and Mumford, and they are trigonal or smooth plane quintics.

We remark that in the appendix of [Gre84], Green and Lazarsfeld proved that if $p \geq \mathrm{Cliff}(C)$ then $K_{p,2}(C, \omega_C) \neq 0$. A first proof of Conjecture 2 was given by Voisin for $[C] \in M_g$ general ([Voi02], [Voi05]). More recent proofs can be found in [AFP$^+$19] and [Kem25].
Despite the many advances made over the years, the conjecture in its full generality is still open.

0.6 Prym varieties and the Prym map

Brill–Noether theory focuses on the canonical morphism $\varphi_{\omega_C} : C \to \mathbb{P}^{g-1}$, which is an embedding as soon as the curve is not hyperelliptic. There are also other morphisms that it is natural to study in the context of algebraic curves. For instance, given a smooth curve C of genus g, one can consider a non-trivial two torsion line bundle $\eta \in \mathrm{Pic}^0(C)$, which defines an étale double cover ([ACGH85, Appendix B, 2.13])

$$\pi : \tilde{C} \to C.$$

Bearing in mind that the Jacobian variety $JC = H^0(C, \omega_C)^\vee / H_1(C, \mathbb{Z})$ parametrises classes of linear equivalence of divisors of degree zero, one may define the so-called *norm map* induced by π as

$$\mathrm{Nm}_\pi : \qquad J\tilde{C} \to JC$$
$$[\sum_i n_i p_1] \mapsto [\sum_i n_i \pi(p_i)]$$

The kernel of Nm_π is not connected, but since Nm_π is a group homomorphism, the connected component containing the zero element is a subgroup of $J\tilde{C}$, so one may define the *Prym variety* of π as

$$P(\pi) := (\ker(\mathrm{Nm})_\pi)^0 \subset J\tilde{C}.$$

The principal polarization Θ on $J\tilde{C}$ restricts to a polarization Ξ on $P(\pi)$, and set-theoretically,

$$\Xi = \{L \in P(\pi) \mid h^0(\tilde{C}, L) \geq 2\}.$$

Thus, $(P(\pi), \Xi)$ is a principally polarized abelian subvariety of $J\tilde{C}$, and, as such,

$$\dim(P(\pi)) = \dim J\tilde{C} - \dim JC = 2g - 1 - g.$$

Note that if one consider the involution ι on $\tilde{C}$, it induces an automorphism ι^* on $J\tilde{C}$. This implies that the Prym variety $P(\pi)$ can be also described as $P(\pi) = \mathrm{Im}(\mathrm{Id} - \iota^*) \subset J\tilde{C}$. The moduli space

$$\mathcal{R}_g := \{(C, \eta) \mid [C] \in M_g,\ \eta \in \mathrm{Pic}^0(C)[2] \setminus \{\mathcal{O}_C\}\}$$

of dimension $3g - 3$ and parametrizing Prym curves, provides a bridge between M_g and the moduli space of abelian varieties $\mathcal{A}_{g-1}$. Indeed, the following correspondance exists:

$$\begin{array}{ccc} & \mathcal{R}_g & \\ {}^{f}\swarrow & & \searrow^{Pr_g} \\ M_g & & \mathcal{A}_{g-1} \end{array}$$

where f is the forgetful map, and Pr_g is the Prym map sending $(C, \eta) \in \mathcal{R}_g$ to $(P(\pi), \Xi) \in \mathcal{A}_{g-1}$. There has always been a great deal of research activity around Pr_g. To mention just one example, the *Prym–Torelli problem* addresses the injectivity of the Prym map. In [DS81] Donagi and Smith proposed a method of computing the degree of the Prym map, and showed that for $g \leq 6$, a line bundle η cannot be determined by its Prym variety (in [Don81] examples for any genus ≤ 6 are provided). In particular, they computed that, for $g = 6$, the degree of Pr_g is 27. For a general $(C, \eta) \in \mathcal{R}_g$, Kanev applied the Donagi–Smith method for arbitrary g, and showed that if $g \geq 9$, then the Prym map has degree one ([Kan83]), and, independently and almost simultaneously, Friedman and Smith proved that for general curves of genus $g \geq 7$, Pr_g is birational onto its image ([FS82]). However, the Prym map is not injective ([Don92], [Bea77]; in [IL12] it is also proved the noninjectivity of the Prym map for curves of arbitrary high Clifford index), and the study of the non-injectivity locus is still a famous open problem. We remark that in order to check whether Pr_g is generically finite, one studies the injectivity of the differential dPr_g at a generic point $(C, \eta) \in \mathcal{R}_g$ (equivalent to the surjectivity of its dual $dPr_g^\vee$). Repeated

$$dPr_{g(C,\eta)} \colon H^0(C, \omega_C^{\otimes 2})^\vee \to (\mathrm{Sym}^2 H^0(\omega_C \otimes \eta))^\vee, \tag{3}$$

which is subject of the so-called *infinitesimal Prym–Torelli problem*. On the one hand, Formula (3) arises because the forgetful map $f \colon \mathcal{R}_g \to M_g$ is finite, so the dual of the tangent space satisfies

$$(\mathcal{T}_{(C,\eta)}\mathcal{R}_g)^\vee \simeq (\mathcal{T}_{(C)}M_g)^\vee \simeq H^0(C, \omega_C^{\otimes 2}).$$

On the other hand, since $\mathcal{T}_0\Big(P(\pi)\Big) \simeq H^0(\omega_C \otimes \eta)^\vee$ where $0 \in P(\pi)$, then

$$\Big(\mathcal{T}_{(P(\pi),\Xi)}\mathcal{A}_{g-1}\Big)^\vee \simeq \mathrm{Sym}^2\mathcal{T}_0(P(\pi))$$

(cf. [BO22, p. 9]).

0.7 Syzygies of Prym-canonical curves

We now focus on the line bundle $\omega_C \otimes \eta$ appearing in (3), known as the *Prym–canonical bundle*, and often considered in literature. When it is base-point free, it defines the *Prym-canonical morphism* $\varphi_\eta \colon C \to \mathbb{P}^{g-2}$; we refer to the first section of Chapter 1 for preliminary results about the base-point freeness of $|\omega_C \otimes \eta|$, and the birationality of φ_η. Since property (N_0) (see the Section 0.5 for the notation) is about normal generation, we recall the following results by Green and Lazarsfeld, stating them only in the case of Prym-canonical line bundles.

Theorem 0.7. *[GL86, Thm 3] Let (C, η) be a smooth irreducible Prym curve of genus $g \geq 7$, and assume that $\omega_C \otimes \eta$ is very ample. Then, $\omega_C \otimes \eta$ fails to be normally generated if and only if there exists an integer $1 \leq n \leq g-4$ and an effective divisor D on C of degree $d \geq 2n+2$ such that*

i. $H^1(C, \omega_C^{\otimes 2}(-D)) = 0$ *and*

ii. *D spans a n-plane $\Lambda \subseteq \mathbb{P}^{g-2}$ in which D fails to impose independent conditions on quadrics.*

Condition *ii.* is equivalent to $H^1(\Lambda, \mathcal{I}_{D/\Lambda}(2)) \neq 0$, where $\mathcal{I}_{D/\Lambda}(2)$ is the ideal sheaf of D in Λ. For the "⇒" direction of the proof, the idea is to give an interpretation of the failure of normal generation in terms of extensions of line bundles. First of all, since $\mathrm{Ext}^1(\omega_C \otimes \eta, \eta) \neq \emptyset$, there is at least a nontrivial extension

$$0 \to \eta \to E \to \omega_C \otimes \eta \to 0,$$

where E is a rank 2 vector bundle such that $\det(E) = \omega_C$ and $h^0(E) = h^0(\omega_C \otimes \eta) + h^0(\eta)$. Then, one produces a sub-bundle A of E of suitably large degree, and by degree reasons, it is proved that A is of the form $\omega_C \otimes \eta(-D)$ for some effective divisor D on C. Finally, it is shown that D satisfies the statements i and ii.
In the same article the authors also proved the following:

Theorem 0.8. *[GL86, Thm 1] Let (C, η) be a smooth irreducible Prym curve, and assume $\omega_C \otimes \eta$ very ample. If*

$$\mathrm{Cliff}(C) \geq 2g + 2,$$

then $\omega_C \otimes \eta$ is normally generated.

Theorem 0.9. *[GL86, Thm 2.1] Assume that the curve C has genus $g > 26$, Clifford index 2 and is not bielliptic. Also assume that the Prym-canonical bundle $\omega_C \otimes \eta$ is very ample. Then, the Prym-canonical bundle is not normally generated if and only if the Prym-canonical curve $\varphi_\eta(C) \subset \mathbb{P}^{g-2}$ has a 4-secant line cutting out a divisor which moves in a g^1_4.*

The above results show that the syzygies of a Prym-canonical curve $\varphi_\eta(C)$ are affected both by the Brill–Noether behaviour of C and by the secant varieties to $\varphi_\eta(C)$. Insired by Theorem 0.9, Lange and Sernesi studied in [LS96] the ideal of the Prym-canonical curve. In particular, given a Prym-canonical curve $C \subseteq \mathbb{P}^{g-2}$ and denoting by Q the quadrics in $\mathbb{P}^{g-2}$, they proved the following results:

Corollary 0.10. *[LS96, Cor. 2.3] If $\mathrm{Cliff}(C) \geq 5$ then $\varphi_\eta(C) = \bigcap_{Q \supset \varphi_\eta(C)} Q$.*

Corollary 0.11. *[LS96, Cor. 2.5] If $\mathrm{Cliff}(C) = 4$ then $\varphi_\eta(C) = \bigcap_{Q \supset \varphi_\eta(C)} Q$ if and only if the Prym-canonical curve $\varphi_\eta(C)$ has no trisecant lines.*

In [FL10], given a Prym curve $(C, \eta) \in \mathcal{R}_g$, the authors used syzygies of the Prym-canonical curve to stratify $\mathcal{R}_g$. In particular, the strata consist of the Prym curves $(C, \eta) \in \mathcal{R}_g$ such that their Prym-canonical model fails the property (N_i) introduced in the previous section. In symbols,

$$\mathcal{U}_{g,i} := \{(C, \eta) \in \mathcal{R}_g \mid K_{i,2}(C, \omega_C \otimes \eta) \neq \emptyset\}.$$

They proved that for $g < 2i + 6$ and for any $(C, \eta) \in \mathcal{R}_g$, $K_{i,2}(C, \omega_C \otimes \eta) = 0$ ([FL10, Prop. 3.1]), and they formulated a conjecture analogous to Green's conjecture. The *Prym–Green's conjecture* is the following:

Conjecture 3. *For a general $(C, \eta) \in \mathcal{R}_g$ and $g \geq 2i+6$, one has that $K_{i,2}(C, \omega_C \otimes \eta) = 0$. Equivalently, the Prym-canonical curve $C \subseteq \mathbb{P}^{g-2}$ satisfies the Green- Lazarsfeld property (N_i) whenever $g \geq 2i + 6$. For $g = 2i + 6$, the locus $\mathcal{U}_{2i+6,i}$ is an effective divisor on $\mathcal{R}_{2i+6}$.*

A few years later, the conjecture was studied in depth in [CEFS13]. The authors generalized the statement of the Prym–Green conjecture to l-torsion line bundles η and to *paracanonical* curves, that is, curves embedded into $\mathbb{P}^{g-2}$ with $\omega_C \otimes \eta$. The statement they gave is the following (we remark that the moduli space $\mathcal{R}_{g,l}$ parametrises level l curves, that is, triples (C, L, ϕ) where C is a smooth curve equipped with a line bundle L and a trivialisation $\phi \colon L^{\otimes l} \to \mathcal{O}_C$):

Conjecture 4. *[CEFS13, Conj. A] For a general level l curve $(C,\eta) \in \mathcal{R}_{g,l}$ of genus $g \geq 6$, the homogeneous coordinate ring of the paracanonical curve has, in even genus, a pure resolution and the paracanonical curve satisfies property $(N_{\frac{g}{2}-2,1})$, that is,*

$$K_{\frac{g}{2}-2,1}(C,\omega_C \otimes \eta) = 0,$$

whereas in odd genus

$$K_{\frac{g-3}{2},1}(C,\omega_C \otimes \eta) = 0 \quad \text{and} \quad K_{\frac{g-7}{2},1}(C,\omega_C \otimes \eta) = 0.$$

Using nodal curves, the authors showed computationally that the Prym-Green conjecture holds true for all even genera with $g \leq 18$ and $l \leq 5$ over a field of characteristic 0, with possible exceptions for the cases $(g,l) = (8,2)$ or $(16,2)$ ([CEFS13, Prop. 4.4]). For $l = 2$, namely when C is a Prym-canonical curve, Farkas and Kemeny verified the conjecture for odd genus by specialization to Nikulin surfaces, namely $K3$ surfaces X equipped with a double cover $\tilde{X} \to X$ branched along eight disjoint rational curves ([FK16, Thm 1.1]). They also verified the conjecture for general paracanonical curves of odd genus of all but finitely many levels $l \geq \sqrt{\frac{g+2}{2}}$ ([FK17, Thm 1.1]), and they proved a weaker version of the conjecture for even genus ([FK17, Thm 0.2]), that is, a general level l curve (C,η) of genus ≥ 8 and with $l = 2$ or $l \geq \sqrt{\frac{g+2}{2}}$, is such that $K_{\frac{g}{2}-4,2}(C,\omega_C \otimes \eta) = 0$ and $K_{\frac{g}{2}-1,1}(C,\omega_C \otimes \eta) = 0$. The geometrical reasons for the failure of the conjecture for $l = 2$ and $g = 2^n \geq 8$ are still not understood.
Conjecture 3 only concerns general Prym curves, so if one is interested in the syzygies of an arbitrary curve $(C,\eta) \in \mathcal{R}_g$, it is natural to expect that they depend not only on the Clifford index, that by Riemann–Roch can be considered as computed with respect to the canonical bundle, but also on another invariant that depends on the secant varieties $V_e^{e-f}(\omega_C \otimes \eta)$ associated with the Prym-canonical bundle (see Section 0.2). This provides part of the motivation for the present thesis, where we define a new invariant for Prym-canonical curves, the *Prym-canonical Clifford index*, computed with respect to $\omega_C \otimes \eta$. We refer to Chapter 1.2 for the definition and to Chapter 1.5 for its relation to secant varieties.

0.8 Main results and organization of the thesis

In Chapter 1 we define the Prym-canonical Clifford index and the Prym-canonical Clifford dimension. Given a Prym curve $(C,\eta) \in \mathcal{R}_g$ and a line bundle $L \in \mathrm{Pic}(C)$ such that $h^0(L) \geq 1$, $h^0(L \otimes \eta) \geq 1$, then

$$\mathrm{Cliff}_\eta(L) := h^0(\omega_C \otimes \eta) - h^0(L) - h^0(\omega_C \otimes L^\vee) + 1,$$

and the Prym-canonical Clifford index of (C,η) is the minimum $\mathrm{Cliff}_\eta(L)$ achieved by all the line bundles L on C of degree $\leq g-1$ such that $h^0(L)\geq 1$, $h^0(L\otimes\eta)\geq 1$. We prove that if L computes the Prym-canonical Clifford index, that is, $\mathrm{Cliff}_\eta(L)=\mathrm{Cliff}_\eta(C)$, then L and $L\otimes\eta$ have no base points in common.
We also define an analogue of the Clifford dimension. The Prym-canonical Clifford dimension of C is defined as

$$\mathrm{dimCliff}_\eta(C):=\min\Big\{(r,r')\;\Big|\;\exists\, L\in \mathrm{Pic}(C),\ \deg(L)\leq g-1,\ \ r=r(L),\ r'=r(L\otimes\eta),\ \mathrm{Cliff}_\eta(C)=\mathrm{Cliff}_\eta(L)\Big\},$$

where the minimum is taken with respect to the lexicographic order on $\mathbb{Z}^2$. In the classical case, if L computes the Clifford index of C, that is, $\mathrm{Cliff}(L)=\mathrm{Cliff}(C)$, then both L and $\omega_C\otimes L^\vee$ are base-point free. For the base-point freeness of L and $L\otimes\eta$ it is necessary that L computes the Prym-canonical Clifford dimension of C, that is, $\mathrm{dimCliff}_\eta(C)=(r(L),r(L\otimes\eta))$.
This fact can be generalized in terms of the secant varieties $V_e^{e-f}(\omega_C\otimes\eta)$ parametrising effective divisors of degree e which impose at most $e-f$ independent conditions on $\omega_C\otimes\eta$. We show the following:

Lemma. *Let L be a line bundle computing the Prym-canonical Clifford dimension of (C,η). Then, for every positive integers $1\leq f<e\leq\min\{\deg L, r(L)+f\}$ and $1\leq f'<e\leq\min\{\deg L, r(L\otimes\eta)+f'\}$ such that $f+f'\geq e$, we have*

$$V_e^{e-f}(L)\cap V_e^{e-f'}(L\otimes\eta)=\emptyset.$$

Proof. See Lemma 1.11. □

As an application, we get that

Lemma. *For every Prym curve (C,η) of genus $g\geq 2$, one has*

$$\mathrm{dimCliff}_\eta(C)\neq(1,1).$$

Proof. See Lemma 1.12. □

This yields that the Prym-canonical Clifford dimension of a curve (C,η) can be either $(0,0)$, or strictly greater than $(1,1)$, as we prove also that the case $\mathrm{dimCliff}_\eta(C)=(0,r')$, with $r'\geq 1$, cannot occur. The main result is the following (for the proof see Theorem 1.15):

Theorem (Prym-Clifford's Theorem)**.** *Let (C,η) be a Prym curve of genus $g \geq 2$. Then*

$$\mathrm{Cliff}_\eta(C) \geq 0$$

and equality holds if and only if $\omega_C \otimes \eta$ has some base points, that is, when C is hyperelliptic and $\eta = \mathcal{O}_C(p-q)$ with p,q ramification points of the g^1_2.

The "if" direction of the equality in the statement of the theorem relies on the following lemma:

Lemma 1. *Let L be a line bundle contributing to the Prym-canonical Clifford index of (C,η) such that* $\mathrm{deg}L \leq g-1$, $h^0(L) \geq 2$ *and* $h^0(L \otimes \eta) \geq 2$. *Then,*

$$\mathrm{Cliff}_\eta(L) + \mathrm{Cliff}_\eta(L \otimes \eta) = 2\mathrm{Cliff}_\eta(L) = \mathrm{Cliff}(L) + \mathrm{Cliff}(L \otimes \eta) - 2.$$

In particular, one gets

$$\mathrm{Cliff}_\eta(L) \geq \mathrm{Cliff}(C) - 1.$$

Proof. See Lemma 1.10. □

At the end of the chapter, we introduce another invariant. Given $\pi\colon \tilde{C} \to C$ the étale double cover induced by η, and ι the covering involution on $\tilde{C}$, we define the ι-invariant Clifford index of $\tilde{C}$ as

$$\mathrm{Cliff}(\tilde{C})^\iota := \min\left\{ \frac{\mathrm{Cliff}(\tilde{L})}{2} \;\middle|\; \tilde{L} \in \mathrm{Pic}(\tilde{C})^\iota,\ \deg(\tilde{L}) \leq g(\tilde{C}) - 1, h^0(\tilde{L}) \geq 2 \right\},$$

where $\mathrm{Pic}(\tilde{C})^\iota$ parametrizes the ι-invariant line bundles on $\tilde{C}$

$$\mathrm{Pic}(\tilde{C})^\iota := \{\tilde{L} \in \mathrm{Pic}(\tilde{C}) \mid \iota^*\tilde{L} \simeq \tilde{L}\} = \{\pi^*L \in \mathrm{Pic}(\tilde{C}\ ),\ L \in \mathrm{Pic}(C)\}.$$

We obtain the following result, which shows that $\tilde{C}$ and the Prym-canonical Clifford index of C are related.

Proposition. *Let (C,η) be a smooth Prym curve. Let $\pi\colon \tilde{C} \to C$ be the étale double cover induced by η and denote by ι the covering involution on $\tilde{C}$. Then, the following holds:*

$$\mathrm{Cliff}(\tilde{C})^\iota = \min\{\mathrm{Cliff}_\eta(C),\ \mathrm{Cliff}(L) + r(L) \mid L \in \mathrm{Pic}(C), \deg(L) \leq g-1, h^0(L) \geq 2\}.$$

If moreover C has Clifford dimension 1*, then:*

$$\mathrm{Cliff}(\tilde{C})^\iota = \min\{\mathrm{Cliff}_\eta(C),\ \mathrm{Cliff}(C) + 1\}.$$

Proof. See Proposition 1.17. □

In Chapter 2 we classify curves $(C, \eta) \in \mathcal{R}_g$ such that $1 \leq \mathrm{Cliff}_\eta(C) \leq 2$, and we compute the Prym-canonical Clifford dimension of bielliptic curves, that is, double covers of an elliptic curve.

Theorem. *Let (C, η) be a smooth Prym curve of genus $g \geq 3$. Then, the equality $\mathrm{Cliff}_\eta(C) = 1$ holds if and only if $\omega_C \otimes \eta$ is base point free but not very ample, that is, C has a g^1_4 and $\eta = \mathcal{O}_C(p + q - x - y)$, with $2(p + q) \sim 2(x + y)$.*

The "if" part of the statement is trivial. The "only if" part is equivalent to the statement that $\mathrm{Cliff}_\eta(C) = 1$ implies $\mathrm{dimCliff}_\eta(C) = (0, 0)$. Indeed, if the Prym-canonical Clifford dimension of C is $(0, 0)$ and if $L \in \mathrm{Pic}(C)$ computes it, then $\deg(L) = 2$ and $L = \mathcal{O}_C(D)$ where D is a degree 2 divisor on C. The Riemann–Roch Theorem and the assumption $\mathrm{Cliff}_\eta(L) = 1$ imply that the Prym-canonical morphism identifies the points of D , and thus the statement follows. To prove that $\mathrm{dimCliff}_\eta(C) = (0, 0)$, we take a line bundle L computing the Prym-canonical Clifford dimension of (C, η) and, by contradiction, assume $2 \leq r(L) \leq r(L \otimes \eta)$. By Lemma 1, we obtain that $0 \leq \mathrm{Cliff}(C) \leq 2$ and $\mathrm{Cliff}(L) + \mathrm{Cliff}(L \otimes \eta) = 4$. We organize the proof by considering the cases arising from this inequality. In order to rule them out, we make use of known results (that we recall in Section 1) concerning linear series on curves with gonality 3 ([MS86]), 4 ([CM00]), 5 ([Par02]). We also need to study linear series on plane curves of degree ≤ 6 having double points as singularities. For a detailed proof, see Theorem 2.19. We also prove the following result:

Theorem. *Let (C, η) be a smooth Prym curve of genus $g \geq 4$. Then, $\mathrm{Cliff}_\eta(C) = 2$ if and only if the Prym-canonical morphism is an embedding and $\varphi_\eta(C)$ has a trisecant line.*

The strategy is the same, but in this case, Lemma 1 gives $0 \leq \mathrm{Cliff}(C) \leq 3$ and we have much more cases to exclude. The proof is therefore significantly more demanding (see Theorem 2.21).
Finally, as concerns bielliptic curves, we obtain the following theorem (see Theorem 2.12):

Theorem. *If C is a bielliptic curve, then $\mathrm{dimCliff}_\eta(C) = (0, 0)$.*

Chapter 3 is devoted to the study of hyperelliptic and general curves. Due to a result by Verra ([Ver13, Lemma 4.3]), given a hyperelliptic curve C of genus g, for any $\eta \in \mathrm{Pic}^0(C)[2] \setminus \{\mathcal{O}_C\}$, there exists an integer $1 \leq k \leq \lfloor \frac{g+1}{2} \rfloor$ such that η can be written as

$$\eta = \mathcal{O}_C(w_1 + \ldots + w_k - w_{k+1} - \ldots - w_{2k}), \quad (4)$$

where $w_1, \dots, w_{2k}$ are distinct Weierstrass points. Surprisingly, the Prym-canonical Clifford index of C depends on the integer k. Our result is the following (see Theorem 3.2):

Theorem. *If C is hyperelliptic and η can be written as in* (4) *for some $k \leq \lfloor \frac{g+1}{2} \rfloor$, then*

$$\mathrm{Cliff}_\eta(C) = k - 1 \quad \textit{and} \quad \mathrm{dimCliff}_\eta(C) = (0, 0).$$

Indeed, we firstly consider the line bundle $A = \mathcal{O}_C(w_1 + \dots w_k)$ and we compute that $h^0(A) = h^0(A \otimes \eta) = 1$. We then take a line bundle L computing the Prym-canonical Clifford dimension of (C, η) and prove that $L = A$ by showing that if $D \in |L|$ and $D' \in |L \otimes \eta|$, then D and D' are sums of Weierstrass points.
We then compute the rational normal scroll S containing a hyperelliptic Prym-canonical curve $\varphi_\eta(C)$. We find that the scroll is of type $S = S(g - 1 - k, k - 2)$. In [Par10], Park proved that the syzygies of C can be encoded by the rational normal scroll $S \supset C$. Given $(C, \eta) \in \mathcal{R}_g$, for $k \geq 2$ and $p \geq 1$, one may define the property $(N_{k,p})$ of (C, η) as the vanishing $\beta_{i,j}(C) = 0$ for $1 \leq i \leq p$ and all $j \geq k$. This is a generalization of the property (N_i) mentioned in Section 0.5. As a consequence of Theorem [Par10, Thm 1.3], we get the following corollary (see Corollary 3.13):

Corollary. *Let C be a hyperelliptic curve of genus g and let $\eta = \mathcal{O}_C(w_1 + \dots + w_k - w_{k+1} - \dots - w_{2k})$ with $w_1, \dots, w_{2k}$ Weierstrass points and $3 \leq k \leq \lfloor \frac{g+1}{2} \rfloor$. Set*

$$\nu := \begin{cases} 5 & \text{for } k = 3 \\ 4 & \text{for } k = 4 \\ 3 & \text{for } k \geq 5 \end{cases} \qquad \text{and} \quad p = \nu(k-2) - 2k + 1.$$

Then:

i. the Castelnuovo–Mumford regularity of $\varphi_\eta(C)$ is $\nu + 1$;

ii. $\varphi_\eta(C)$ satisfies $N_{\nu,p}$ but fails property $N_{\nu,p+1}$.

In Section 2 we compute the Prym-canonical Clifford index of a general curve $(C, \eta) \in \mathcal{R}_g$. The surjectivity of the difference map $\phi_d \colon C_d \times C_d \to \mathrm{Pic}^0(C)$ for $d \geq \frac{g}{2}$, implies the inequality $\mathrm{Cliff}_\eta(C) \leq \lfloor \frac{g-1}{2} \rfloor$. In analogy with the Clifford index, firstly we verify that:

Proposition. *The function*

$$(C, \eta) \to \mathrm{Cliff}_\eta(C)$$

is lower semicontinuous in families of Prym curves.

Proof. See Proposition 3.15. □

Then, by specialization to hyperelliptic Prym curves (C, η) with maximal $k = \lfloor \frac{g+1}{2} \rfloor$, we obtain the following:

Theorem. *Let $(C, \eta) \in \mathcal{R}_g$ be a general curve. Then,*

$$\mathrm{Cliff}_\eta(C) = \left\lfloor \frac{g-1}{2} \right\rfloor.$$

Proof. See Theorem 3.16. □

Chapter 4 is based on the definition of Prym-exceptional curves, that is, curves C with Prym-canonical Clifford dimension

$$\mathrm{dimCliff}_\eta(C) \gneq (0,0).$$

Since we proved that $\mathrm{dimCliff}_\eta(C) \neq (1,1)$ and $\mathrm{dimCliff}_\eta(C) \neq (0, r')$ with $r' \geq 1$, then C is Prym-exceptional if $\mathrm{dimCliff}_\eta(C) \gneq (1,1)$. We investigate them on a special class of $K3$ surfaces, namely, Nikulin surfaces. They are widely used in the literature in the study of Prym curves (among the works on the topic are [FV12], [FV16], [FV24], [KLCV21], [KLCV20]), and very recently, Farkas and Lelli Chiesa used them to discuss a more refined version ([FLC25, Thm 1.2]) of Theorem [FLC25, Thm 1.1], where they establish a necessary and sufficient condition for the emptiness of the secant varieties associated to a Prym-canonical curve C.
A Nikulin surface is a $K3$ surface S endowed with a nontrivial double cover $\pi\colon \tilde{S} \to S$ with a branch divisor $N := N_1 + \ldots + N_8$ consisting of eight disjoint smooth (-2)-curves on S. They provide a natural candidate for a nontrivial line bundle of torsion 2 on a curve $C \subset S$, that is $\mathcal{O}_C(M)$ where $2M \sim N_1 + \ldots + N_8$, and so $\omega_C \otimes \eta \simeq \mathcal{O}_C(C - M)$.
In the first section, we consider Nikulin surface S of Picard rank 10. In particular, we take S such that $\mathrm{Pic}(S) = \mathcal{N} \oplus_\perp (\mathbb{Z}[D] \oplus \mathbb{Z}(\Gamma))$, where D and Γ are divisors such that $D^2 = 2r - 2$ with $r \geq 2$, $D \cdot \Gamma = 1$, $\Gamma^2 = -2$, and $C \sim 2D + \Gamma$, and $\mathcal{N}$ is the even lattice of rank 8 generated by $N_1, \ldots, N_8$ and M. First of all, we exhibit a new example of exceptional curves with respect to the Clifford index (see Proposition 4.6).

Proposition. *Let S be a Nikulin surface such that $\mathrm{Pic}(S) = \mathcal{N} \oplus_\perp (\mathbb{Z}[D] \oplus \mathbb{Z}(\Gamma))$ with D and Γ as above, and consider the line bundle $L \sim 2D + \Gamma \in \mathrm{Pic}(S)$. Then, the linear system $|L|$ contains exceptional curves in the classical sense.*

To prove it, we apply a result by Knutsen ([Knu09, Prop. 4.1]), in which he relaxed the hypothesis of Theorem [ELMS89, Thm 4.3], and proved that it is enough to show that to there is no line bundle B on S satisfying $0 \leq B^2 \leq D^2 - 1$ and $0 < B.L - B^2 \leq D^2$. Then, we compute that $\mathrm{Cliff}_\eta(\mathcal{O}_C(D)) = 2r - 2$, and we ask wheteher $\mathcal{O}_C(D)$ computes the Prym-canonical Clifford dimension of C, implying that (C, η) is Prym-exceptional. However, because of the surjectivity of the difference map $\phi_d \colon C^d \times C^d \to \mathrm{Pic}^0(C)$ for $d \geq 2r - 1$, this is not the case.

In the second section, we take into account Nikulin surfaces S with Picard rank 9, in particular $\mathrm{Pic}(S) = \mathcal{N} \oplus_\perp \mathbb{Z}[D]$ and construct examples of Prym curves which do not have maximal Prym-canonical Clifford index. Indeed, if $C \sim 2D$, then $g(C) = 4r - 3$, and we compute that $\mathrm{Cliff}_\eta(\mathcal{O}_C(D)) = 2r - 3 < \frac{g-1}{2}$. Once again, the divisor $\mathcal{O}_C(D)$ fails to compute the Prym-canonical Clifford dimension of C.

Chapter 1

The Prym-canonical Clifford index

1.1 Preliminaries

Unless otherwise indicated, (C, η) will be a Prym curve of genus $g \geq 2$, that is, C is a smooth genus g curve over $\mathbb{C}$ and $\eta \in \mathrm{Pic}^0(C)$ is a nontrivial 2-torsion line bundle ($\mathrm{Pic}^0(C)[2] \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2g}$, so an η always exists). Since the trivial bundle $\mathcal{O}_C$ is the only degree zero bundle with one section, by Riemann-Roch $h^0(C, \omega_C \otimes \eta) = g - 1$, so the Prym-canonical bundle $\omega_C \otimes \eta$ defines a morphism $\varphi_\eta \colon C \to \mathbb{P}^{g-2}$, called the Prym-canonical morphism. Below we present some preliminaries.

Lemma 1.1. *([CDGK20, Lem. 2.1]) Let (C, η) be a Prym curve of genus $g \geq 3$. Then:*

i. p is a base point of $|\omega_C \otimes \eta|$ if and only if $|p + \eta| \neq \emptyset$. This happens if and only if C is hyperelliptic and $\eta \sim \mathcal{O}_C(p - q)$, with p and q ramification points of the g^1_2. In particular, p and q are the only base points;

ii. if $|\omega_C \otimes \eta|$ is base point free, then it does not separate p and q (possibly infinitely near) if and only if $|p + q + \eta| \neq \emptyset$. This happens if and only if C has a g^1_4 and $\eta \sim \mathcal{O}_C(p + q - x - y)$, where $2(x + y)$ and $2(p + q)$ are members of the g^1_4. In particular, also x and y are not separated by $|\omega_C \otimes \eta|$.

Recalling that $p \in C$ is a base point for $|\omega_C \otimes \eta|$ if and only if $h^0(C, \omega_C \otimes \eta(-p)) = h^0(C, \omega_C \otimes \eta)$, and that the morphism φ_η does not separate $p, q \in C$ if and only if $h^0(C, \omega_C \otimes \eta(-p - q)) = h^0(C, \omega_C \otimes \eta(-p))$, the statements easily follow from Riemann-Roch and Serre duality.

Corollary 1.2. *([CDGK20, Cor. 2.2]) Suppose the Prym-canonical system is base-point free and C has genus $g \geq 4$. If $\varphi_\eta \colon C \to \mathbb{P}^{g-2}$ is not birational onto its image,* *then it is of degree 2 onto a smooth elliptic curve $E \subseteq \mathbb{P}^{g-2}$ (hence, C is bielliptic). Moreover, η is the pullback of a nontrivial 2-torsion line bundle on E.*

When $g = 2$, the Prym-canonical system $|\omega_C \otimes \eta|$ cannot be base-point free, as φ_η sends C to a point. Analogously, for $g = 3$, φ_η cannot be birational onto its image since C is sent to $\mathbb{P}^1$.

Remark 1.3. By the previous corollary, if φ_η is not birational, then it factors as

$$\begin{array}{ccc} C & \xrightarrow{f} & E \\ {\scriptstyle \varphi_\eta}\downarrow & \swarrow & \\ \mathbb{P}^{g-2} & & \end{array}$$

where f is a degree 2 cover on a smooth elliptic curve E. In particular, every divisor in $|\omega_C \otimes \eta|$ is the pullback of a hyperplane section of $E \subseteq \mathbb{P}^{g-2}$.

The above results appear also in the classical book [CD89, Ch. I, Sec. 6]

1.2 Definition of $\mathrm{Cliff}_\eta(C)$

We define a new Clifford index, that we use to study Prym-canonical curves. Given a line bundle $L \in \mathrm{Pic}(C)$, Riemann-Roch formula yields that its Clifford index can be written as

$$\mathrm{Cliff}(L) = h^0(\omega_C) - h^0(L) - h^0(\omega_C \otimes L^\vee) + 1,$$

so one may think that the Clifford index as being computed with respect to the canonical bundle. We replace ω_C with the Prym-canonical bundle.

Definition 1.4. *Let (C, η) be a Prym curve of genus $g \geq 2$. Given $L \in \mathrm{Pic}(C)$ such that $h^0(L) \geq 1$, $h^0(L \otimes \eta) \geq 1$, the* ***Prym-canonical Clifford index of** L **with respect to** η is*

$$\mathrm{Cliff}_\eta(L) := h^0(\omega_C \otimes \eta) - h^0(L) - h^0(\omega_C \otimes \eta \otimes L^\vee) + 1.$$

Denoting by $r(L)$ the dimension of the linear system $|L|$, the previous definition can be rewritten as

$$\mathrm{Cliff}_\eta(L) = r(\omega_C \otimes \eta) - r(L) - r(\omega_C \otimes \eta \otimes L^\vee).$$

Definition 1.5. *The* ***Prym-canonical Clifford index of*** (C,η) *is*

$$\begin{aligned}\mathrm{Cliff}_\eta(C) &:= \min\{\mathrm{Cliff}_\eta(L)\ |h^i(L)\geq 1, h^i(L\otimes\eta)\geq 1,\ i=0,1 \\ &= \min\{\mathrm{Cliff}_\eta(L)\ |\deg(L)\leq g-1, h^0(L)\geq 1, h^0(L\otimes\eta)\geq 1\}.\end{aligned} \tag{1.1}$$

A line bundle L is said to **contribute to the Prym-canonical Clifford index** *if both $h^i(L)\geq 1$ and $h^i(L\otimes\eta)\geq 1$ for $i=0,1$, and to* **compute the Prym-canonical Clifford index** *if moreover* $\mathrm{Cliff}_\eta(C)=\mathrm{Cliff}_\eta(L)$.

The second equality in (1.1) follows from the fact that $\mathrm{Cliff}_\eta(L)=\mathrm{Cliff}_\eta(\omega_C\otimes\eta\otimes L^\vee)$ and that any effective line bundle of degree $\leq g-1$ is special. Requiring $\deg(L)\leq g-1$ as in the second line of (1.1) thus avoids considering both L and $\omega_C\otimes\eta\otimes L^\vee$.
Also note that, by Riemann-Roch, we obtain

$$\begin{aligned}\mathrm{Cliff}_\eta(L) &= \deg(L)-h^0(L)-h^0(L\otimes\eta)+1 \\ &= \mathrm{Cliff}_\eta(L\otimes\eta) \\ &= \mathrm{Cliff}_\eta(\omega_C\otimes L^\vee).\end{aligned}$$

We recall that in the classical sense, a line bundle L of degree $d\leq g-1$ is said to contribute to the Clifford index of C if $h^0(L)\geq 2$, or equivalently, there exist two effective divisors $D,D'\in|L|$ such that the trivial bundle $\mathcal{O}_C\simeq\mathcal{O}_C(D-D')$. Our conditions $h^0(L)\geq 1$ and $h^0(L\otimes\eta)\geq 1$ in Definition 1.5 are a natural generalization as they are equivalent to the existence of two effective divisors $D\in|L|$ and $D'\in|L\otimes\eta|$ such that $\eta\simeq\mathcal{O}_C(D-D')$.

Remark 1.6. A first upper bound on the Prym-canonical Clifford index of C is a consequence of the surjectivity of the difference map

$$\begin{aligned}\phi_d\colon C_d\times C_d &\to \mathrm{Pic}^0(C) \\ (D,D') &\mapsto \mathcal{O}_C(D-D')\end{aligned}$$

that occurs for all $d\geq\lfloor\frac{g+1}{2}\rfloor$ (cf. [ACGH85, ch. 5]), so that it is always possible to write $\eta=\mathcal{O}_C(D-D')$ with D,D' effective of degree $\lfloor\frac{g+1}{2}\rfloor$. It follows that

$$\mathrm{Cliff}_\eta(C)\leq\mathrm{Cliff}_\eta(\mathcal{O}_C(D))=\left\lfloor\frac{g+1}{2}\right\rfloor-1=\left\lfloor\frac{g-1}{2}\right\rfloor.$$

We now compute some examples, and we firstly consider the cases described in the statement of Lemma 1.1.

Example 1.6.1. Let C be hyperelliptic and $\eta = \mathcal{O}_C(p-q)$ with p, q ramification points of the g^1_2. Setting $L := \mathcal{O}_C(p)$, we have $L \otimes \eta \simeq \mathcal{O}_C(2p-q) \simeq \mathcal{O}_C(q)$ (as $\eta \simeq \eta^\vee$), so that $h^0(L) = h^0(L \otimes \eta) = 1$ and $\mathrm{Cliff}_\eta(L) = 0$.

Example 1.6.2. Assume that the Prym-canonical system $|\omega_C \otimes \eta|$ is base point free but the Prym-canonical morphism is not birational, that is equivalent to $\eta = \mathcal{O}_C(p+q-x-y)$ with $2(p+q) \sim 2(x+y)$. Fix $L := \mathcal{O}_C(p+q)$ so that $L \otimes \eta \simeq \mathcal{O}_C(2p+2q-x-y) \simeq \mathcal{O}_C(x+y)$. We claim that $h^0(L) = 1$. Indeed, if L were a g^1_2, denoting by ι the hyperelliptic involution, we may write $\eta \simeq \mathcal{O}_C(\iota(x)-y)$ and we would get a contradiction because, by Lemma 1.1, $\omega_C \otimes \eta$ would have base points. Analogously, one shows that $h^0(L \otimes \eta) = 1$ and thus $\mathrm{Cliff}_\eta(L) = 1$. Theorem 1.15 will then imply $\mathrm{Cliff}_\eta(C) = 1$.

Example 1.6.3. The previous example covers also the case of C bielliptic (i.e., there exists a double cover $f: C \to E$ to an elliptic curve E) with a 2 torsion line bundle on C of the form $\eta = f^*(\epsilon)$ for some $\epsilon \in \mathrm{Pic}^0(E)[2] \setminus \{\mathcal{O}_E\}$. Indeed, since the Abel-Jacobi map $E \to \mathrm{Pic}^0(E)$ is an isomorphism, we may write $\epsilon \simeq \mathcal{O}_E(x-y)$ for some $x, y \in E$ such that $2x \sim 2y$, and $\eta \simeq \mathcal{O}_C(x_1+x_2-y_1-y_2)$, where x_1, x_2 (respectively, y_1, y_2) are the inverse images under f of x (resp., y) so that $2(x_1+x_2) \sim 2(y_1+y_2)$ holds. Thus, $\mathrm{Cliff}_\eta(C) = 1$.

1.3 Definition of $\mathrm{dimCliff}_\eta(C)$ and some useful results

Since in the definition of the Prym-canonical Clifford index of a line bundle $L \in \mathrm{Pic}(C)$ both $h^0(C, L)$ and $h^0(C, L \otimes \eta)$ appear, one is led naturally to define the Prym-canonical Clifford dimension of C as the minimal couple $(r(L), r(L \otimes \eta))$, where L computes $\mathrm{Cliff}_\eta(C)$.

Definition 1.7. *The **Prym-canonical Clifford dimension of C** is defined as*

$$\mathrm{dimCliff}_\eta(C) := \min\left\{(r, r') \;\middle|\; \exists\, L \in \mathrm{Pic}(C),\ \deg(L) \leq g-1,\ \ r = r(L),\ r' = r(L \otimes \eta),\right.$$
$$\left. \mathrm{Cliff}_\eta(C) = \mathrm{Cliff}_\eta(L)\right\}.$$

where the minimum is taken with respect to the lexicographic order on $\mathbb{Z}^2$.

We say that $L \in \mathrm{Pic}(C)$ **computes the Prym-canonical Clifford dimension** *if $\mathrm{Cliff}_\eta(C) = \mathrm{Cliff}_\eta(L)$ and $\mathrm{dimCliff}_\eta(C) = (r(L), r(L \otimes \eta))$.*

Since the order is lexicographic and $\mathrm{Cliff}_\eta(L) = \mathrm{Cliff}_\eta(L \otimes \eta)$ for any L, it easily follows that $\mathrm{dimCliff}_\eta(C) = (s, s')$ for some integers s, s' so that $0 \leq s \leq s'$.

Lemma 1.8. *For every Prym curve* (C, η) *and any* $r' \geq 1$*, one has*

$$\dim\mathrm{Cliff}_\eta(C) \neq (0, r').$$

Proof. By contradiction, assume $\dim\mathrm{Cliff}_\eta(C) = (0, r')$ for some $r' \geq 1$. Let L be a line bundle computing the Prym-canonical Clifford dimension of C and take $D = q_1 + \ldots + q_d \in |L|$. Since $h^0(L) = 1$, all the q_i are base points of L and, setting $L_1 = L(-q_1)$, we have $h^0(L_1) = 1$ and $h^0(L_1 \otimes \eta) = h^0(L \otimes \eta(-q_1)) \geq r' - 1 \geq 0$. We get

$$\begin{aligned}\mathrm{Cliff}_\eta(L_1) &= \deg(L_1) - h^0(L_1) - h^0(L_1 \otimes \eta) + 1 \\ &\leq \deg(L) - 1 - h^0(L) - h^0(L \otimes \eta) + 1 + 1 \\ &= \mathrm{Cliff}_\eta(L) = \mathrm{Cliff}_\eta(C),\end{aligned}$$

and thus equality holds. In conclusion, we have $\mathrm{Cliff}_\eta(L_1) = \mathrm{Cliff}_\eta(C)$ and $(r(L_1), r(L_1 \otimes \eta)) = (r(L), r(L \otimes \eta) - 1) < (r(L), r(L \otimes \eta))$ in contradiction with the assumption that L computes the Prym-canonical Clifford dimension of C. □

In the classical sense, a line bundle $L \in \mathrm{Pic}(C)$ computing the Clifford index of the curve C is such that both L and $\omega_C \otimes L^\vee$ are base point free. Regarding the Prym-canonical Clifford index, in order to ensure the base-point freeness of both L and $L \otimes \eta$, it is necessary to require that L compute the Prym–Clifford dimension of C. The following holds:

Lemma 1.9. *Let* L *be a line bundle computing the Prym-canonical Clifford index of* (C, η)*. Then,* L *and* $L \otimes \eta$ *have no base points in common.*
If moreover $\dim\mathrm{Cliff}_\eta(C) \geq (1, 1)$ *and* L *computes the Prym-canonical Clifford dimension of* (C, η)*, then both* L *and* $L \otimes \eta$ *are base point free.*

Proof. By contradiction, assume p is a base point of both L and $L \otimes \eta$ and set $L_1 := L(-p)$. Since $h^0(L_1) = h^0(L)$ and $h^0(L_1 \otimes \eta) = h^0(L \otimes \eta)$, the line bundle L_1 contributes to the Prym-canonical Clifford index and one easily computes that $\mathrm{Cliff}_\eta(L_1) = \mathrm{Cliff}_\eta(L) - 1$, in contradiction with the equality $\mathrm{Cliff}_\eta(C) = \mathrm{Cliff}_\eta(L)$.
Assume $\dim\mathrm{Cliff}_\eta(C) = (r(L), r(L \otimes \eta)) \geq (1, 1)$ and p a base point of L. Setting $L_1 = L(-p)$, $h^0(L_1) = h^0(L)$ and $h^0(L_1 \otimes \eta) = h^0(L(-p) \otimes \eta) \geq h^0(L \otimes \eta) - 1 \geq 1$. We get

$$\begin{aligned}\mathrm{Cliff}_\eta(L_1) &= \deg(L_1) - h^0(L_1) - h^0(L_1 \otimes \eta) + 1 \\ &\leq \deg(L) - 1 - h^0(L) - h^0(L \otimes \eta) + 1 + 1 \\ &= \mathrm{Cliff}_\eta(L) = \mathrm{Cliff}_\eta(C),\end{aligned}$$

and thus equality holds. In conclusion, the line bundle L_1 computes the Prym-canonical Clifford index and $(r(L_1), r(L_1 \otimes \eta)) = (r(L), r(L \otimes \eta) - 1) < (r(L), r(L \otimes \eta))$ in contradiction with the assumption that L computes the Prym-canonical Clifford dimension. Analogously, one shows that $L \otimes \eta$ is base-point free. □

The next result expresses a relation between $\mathrm{Cliff}(C)$ and $\mathrm{Cliff}_\eta(C)$, and it will be used in the proofs of Theorems 1.15, 2.19 and 2.21.

Lemma 1.10. *Let L be a line bundle contributing to the Prym-canonical Clifford index of (C, η) such that $\deg L \leq g - 1$, $h^0(L) \geq 2$ and $h^0(L \otimes \eta) \geq 2$. Then,*

$$\begin{aligned} \mathrm{Cliff}_\eta(L) + \mathrm{Cliff}_\eta(L \otimes \eta) &= 2\mathrm{Cliff}_\eta(L) \\ &= \mathrm{Cliff}(L) + \mathrm{Cliff}(L \otimes \eta) - 2. \end{aligned} \tag{1.2}$$

In particular, one gets

$$\mathrm{Cliff}_\eta(L) \geq \mathrm{Cliff}(C) - 1. \tag{1.3}$$

Proof. Both L and $L \otimes \eta$ contribute to the Clifford index of C. The inequality follows directly from the definition. □

We remark that given a general Prym curve (C, η), inequality (4.6) reads like

$$\mathrm{Cliff}_\eta(L) \geq \left\lfloor \frac{g-1}{2} \right\rfloor - 1.$$

Since in the next Lemma we will make use of secant varieties, we recall their definition. Given a line bundle $L \in \mathrm{Pic}^d(C)$ with $h^0(C, L) = r + 1$ and positive integers $0 \leq f < e$, one introduces the variety

$$V_e^{e-f}(L) := \{Z \in C_e \mid h^0(C, L(-Z)) \geq r + 1 - e + f\}$$

of effective divisors Z of degree e that fail to impose f independent conditions on $|L|$. In Section 0.2, it is discussed how they are connected with $\mathrm{Cliff}_\eta(C)$.

Lemma 1.11. *Let L be a line bundle computing the Prym-canonical Clifford dimension of (C, η). Then, for every positive integers $1 \leq f < e \leq \min\{\deg L, r(L) + f\}$ and $1 \leq f' < e \leq \min\{\deg L, r(L \otimes \eta) + f'\}$ such that $f + f' \geq e$, we have*

$$V_e^{e-f}(L) \cap V_e^{e-f'}(L \otimes \eta) = \emptyset.$$

Proof. By contradiction, assume there exists an effective divisor $D \in V_e^{e-f}(L) \cap V_e^{e-f'}(L \otimes \eta)$. Then,

$$\begin{aligned} h^0(L(-D)) &\geq h^0(L) - e + f \geq 1 \\ h^0(L \otimes \eta(-D)) &\geq h^0(L \otimes \eta) - e + f' \geq 1, \end{aligned} \tag{1.4}$$

so $L(-D)$ contributes to the Prym-canonical Clifford index. Since L computes the Prym-canonical Clifford dimension of C, we have $\mathrm{Cliff}_\eta(L) < \mathrm{Cliff}_\eta(L(-D))$, or equivalently,

$$h^0(L) + h^0(L \otimes \eta) > e + h^0(L(-D)) + h^0(L \otimes \eta(-D)),$$

and the inequalities (1.4) would led to the contradiction

$$h^0(L) + h^0(L \otimes \eta) > h^0(L) + h^0(L \otimes \eta) + (f + f' - e).$$

□

The above result can be used to prove the following:

Lemma 1.12. *For every Prym curve (C, η) of genus $g \geq 2$, one has*

$$\mathrm{dimCliff}_\eta(C) \neq (1, 1).$$

Proof. Let L be a line bundle computing the Prym-canonical Clifford dimension of C and, by contradiction, assume $\mathrm{dimCliff}_\eta(C) = (r(L), r(L \otimes \eta)) = (1, 1)$. Both L and $L \otimes \eta$ are base point free by Lemma 1.9 and thus define a morphism

$$f \colon C \xrightarrow{|L|, |L \otimes \eta|} \mathbb{P}^1 \times \mathbb{P}^1.$$

Lemma 1.11 then forces f to be an embedding. Indeed, if this were not the case, we would find a degree 2 divisor D such that $h^0(L(-D)) = h^0(L \otimes \eta(-D)) = 1$, that is, $D \in V_2^1(L) \cap V_2^1(L \otimes \eta)$; this contradicts Lemma 1.11 for $e = 2$ and $f = f' = 1$.

We have thus reduced to exclude the case where f embeds $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ as a curve of bidegree (d, d) with $d = \deg L$. The cohomology of the following short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2 - d, -2 - d) \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, -2) \to \mathcal{O}_C \to 0$$

yields $h^1(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2 - d, -2 - d)) \geq 1$, as $h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, -2)) = 0$ and $h^0(\mathcal{O}_C) = 1$. However, by the Künneth formula, that is,

$$h^i(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)) = \bigoplus_{p+q=i} h^p(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) h^q(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b))$$

one gets that $h^1(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2 - d, -2 - d)) = 3$ if $d = 2$ and $h^1(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2 - d, -2 - d)) = 0$ otherwise. This is a contradiction, as by assumption $g \geq 2$ and thus, by the Riemann–Roch formula, $2 = h^0(L) \lneq d$. □

Lemma 1.11 also has the following corollary.

Corollary 1.13. *Assume that* $\dim\mathrm{Cliff}_\eta(C) = (1,2)$ *and let* L *be a line bundle computing the Prym-canonical Clifford dimension of* (C,η)*. Then, no fiber of* ϕ_L *contains a degree* 3 *divisor lying in the intersection of* $\phi_{L\otimes\eta}(C) \subset \mathbb{P}^2$ *with a line. Furthermore,* $\phi_{L\otimes\eta}$ *does not identify any pair of points in a fiber of* ϕ_L*.*

Proof. Since $r(L\otimes\eta) = 2$, we may apply Lemma 1.11 for $1 \le e - f' \le 2$ and $e - f = 1$. Consider the case $e - f' = 2$. The hypothesis of Lemma 1.11 are satisfied when $f + f' \ge e$, that is, $e \ge 3$, in which case we obtain that $V_e^1(L) \cap V_e^2(L\otimes\eta) = \emptyset$, and the statement follows taking $e = 3$.

We now consider the case where $e - f' = e - f = 1$, so that $f + f' \ge e$ as soon as $e \ge 2$. Lemma 1.11 then yields $V_e^1(L) \cap V_e^1(L\otimes\eta) = \emptyset$ and the last part of the statement follows choosing $e = 2$. □

Remark 1.14. Given a smooth Prym curve (C,η) of genus g, the secant variety associated with $\omega_C\otimes\eta$ encodes $h^0(\mathcal{O}_C(D)\otimes\eta)$ for any $D \in V_e^{e-f}(\omega_C\otimes\eta)$. Indeed, the Riemann-Roch formula yields that the condition

$$D \in V_e^{e-f}(\omega_C\otimes\eta) = \{D \in C_e \mid h^0(C, \omega_C\otimes\eta(-D)) \ge g - 1 - e + f\}$$

is equivalent to $h^0(\mathcal{O}_C(D)\otimes\eta) \ge f$. Therefore, an effective line bundle $L = \mathcal{O}_C(D)$ of degree $e \le g-1$ contributes to the Prym-canonical Clifford index of (C,η) if and only if $D \in V_e^{e-1}(\omega_C\otimes\eta)$. Moreover, one has the following inclusions:

$$V_e^{e-1}(\omega_C\otimes\eta) \supseteq V_e^{e-2}(\omega_C\otimes\eta) \supseteq \ldots \supseteq V_e^1(\omega_C\otimes\eta),$$

and if $D \in V_e^{e-f}(\omega_C\otimes\eta)$, then

$$\begin{aligned} \mathrm{Cliff}_\eta(\mathcal{O}_C(D)) &= e - h^0(\mathcal{O}_C(D)) - h^0(\mathcal{O}_C(D)\otimes\eta) + 1 \\ &= e - h^0(\mathcal{O}_C(D)) - f + 1 \le e - f. \end{aligned} \tag{1.5}$$

In particular, if the Prym-canonical Clifford dimension is $(0,0)$, then $\mathrm{Cliff}_\eta(C) = e_0 - 1$ where e_0 is the minimal e such that $V_e^{e-1}(\omega_C\otimes\eta)$ is nonempty.

1.4 The Prym-Clifford's Theorem and proof

We recall that a useful consequence of the geometric version of Riemann–Roch is that given a hyperelliptic curve C of genus $g \ge 2$, then every special linear series g^r_d on C is of the form $rg^1_2 + p_1 + \ldots + p_{d-2r}$, where all the p_i's are distinct ([ACGH85, p. 41]). In the proof of the following theorem we will also use the fact that given $L \in \mathrm{Pic}(C)$ computing the Clifford index of C when $\mathrm{Cliff}(C) = 1$, then L is either a g^1_3, or a very ample g^2_5 (see Remark 2.4).

Theorem 1.15. *Let (C, η) be a Prym curve of genus $g \geq 2$. Then*

$$\mathrm{Cliff}_\eta(C) \geq 0$$

and equality holds if and only if $\omega_C \otimes \eta$ has some base points, that is, when C is hyperelliptic and $\eta = \mathcal{O}_C(p - q)$ with p, q ramification points of the g^1_2.

Proof. The argument is similar to the proof of the classical Clifford's theorem in [Har77] with appropriate modifications.
Consider $L \in \mathrm{Pic}(C)$ such that $\mathrm{Cliff}_\eta(L) = \mathrm{Cliff}_\eta(C)$. Since $r(L) \geq 0$ and $r(\omega_C \otimes \eta \otimes L^\vee) \geq 0$, the inequality follows from

$$r(L) + r(\omega_C \otimes \eta \otimes L^\vee) \leq r(\omega_C \otimes \eta)$$

(see [Har77, Lemma 5.5, ch. IV]). Now we prove the second part of the statement.

When C is hyperelliptic and η is as in the statement, the equality $\mathrm{Cliff}_\eta(C) = 0$ follows from Example 1.6.1.

Viceversa, let $\mathrm{Cliff}_\eta(C) = 0$ and let L be a line bundle computing the Prym-canonical Clifford dimension of (C, η). If $r(L) = 0$, then we have

$$r(\omega_C \otimes \eta \otimes L^\vee) = r(\omega_C \otimes \eta),$$

that is, the only effective divisor in $|L|$ lies in the base locus of $|\omega_C \otimes \eta|$. Hence, in this case the conclusion follows from Lemma 1.1. We now suppose $r(L) \geq 1$; since L computes the Prym-canonical Clifford dimension of (C, η), then $r(L \otimes \eta) \geq r(L) \geq 1$. Lemma 1.10 then yields

$$\mathrm{Cliff}(L) + \mathrm{Cliff}(L \otimes \eta) = 2.$$

Since $r(L \otimes \eta) \geq r(L)$, then $\mathrm{Cliff}(L) \geq \mathrm{Cliff}(L \otimes \eta)$; we obtain that $0 \leq \mathrm{Cliff}(L \otimes \eta) \leq 1$. If $\mathrm{Cliff}(L \otimes \eta) = 0$ and $\mathrm{Cliff}(L) = 2$, then C is hyperelliptic, $L \otimes \eta$ is a multiple of the g^1_2 and L is a multiple of the g^1_2 plus 2 base points in contradiction with Lemma 1.8.

If $\mathrm{Cliff}(L) = \mathrm{Cliff}(L \otimes \eta) = 1$, then C cannot be hyperelliptic because otherwise L and $L \otimes \eta$ would have base points. Hence, $\mathrm{Cliff}(C) = 1$ and L and $L \otimes \eta$ are either both of type g^1_3 or both very ample g^2_5. The former case contradicts Lemma 1.12. The latter case can be excluded thanks to the unicity of a g^2_d on a smooth plane curve of degree $d \geq 4$ ([ACGH85, p. 56]). □

Given a Prym curve (C, η) of genus 2, the Prym-canonical system $|\omega_C \otimes \eta|$ has base points as it has dimension 0; this agrees with the fact that every 2-torsion line bundle on a genus 2 curve can be written as $\eta = \mathcal{O}_C(p - q)$ where p and q are Weierstrass points. In particular, $\mathrm{Cliff}_\eta(C) = 0$ for every Prym curve (C, η) of genus 2.

1.5 Some remarks on the double cover of C

Let $\pi\colon \tilde{C} \to C$ be the étale double cover induced by η and let ι be the covering involution on $\tilde{C}$. The ι-invariant line bundles on $\tilde{C}$ are parametrized by

$$\begin{aligned}\mathrm{Pic}(\tilde{C})^{\iota} :&= \{\tilde{L} \in \mathrm{Pic}(\tilde{C}) \mid \iota^*\tilde{L} \simeq \tilde{L}\}\\ &= \{\pi^*L \in \mathrm{Pic}(\tilde{C}\ ),\ L \in \mathrm{Pic}(C)\}.\end{aligned}$$

We introduce the following definition.

Definition 1.16. *The* ***ι-invariant Clifford index of*** $\tilde{C}$ *is*

$$\mathrm{Cliff}(\tilde{C})^{\iota} := \min\left\{\frac{\mathrm{Cliff}(\tilde{L})}{2}\ \middle|\ \tilde{L} \in \mathrm{Pic}(\tilde{C})^{\iota},\ \deg(\tilde{L}) \leq g(\tilde{C}) - 1, h^0(\tilde{L}) \geq 2\right\}.$$

We say that a line bundle $\tilde{L} \in \mathrm{Pic}(\tilde{C})$ contributes to the ι-invariant Clifford index of $\tilde{C}$ if it is ι-invariant and contributes to the Clifford index of $\tilde{C}$.
By the Push-Pull Formula, a ι-invariant line bundle $\tilde{L} = \pi^*L$ satisfies

$$h^0(\pi^*L) = h^0(L) + h^0(L \otimes \eta),$$

and thus $\tilde{L} = \pi^*L$ contributes to the ι-invariant Clifford index of $\tilde{C}$ if and only if one of the following occurs:

a. L contributes to the Prym-canonical Clifford index of (C, η);

b. one between L and $L \otimes \eta$ contributes to the Clifford index of C.

We thus obtain the following result relating the Prym-canonical Clifford index of $\tilde{C}$ and the ι-invariant Clifford index of $\tilde{C}$.

Proposition 1.17. *Let (C, η) be a smooth Prym curve. Let $\pi\colon \tilde{C} \to C$ be the étale double cover induced by η and denote by ι the covering involution on $\tilde{C}$. Then, the following holds:*

$$\mathrm{Cliff}(\tilde{C})^{\iota} = \min\{\mathrm{Cliff}_{\eta}(C),\ \mathrm{Cliff}(L) + r(L) \mid L \in \mathrm{Pic}(C), \deg(L) \leq g - 1, h^0(L) \geq 2\}.$$

If moreover C has Clifford dimension 1, then:

$$\mathrm{Cliff}(\tilde{C})^{\iota} = \min\{\mathrm{Cliff}_{\eta}(C),\ \mathrm{Cliff}(C) + 1\}.$$

Proof. Let $\tilde{L} = \pi^* L$ be a line bundle contributing to the ι-invariant Clifford index of $\tilde{C}$. Without loss of generality, we can assume $h^0(L) \geq h^0(L \otimes \eta)$. If L contributes to the Prym-canonical Clifford index of (C, η), or equivalently, $h^0(L \otimes \eta) \neq 0$, one easily computes:

$$\mathrm{Cliff}(\tilde{L}) = 2\mathrm{Cliff}_\eta(L). \tag{1.6}$$

On the other hand, if $h^0(L \otimes \eta) = 0$, then L contributes to the Clifford index of C and

$$\mathrm{Cliff}(\tilde{L}) = 2\deg(L) - 2h^0(L) + 2 = 2\mathrm{Cliff}(L) + 2r(L) \geq 2\mathrm{Cliff}(C) + 2.$$

If moreover C has Clifford dimension 1, the lower bound $\mathrm{Cliff}(C) + 1$ is reached by choosing a pencil computing the gonality, and this finishes the proof. □

Example 1.17.1. Let C be hyperelliptic and $\eta = \mathcal{O}_C(w_1 - w_2)$ (see (4)). Then, by Theorem 3.2, the Prym-canonical Clifford index of C is 0, and $\mathrm{Cliff}(C) + 1 = 1$, so that $\mathrm{Cliff}_\eta(\tilde{C})^\iota = \mathrm{Cliff}_\eta(C) = 0$. Indeed, such $\tilde{C}$ is hyperelliptic as $\pi^*\mathcal{O}_C(w_1) = g^1_2$.

Example 1.17.2. Assume now C hyperelliptic and $\eta = \mathcal{O}_C(w_1 + \ldots + w_k - w_{k+1} - \ldots - w_{2k})$ with $k \geq 3$ (see (4)). Then, by Theorem 3.2, the Prym-canonical Clifford index of C is $\mathrm{Cliff}_\eta(C) \geq 2$, and $\mathrm{Cliff}(C) + 1 = 1$. Hence, $\mathrm{Cliff}_\eta(\tilde{C})^\iota = \mathrm{Cliff}(C) + 1 = 1$, and it is computed by $\tilde{L} = \pi^* g^1_2$.

Corollary 1.18. *Let (C, η) be a smooth Prym curve, and assume that $\omega_C \otimes \eta$ is base-point free. Then,*

$$\mathrm{Cliff}(\tilde{C}) \leq 2\mathrm{Cliff}_\eta(C).$$

Proof. The statement immediately follows by considering in (1.6) a line bundle L computing the Prym-canonical Clifford dimension of C, namely, $\dim\mathrm{Cliff}_\eta(C) = (r(L), r(L \otimes \eta))$. □

We remark that Proposition 1.17 provides an alternative proof of the Prym-canonical Clifford Theorem. Indeed, if $\mathrm{Cliff}_\eta(C) = 0$, then $\mathrm{Cliff}(\tilde{C})^\iota = 0$, which is equivalent to $\tilde{C}$ hyperelliptic with the $g^1_2 = \pi^*\mathcal{O}_C(p)$, for some $p \in C$ such that $\mathcal{O}_C(p) \otimes \eta$ is effective. It follows that $\eta = \mathcal{O}_C(p - q)$.

Chapter 2

Classification of curves with low Prym-canonical Clifford index

Theorem 1.15 characterizes curves with Prym-canonical Clifford index 0. In this chapter, we discuss the case of bielliptic curves, and we classify curves $(C, \eta) \in \mathcal{R}_g$ such that $1 \leq \mathrm{Cliff}_\eta(C) \leq 2$. Firstly, we present some preliminary results concerning the classification of curves via the Clifford index, as well as a description of linear series on trigonal, tetragonal, and 5-gonal curves. We will make use of them for the proofs of the main results of this Chapter, namely, Theorems 2.19 and 2.21.

2.1 Linear series on curves of gonality $3, 4, 5$

In [CM91], Coppens and Martens use secant varieties to prove the following result (we recall that a curve $C \subset \mathbb{P}^r$ is said to be *non-degenerate* if it is not contained in any hyperplane, and when C is smooth, it is *linearly normal* if the hypersurfaces of degree one cut out the complete linear series $|\mathcal{O}_C(1)|$):

Theorem 2.1. *Any reduced irreducible non-degenerate and linearly normal curve C of degree $d \geq 4r - 7$ in $\mathbb{P}^r$ with $r \geq 2$, has a $(2r-3)$-secant $(r-2)$-plane.*

Besides applying Theorem 2.1 to obtain that an exceptional curve in the classical sense of gonality k has Clifford index $k-3$, and to improve Clifford's theorem ([CM91, Thm. B]), they also use it to determine the maximal degree of all linear series of degree $d \leq g-1$ on C which compute the Clifford index of a curve.

Theorem 2.2. *[CM91, Thm C] Let C be a curve of genus g and Clifford index γ. Any g^r_d on C with $d \leq g-1$ computing γ has degree $d \leq 2(\gamma+2)$ unless C is hyperelliptic or bielliptic.*

They also derive the following corollary:

Corollary 2.3. *[CM91, Cor. 3.2.5] Let C be a curve of genus $g > 2\mathrm{Cliff}(C) + 4$ (respectively, $g > 2\mathrm{Cliff}(C) + 5$) if $\mathrm{Cliff}(C)$ is odd (resp., even). If there is a g^r_d with $d \leq g - 1$ computing the Clifford index of C, then $d \leq 3(\mathrm{Cliff}(C) + 2)/2$ unless C is hyperelliptic or bielliptic.*

Hyperelliptic curves are studied in Chapter 3, while the case of bielliptic curves is covered by Corollary 2.13.

Remark 2.4. Let L be a line bundle of degree $d \leq g - 1$ computing the Clifford index of C. The above results yield that, if $\mathrm{Cliff}(C) = 1$, then L is either a g^1_3 or a g^2_5; since the arithmetic genus of a plane quintic is 6, our requirement on d implies $g = 6$ in the latter case (that is, the g^2_5 is automatically very ample).
Analogously, if $\mathrm{Cliff}(C) = 2$, then L is of type either g^1_4, or g^2_6, or g^3_8, and the latter case may only occur for $g = 9$. If we further require that L computes the Clifford dimension of C, we can exclude that L is a g^3_8 because curves of Clifford dimension 3 are complete intersections of two cubics in $\mathbb{P}^3$ and have genus 10 (cf. [ELMS89]). We recover the well-known fact that, if $\mathrm{Cliff}(C) = 2$, then C is either tetragonal or a smooth plane sextic.
Finally, if $\mathrm{Cliff}(C) = 3$, then any line bundle L of degree $d \leq g - 1$ computing the Clifford index is forced to be either a g^1_5, or a g^2_7, or, only when $g = 10$, a g^3_9. If we further require that L computes the Clifford dimension of C, we recover that, if $\mathrm{Cliff}(C) = 3$, then either C is 5-gonal, or $g = 15$ and C is a smooth plane septic, or $g = 10$ and C is a complete intersection of two cubics in $\mathbb{P}^3$.

We now consider trigonal curves, i.e., nonhyperelliptic curves admitting a degree 3 cover of $\mathbb{P}^1$, and we recall the definition of the so-called Maroni invariant. It occurs already in classical texts ([Chr78, p. 261, 291], [HL02, sec. 4-5]), and it was studied by Maroni in [Mar46]. If $\pi\colon C \to \mathbb{P}^1$ is the 3−sheeted covering induced by the g^1_3, then $\pi_*\omega_C = \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \mathcal{O}_{\mathbb{P}^1}(e_2) \oplus \mathcal{O}_{\mathbb{P}^1}(e_3)$ for some integers $e_1 \geq e_2 \geq e_3$, and $H^0(C, \omega_C \otimes kg^1_3) \simeq H^0(\mathbb{P}^1, \pi_*\omega_C \otimes \mathcal{O}_{\mathbb{P}^1}(k)) \simeq \bigoplus_{i=1}^3 H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(e_i + k))$ (see [MS86, p.172]). Thus, the integer e_1 is the maximum r such that $\omega_C \otimes (rg^1_3)^\vee$ is effective. The Maroni invariant of C can be defined as the integer

$$m := g - 2 - e_1 \tag{2.1}$$

(see [MS86, 1.1, p. 172]). It satisfies the following (cf. [MS86, (1.1)] and [BC81, p.96]):

$$\frac{g-4}{3} \leq m \leq \frac{g-2}{2}, \quad m - g \equiv 0 \bmod 2. \tag{2.2}$$

Geometrically, Maroni introduced the invariant m as follows. Given a trigonal curve C of genus g, one can consider its canonical model in $\mathbb{P}^{g-1}$; and if $p_1+p_2+p_3$ is a divisor of the g^1_3, then p_1, p_2, p_3 lie on the same line in $\mathbb{P}^{g-1}$. The union of all these lines is a smooth rational normal scroll S of minimal degree $g-2$. By construction, S is the Hirzebruch surface $\Sigma_e \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(m))$, where $e=a-m$ (we recall that a Hirzebruch surface is a geometrically ruled surface over $\mathbb{P}^1$, and it is isomorphic to $\mathbb{F}_n := \mathbb{P}(\mathcal{O}\oplus\mathcal{O}(n))$ for some positive n). The Picard group of S is generated by two curves C_0, R, where R is a fiber of the projection map $\pi\colon S\to\mathbb{P}^1$, while C_0 is a minimal section of S, and they satisfy

$$C_0^2=-e, \quad C_0\cdot R=1, \quad R^2=0.$$

Since $H\sim C_0+aR$ is a hyperplane section of S, one may also view the Maroni invariant m as $m=a-e=C_0\cdot H=\deg(C_0)$ (this construction can be found in [Sch86]).
As concerns the Brill–Noether theory of a trigonal curve C, one may define the varieties

$$U^r_d := \left\{ g^r_d \;\middle|\; \exists\, D_{d-3r}\in C_{d-3r} \text{ such that } g^r_d(-D_{d-3r}) = r g^1_3 \right\}$$

if $d-3r\geq 0$,

$$U^r_d := \emptyset$$

otherwise, and

$$V^r_d := \left\{ l\in G^r_d(C) \;\middle|\; \exists\, D_{2d-g-3r+1}\in C_{2d-g-3r+1} \text{ such that } l=\omega_C-(g^1_3+D_{2d-g-3r+1}) \right\}$$

if $2d-g-3r+1\geq 0$,

$$V^r_d := \emptyset$$

otherwise. One easily verify that $\dim U^r_d \geq \dim V^r_d$ for $d\leq g-1$, with the equality occurring when $d=g-1$. Also, $U^r_d\cup V^r_d\subseteq W^r_d=\{L\in \mathrm{Pic}^n(C)\mid h^0(C,L)\geq r+1\}$.
Another result due to Maroni concerning the varieties defined above is

Proposition 2.5. *[MS86, Prop. 1, p.173] For $d<g$ and $r\geq 1$ we have:*

1. *$W^r_d(C)=U^r_d\cup V^r_d$;*
2. *if $U^r_d\neq\emptyset$, then it is an irreducible component of $W^r_d(C)$;*
3. *if $V^r_d\neq\emptyset$, then also $U^r_d\neq\emptyset$, and V^r_d is an irreducible component of W^r_d different from U^r_d if and only if $g-d+r-1\leq m$, where m is the Maroni invariant.*

The above result yields:

Corollary 2.6. *[MS86, Cor. 2, p. 175] Let C be a smooth irreducible projective trigonal curve of genus g and L a line bundle on C of degree $0 \leq d < g$. Then $h^0(L) \leq \frac{d}{3}+1$, and equality holds if and only if either $L = \frac{d}{3}g^1_3$, or $d = g-1$ and $L = \omega_C - (\frac{d}{3}g^1_3)$. Furthermore, in the latter case we have $\omega_C - (\frac{d}{3}g^1_3) = \frac{d}{3}g^1_3$ if and only if the Maroni invariant m is minimal (i.e., $m = \frac{g-4}{3}$).*

Corollary 2.7. *[MS86, Cor.3, p.175] Assume that C is a trigonal curve, and let L be a globally generated line bundle of degree $d > 3$ on C with $h^0(L) = 2$. Then, $L^{\otimes 2}$ is nonspecial.*

As regards linear series on a tetragonal curve C, that is, a curve admitting a morphism $C \to \mathbb{P}^1$ of minimal degree 4, once a g^1_4 is fixed, Coppens and Martens described linear series on C with respect to this fixed g^1_4 as follows.

Corollary 2.8. *[CM00, Cor. 1.10] Let C be a tetragonal curve. Fix a g^1_4 on C. Then any base-point free g^r_d on C such that both $r \geq 1$ and $r' := g-d+r-1 \geq 1$ is, with respect to the fixed g^1_4, of one of the following types:*

1. *$g^r_d = rg^1_4$;*
2. *the residual of the g^r_d is $r'g^1_4$ plus an effective divisor F;*
3. *$g^r_d = (r-1)g^1_4 + F$, for some effective divisor F;*
4. *$g^r_d = l + kg^1_4$, with l of type (3) and $\dim l = r - 2k \geq 1$.*

On the other hand, if the gonality of the curve C is 5, then

Lemma 2.9. *[Par02, Lemma 1] Let C be a $5-$gonal curve. Fix a g^1_5. For any base point free g^r_d on C with both $r \geq 1$ and $r' := r-d+g-1 \geq 1$, denoting by B the base locus of $|\omega_C - g^r_d|$, one of the following holds with respect to the fixed g^1_5:*

1. *$g^r_d = rg^1_5$;*
2. *$g^r_d = (r-1)g^1_5 + E$ for some effective divisor E with $\dim(r-2)g^1_5 + E = r-2$;*
3. *$g^r_d - g^1_5$ has dimension $r-2 \geq 1$ and moreover it is non-trivial (that is, it is base point free with both $h^0 \geq 2$ and $h^1 \geq 2$);*
4. *$\omega_C - g^r_d - B = r'g^1_5$;*
5. *$\omega_C - g^r_d - B = (r'-1)g^1_5 + E$ for some effective divisor E with $\dim(r'-2)g^1_5 + E = r'-2$;*

6. $\omega_C - g^r_d - B - g^1_5$ has dimension $r' - 2 \geq 1$ and moreover it is non-trivial.

We will also make use of the following.
For fixed integers $d > r > 1$, we denote by $\chi(d,r)$ the unique positive integer satisfyng

$$\frac{d-1}{r-1} \leq \chi(d,r) < \frac{d-1}{r-1},$$

and by $\pi(d,r)$ the integer such that

$$\pi(d,r) := \chi(d,r)\Big(\frac{\chi(d,r)-1}{2}(r-1)+\epsilon\Big), \tag{2.3}$$

where $\epsilon = d - 1 - \chi(d,r)(r-1)$ with $1 \leq \epsilon \leq r-1$. If a genus g curve C admits a simple g^r_d (that is, it defines a birational morphism onto its image), then $g \leq \pi(d,r)$. When the upper bound in (2.3) is achieved, if the g^r_d is complete and very ample, then $C \subset \mathbb{P}^r$ is said to be extremal.

Theorem 2.10. *[LM12, Theorem 2.3] Consider two linear series on C of type g^r_d and g^s_m such that the g^r_d is base-point free and simple, and*

$$m \leq kd, \quad s \geq \binom{k+1}{2}(r-1)+k$$

for some integer k, $1 \leq k < \chi(d,r)$. Then,

$$g^s_m = kg^r_d \ \text{ or } \ g \leq \pi(d,r) - \chi(d,r) + k.$$

2.2 Clifford dimension of bielliptic curves

We recall the following definition:

Definition 2.11. *Given $f\colon C \to C'$ a non-trivial covering of smooth curves, a linear series g^r_d on C is* ***induced*** *by C' if $g^r_d = f^*(g^r_{\frac{d}{n}})$, for a $g^r_{\frac{d}{n}}$ on C', where $n = \deg f \geq 2$.*

Coppens, Keem and Martens proved that

Corollary 2.12. *[CKM92, Cor. 2.2.2] Let C be a double covering of a curve C' of genus g'. Then, a base-point free linear series g^r_d on C of degree $d \leq g-1$ is induced by C' if $d \leq g - 2g'$ or if $r \geq 2g'$.*

When C is bielliptic, the above result implies that any base point free g^r_d on C with $d \leq g-2$ or $r \geq 2$ is induced by the elliptic curve E covered by C, that is, is the pullback of a g^{f-1}_f on E with $f = \frac{d}{2}$ (we remark that by the Riemann-Roch formula, every degree m line bundle M on an elliptic curve is such that $h^0(M) = m$). As a consequence, we prove the following:

Theorem 2.13. *If C is a bielliptic curve, then* $\dim\mathrm{Cliff}_\eta(C) = (0,0)$.

Proof. We denote by $\pi\colon C \to E$ the degree 2 map to an elliptic curve E.
Assume that $\dim\mathrm{Cliff}_\eta(C) \neq (0,0)$ and let $L \in \mathrm{Pic}^d(C)$ be a line bundle of degree d that computes the Prym-canonical Clifford dimension of C. Lemma 1.9 implies that both L and $L \otimes \eta$ are base point free and, if either $d \leq g-2$, or $d = g-1$ and $r(L) \geq 2$, Corollary 2.12 yields $L = \pi^* l$ and $L \otimes \eta = \pi^* l'$, where l, l' are complete g^{f-1}_f on E with $f = \frac{d}{2}$. Hence, $\dim\mathrm{Cliff}_\eta(C) = (f-1, f-1)$ and $\mathrm{Cliff}_\eta(C) = \mathrm{Cliff}_\eta(L) = 2f - f - f + 1 = 1$. To get a contradiction, we remark that if $f \geq 2$, one can consider an effective divisor D_k of degree $k < f$ on E, and take $m := l(-D_k)$, $m' := l'(-D_k)$, so that m and m' are complete g^{f-k-1}_{f-k} on E. We get $\eta = \pi^*(m - m')$ and $\mathrm{Cliff}_\eta(\pi^* m) = 1 = \mathrm{Cliff}_\eta(C)$, in contradiction with the assumption that L computes the Prym-canonical Clifford dimension. We conclude that $d = g-1$ and $r(L) = 1$; by Lemma 1.12, $r(L \otimes \eta) \geq 2$, so that $L \otimes \eta = \pi^* l$, where l is a complete g^{f-1}_f on E with $f = (g-1)/2 \geq 3$.
However, the equality $L^{\otimes 2} = \pi^* l^{\otimes 2}$ yields a contradiction, as well. Indeed, it implies that for every divisor $D = p_1 + \cdots + p_{g-1} \in |L|$ the divisor $2D$ is invariant under the covering involution ι, and so it is the sum of $g-1$ fibers of π. Since L is not a pullback, then D is not invariant under ι and thus at least one of the points p_i is a ramification point of π; moving $D \in |L|$, we get infinitely many ramification points which is clearly a contradiction. □

2.3 Singular plane curves

At some point, we deal with all the singularities that may arise on the curve, for this reason we recall the following definitions (we refer to [Har77] and [EH24]). Let $\bar{C}$ be an integral projective scheme of dimension 1, and $\nu\colon C \to \bar{C}$ its normalization. For any $p \in \bar{C}$, we denote by $\delta_p := \mathrm{length}(\tilde{\mathcal{O}}_p/\mathcal{O}_p)$ the δ invariant of the singularity, where $\tilde{\mathcal{O}}_p$ is the integral closure of $\mathcal{O}_p$. One proves that $p_a(\bar{C}) = p_a(C) + \sum_{p \in C} \delta_p$ ([Har77, ch. IV, ex. 1.8, p. 298]). By [Har77, ch. V, ex. 3.9.3] the invariant δ_p can be computed also as $\delta_p = \sum \frac{1}{2} r_q(r_q - 1)$ taken over all infinitely near singular points q lying over p, including p, and where r_q is the multiplicity of q. Moreover,

Corollary 2.14. *[EH24, Cor. 15.11, Ch. 15.1] The δ invariant of any singularity of a plane curve $\bar{C}$ at a point p can be computed as the sum of the numbers $\binom{m_q}{2}$ over all* *infinitely near singular points q, where m_q denotes the multiplicity of the pullback of $\bar{C}$ at q.*

Proposition 2.15. *[EH24, Prop. 2.25, Ch. 2.3] Suppose that $\bar{C}$ is a reduced, irreducible curve and let $\nu\colon C \to \bar{C}$ be its normalization. Let $\mathcal{F} = \nu_*\mathcal{O}_C/\mathcal{O}_{\bar{C}}$. If we set $\delta(\bar{C}) := h^0(\mathcal{F})$, then*

$$p_a(\bar{C}) - p_a(C) = \delta(\bar{C}).$$

It follows that $\delta(\bar{C}) = \sum_{p\in\bar{C}} \delta_p$. Roughly speaking, the invariant $\delta(\bar{C})$ is the number of linear conditions a locally defined function f on C has to satisfy to be the pullback of a function from $\bar{C}$. For instance, if $p \in \bar{C}$ is a node, its preimage in the normalization C consists of two points $q_1, q_2 \in C$. The condition for a function f on C to be the pullback of a function on $\bar{C}$ is that $f(q_1) = f(q_2)$, that is one linear condition, and so $\delta_p = 1$. If $p \in \bar{C}$ is a cusp, then $\nu^*(\mathcal{O}_{\bar{C}}(p))$ consists of one point $q \in C$. The condition to impose to a function f on C is that the derivative $f'(q) = 0$, that is again one condition, and so $\delta_p = 1$.
We emphasize that throughout the proofs of Theorems 2.19 and 2.21, we will deal with plane curves, which are Gorenstein. On the curve $\bar{C}$, the adjoint scheme consists of the singularities, and for plane curves it is the scheme defined by the conductor ideal. We give these definitions in the next result:

Theorem 2.16. *[EH24, Thm 16.17, Ch. 16.8] If $\nu\colon C \to \bar{C}$ is the normalization of a reduced connected projective curve, then the adjoint ideal*

$$\mathcal{U}_{\bar{C}} := \operatorname{ann}_{\mathcal{O}_{\bar{C}}} \frac{\omega_{\bar{C}}}{\nu_*\omega_C}$$

is equal to the conductor ideal

$$\mathcal{I}_{\bar{C}} := \operatorname{ann}_{\mathcal{O}_{\bar{C}}} \frac{\nu_*\mathcal{O}_C}{\mathcal{O}_{\bar{C}}}.$$

Moreover, if $\bar{C}$ is Gorenstein, then

$$\delta(\bar{C}) = \operatorname{length}\frac{\nu_*\mathcal{O}_C}{\mathcal{O}_{\bar{C}}} = \operatorname{length}\frac{\mathcal{O}_{\bar{C}}}{\mathcal{I}_{\bar{C}}}.$$

The conductor ideal is an ideal of both $\bar{C}$ and C (see proof of Theorem [EH24, Thm 15.15, Ch. 15.2] for more details), and it defines a conductor subscheme Δ on $\bar{C}$ of length $\delta(C)$, and another subscheme on C of length $2\delta(C)$.

Theorem 2.17. *[EH24, Thm 15.12, Ch. 15.2] If D is an effective divisor on C that contains Δ, then D is the pullback of a Cartier divisor on $\bar{C}$.*

2.4 Curves with $\mathrm{Cliff}_\eta(C) = 1$

The strategy of the proof of Theorem 2.19 is to assume that $\dim\mathrm{Cliff}_\eta(C) \geq (1,2)$ so that the equality in the statement of Lemma 1.10 holds, and to exclude all possibilities for C arising from it. It will follow that $\dim\mathrm{Cliff}_\eta(C) = (0,0)$, hence the assertion of Theorem 2.19. The proof of Theorem 2.19 relies on what follows.
We recall that a set $Z = \{p_1, \ldots, p_d\}$ of distinct points in $\mathbb{P}^2$ is said to impose independent conditions on curves of degree n if $h^0(\mathbb{P}^2, \mathcal{I}_Z(n)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n)) - d$, where $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^2}$ is the ideal sheaf of the zero-dimensional variety Z.

Corollary 2.18. *[Har77, Ch. V, Cor. 4.4] Let $\mathcal{D}$ be the linear series of plane cubics with assigned base points $p_1, \ldots, p_r$, and assume that no 4 of the p_i are collinear, and no 7 of them lie on a conic. Then we have*

1. *if $r \leq 8$, than $\dim\mathcal{D} = 9 - r$, and*
2. *if $r = 8$, $\dim\mathcal{D} = 1$ and almost every curve in $\mathcal{D}$ is irreducible.*

We are now ready to prove the following theorem:

Theorem 2.19. *Let (C, η) be a smooth Prym curve of genus $g \geq 3$. Then, the equality $\mathrm{Cliff}_\eta(C) = 1$ holds if and only if $\omega_C \otimes \eta$ is base point free but not very ample, that is, C has a g^1_4 and $\eta = \mathcal{O}_C(p + q - x - y)$, with $2(p+q) \sim 2(x+y)$.*

Proof. If $\omega_C \otimes \eta$ is not very ample, then $\mathrm{Cliff}_\eta(C) = 1$ by Example 1.6.2.

Viceversa, assume $\mathrm{Cliff}_\eta(C) = 1$ and let L be a line bundle computing the Prym-canonical Clifford dimension of (C, η) with $\deg(L) \leq g - 1$.

If $\dim\mathrm{Cliff}_\eta(C) = (0,0)$, that is, $r(L) = r(L \otimes \eta) = 0$, then $\deg(L) = 2$ and $L = \mathcal{O}_C(D)$, where D is the only effective divisor in $|L|$. The Riemann-Roch Theorem yields
$$h^0(\omega_C \otimes \eta(-D)) = h^0(L \otimes \eta) - 2 + g - 1 = g - 2,$$
that is, the divisor D fails to impose independent condition to the image of C under the Prym-canonical morphism; equivalently, φ_η identifies the points in D. In particular, $\omega_C \otimes \eta$ is not very ample and the statement follows from Lemma 1.1.

Therefore, we want to rule out the cases with $r(L \otimes \eta) \geq r(L) \geq 1$, where Lemma 1.10 yields
$$\mathrm{Cliff}(L) + \mathrm{Cliff}(L \otimes \eta) = 4,$$
and thus $0 \leq \mathrm{Cliff}(L \otimes \eta) \leq 2$. In particular, one has $0 \leq \mathrm{Cliff}(C) \leq 2$ and, recalling that on a hyperelliptic curve of genus $g \geq 2$ any special linear series g^r_d is a sum of r copies of the g^1_2 plus $d - 2r$ base points ([ACGH85, p. 41]), the case $\mathrm{Cliff}(C) = 0$ can be

excluded because it would force L to have base points, in contradiction with Lemma 1.9. Therefore, from now on we will assume $1 \leq \mathrm{Cliff}(C) \leq 2$. Also note that, by Theorem 2.13, the inequality $r(L \otimes \eta) \geq 1$ implies that C is not bielliptic.

CASE A: $\mathrm{Cliff}(C) = \mathrm{Cliff}(L \otimes \eta) = 1$ and $\mathrm{Cliff}(L) = 3$.

By Remark 2.4, $L \otimes \eta$ is either a g^1_3, or a very ample g^2_5. The former case can be excluded because the inequalities $h^0(L) \geq 2$ and $h^0(L \otimes \eta) \geq 2$ imply

$$1 = \mathrm{Cliff}_\eta(L) = \deg(L) - h^0(L) - h^0(L \otimes \eta) + 1 \leq \deg(L) - 3. \tag{2.4}$$

If $L \otimes \eta$ is a very ample g^2_5, then C has genus 6 and the equality $\mathrm{Cliff}_\eta(L) = 1$ reads as $2 = h^0(L)$ and so $h^0(\omega_C \otimes L^\vee) = 2$ by Riemann-Roch. Take $D = p_1 + \ldots + p_5 \in |\omega_C \otimes L^\vee|$; since $h^0(\omega_C(-D)) = 2$, the points $p_1, \ldots, p_5$ impose one condition less than expected on the canonical linear system $|\omega_C|$. Recalling that $|\omega_C|$ is cut out by plane conics, there is a one-dimensional family $\mathcal{C} = |I_{p_1+\ldots+p_5/\mathbb{P}^2}(2)|$ of conics passing through the five points $p_1, \ldots, p_5$ and any $Q \in \mathcal{C}$ is the union of two lines r and r'. Since $\mathcal{C}$ is 1-dimensional, one of the two lines (let's say r') moves and thus contains at most one of the p_i. We first exclude that four points, for example $p_1, \ldots, p_4$, lie in r and $p_5 \in r'$ by showing that in this case $|L|$ would have a base point in contradiction with Lemma 1.9. Indeed, any conic $Q \in \mathcal{C}$ would be the union of a fixed r and a varying r' though p_5. Any such Q would cut out on the curve 5 more points $q_1, \ldots, q_5$ and, up to reordering, we have that q_1 is the other intersection point of r with C and is thus fixed, while $q_2, \ldots, q_5 \in r'$ vary while varying $Q \in \mathcal{C}$ and thus define a g^1_4. Hence, the linear system $|L| = |\omega_C - p_1 - \ldots - p_5|$ would have a base point at q_1. It remains to treat the case where $p_1 \ldots, p_5$ lie on the same line, but this would lead to the contradiction $L \simeq \mathcal{O}_C(1) \simeq L \otimes \eta$.

CASE B: $\mathrm{Cliff}(C) = 1$ and $\mathrm{Cliff}(L) = \mathrm{Cliff}(L \otimes \eta) = 2$.

We first remark that C cannot be a smooth plane quintic, because otherwise it would have genus $g = 6$ and so any linear series of degree $\leq g - 1$ with Clifford index 2 would be a g^1_4; however, it is not possible that both L and $L \otimes \eta$ are g^1_4 by Lemma 1.12. Therefore, the curve C must be trigonal. By Corollary 2.6, any line bundle of Clifford index 2 on a trigonal curve is either a g^1_4, or a g^2_6. We exclude the former case as it would imply $\mathrm{dimCliff}_\eta(C) = (1, 1)$, in contradiction with Lemma 1.12. In the latter case, Corollary 2.6 implies that the only g^2_6 on C are the linear series $2g^1_3$ and, if $g = 7$, the residual to $2g^1_3$. Since L and $L \otimes \eta$ are distinct as η is non-trivial, we conclude that $g = 7$ and $L = 2g^1_3$ and $L \otimes \eta = \omega_C - 2g^1_3$, or viceversa. However, this yields the contradiction $\omega_C \otimes \eta = 4g^1_3 = \omega_C$, where the second equality again follows from Corollary 2.6 and the fact that the Maroni invariant m of a trigonal genus 7 curve is 1 by (2.2).

CASE C: $\mathrm{Cliff}(C) = \mathrm{Cliff}(L \otimes \eta) = \mathrm{Cliff}(L) = 2$.

The line bundles L and $L \otimes \eta$ define either two g^1_4, or two g^2_6, or two g^3_8 and the genus is 9. The former case cannot occur by Lemma 1.12. In the latter case both the g^3_8 are simple (that is, they define a birational morphism onto the image) as C is neither bielliptic nor hyperelliptic; therefore, we can apply [LM12, Thm 2.3] again reaching the contradiction $\eta \sim \mathcal{O}_C$. Hence, L and $L \otimes \eta$ are g^2_6 and they are simple (as C is not hyperelliptic, bielliptic or trigonal) but not very ample because of the unicity of a g^2_d on a smooth plane curve of degree $d \geq 4$ ([ACGH85, p. 56]). Hence, we can assume that $L \otimes \eta$ is a g^2_6 defining a morphism $f\colon C \to \bar{C} \subseteq \mathbb{P}^2$ that is generically injective but not embedding. Since plane sextics have arithmetic genus 10 and $g \geq d + 1 = 7$, the δ-invariant of $\bar{C}$ satisfies $1 \leq \delta(\bar{C}) \leq 3$. We denote by $A_{\bar{C}}$ the conductor ideal of the normalization $\nu : C \to \bar{C}$ and by $E_{\bar{C}} \subset \bar{C}$ the conductor subscheme, which is defined by $A_{\bar{C}}$ and has length equal to $\delta(\bar{C})$. Since $A_{\bar{C}}$ is also an ideal sheaf of $\nu_*\mathcal{O}_C$, there exists an effective divisor Δ_C on C such that $\deg \Delta_C = 2\delta(C)$ and $A_{\bar{C}} = \nu_*\mathcal{O}_C(-\Delta_C)$. The isomorphism

$$\omega_C \simeq \nu^*\omega_{\bar{C}} \otimes \mathcal{O}_C(-\Delta_C), \tag{2.5}$$

along with the fact that the dualizing sheaf $\omega_{\bar{C}}$ is cut out by cubics by adjunction, yields that the sections of ω_C are cut out by cubics through $E_{\bar{C}}$. We now fix a general element $D = p_1 + \ldots + p_6 \in |L|$; the Riemann-Roch Theorem yields $h^0(\omega_C - p_1 - \ldots - p_6) = 6 - \delta(\bar{C})$. We set $\bar{D} := E_{\bar{C}} + p_1 + \ldots + p_6$, which is a divisor on $\bar{C}$ consisting of $6 + \delta(\bar{C})$ possibly infinitely near points of $\mathbb{P}^2$. Then, there exists a family $\mathcal{D}_{6+\delta(\bar{C})}$ of plane cubics passing through $\bar{D}$ of dimension

$$\dim \mathcal{D}_{6+\delta(\bar{C})} = 5 - \delta(\bar{C}). \tag{2.6}$$

This excludes the case $\delta(\bar{C}) = 3$ because there exists at most a pencil of plane cubics through a 0-dimensional subscheme $\xi \subset \mathbb{P}^2$ of length 9. Indeed, let X_1, X_2 be plane cubics through such a ξ. The short exact sequence

$$0 \to \mathcal{I}_{X_1/\mathbb{P}^2}(3) \to \mathcal{I}_{\xi/\mathbb{P}^2}(3) \to \mathcal{I}_{\xi/X_1}(3) \to 0$$

reads as

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{I}_{\xi/\mathbb{P}^2}(3) \to \mathcal{O}_{X_1} \to 0,$$

hence $h^0(\mathcal{I}_{\xi/\mathbb{P}^2}(3)) = 2$.

Therefore, we must have $\delta(\bar{C}) \in \{1, 2\}$ and, by (2.6) and [Har77, Cor. 4.4, ch. V], the divisor $\bar{D} = E_{\bar{C}} + p_1 + \ldots + p_6$ splits as a sum $\bar{D} = \bar{D}_1 + \bar{D}_2$ of two generalized effective divisors $\bar{D}_1, \bar{D}_2 \subset \bar{C}$ such that either $\bar{D}_1$ is contained in a line and has degree ≥ 4, or $\bar{D}_2$ lies on a conic Q and has degree ≥ 7. The latter case can be excluded because such a conic Q would be fixed and thus the family $\mathcal{D}_{6+\delta(\bar{C})}$ would contain only reducible curves of the form $Q \cap l_t$ where l_t is a line; in particular, $\mathcal{D}_{6+\delta(\bar{C})}$ would have dimension ≤ 2 in contradiction with (2.6). Hence, $\bar{D}_1$ is contained in a fixed line l and has degree $4 \leq d_1 \leq 6$ (where the right hand inequality follows from the fact that $\bar{C}$ is a sextic), and every cubic in the family $\mathcal{D}_{6+\delta(\bar{C})}$ is the union of l with a conic $Q_t \in |I_{\bar{D}_2/\mathbb{P}^2}(2)|$. Note that $\bar{D}_2$ has degree

$$\delta(\bar{C}) \leq d_2 \leq 2 + \delta(\bar{C}) \tag{2.7}$$

and (2.6) can be rewritten as

$$\dim|I_{\bar{D}_2/\mathbb{P}^2}(2)| = 5 - \delta(\bar{C}). \tag{2.8}$$

If $\delta(\bar{C}) = 1$, equations (2.7) and (2.8) yield $d_2 = 1$, $d_1 = 6$ and $D_2 = E_{\bar{C}}$ because $\bar{C}$ is singular at $E_{\bar{C}}$ and otherwise the intersection between $\bar{C}$ and l would be ≥ 7, which is absurd. Analogously, if $\delta(\bar{C}) = 2$, equations (2.7) and (2.8) yield $d_2 = 2$, $d_1 = 6$ and thus again $D_2 = E_{\bar{C}}$. It then follows that $\omega_C(-D) \simeq \nu^*\mathcal{O}_{\bar{C}}(2) \otimes \mathcal{O}_C(-\Delta_C)$, that together with (2.5) yields $L \simeq \mathcal{O}_C(D) \simeq \mathcal{O}_C(1) \simeq L \otimes \eta$, yielding a contradiction. $\square$

We stress that Theorem 2.19 covers also the cases where C is hyperelliptic and $\eta = \mathcal{O}_C(w_1+w_2-w_3-w_4)$ with $w_1, \ldots, w_4$ Weierstrass points (as $2(w_1+w_2) \sim 2(w_3+w_4)$), and where C is bielliptic and η is the pullback of a 2-torsion line bundle on the elliptic curve covered by C as explained in Example 1.6.3. Also, note that a smooth nonhyperelliptic Prym curve (C, η) of genus $g = 3$ such that $\omega_C \otimes \eta$ is base point free has Prym-canonical Clifford index $\mathrm{Cliff}_\eta(C) = 1$. Indeed, the Prym-canonical system $\omega_C \otimes \eta \sim g^1_4$ is not very ample.
As a consequence of Theorem 2.19 and Lemma 1.1, one gets the following:

Corollary 2.20. *Let (C, η) be a smooth Prym curve of genus $g \geq 3$. Then, $\omega_C \otimes \eta$ is very ample if and only if* $\mathrm{Cliff}_\eta(C) \geq 2$.

2.5 Curves with $\mathrm{Cliff}_\eta(C) = 2$

The idea of the proof of Theorem 2.21 is the same as that of the theorem in the previous section.

Theorem 2.21. *Let (C, η) be a smooth Prym curve of genus $g \geq 4$. Then,* $\mathrm{Cliff}_\eta(C) = 2$ *if and only if the Prym-canonical morphism is an embedding and $\varphi_\eta(C)$ has a trisecant line.*

Proof. If D is the divisor cut out by a trisecant line, then $h^0(\mathcal{O}_C(D)) = 1$ and $h^0(\omega_C \otimes \eta(-D)) = g-3$, so that $\mathrm{Cliff}_\eta(\mathcal{O}_C(D)) = 2$.

Viceversa, assume $\mathrm{Cliff}_\eta(C) = 2$ and let L be a line bundle computing the Prym-canonical Clifford dimension of (C, η).

If $\dim\mathrm{Cliff}_\eta(C) = (0,0)$, that is, $r(L) = r(L \otimes \eta) = 0$, then $\deg(L) = 3$. The Riemann-Roch Theorem then implies that the divisor $D \in |L|$ imposes one condition less than expected to the hyperplane sections of the Prym-canonical curve, or equivalently, D is cut out by a trisecant line to $\varphi_\eta(C)$.

We thus want to rule out the cases $h^0(L) \geq 2$, $h^0(L \otimes \eta) \geq 2$, where C is not bielliptic by Theorem 2.13 and we have

$$\mathrm{Cliff}(L) + \mathrm{Cliff}(L \otimes \eta) = 6$$

by Lemma 1.10. In particular, we obtain $0 \leq \mathrm{Cliff}(C) \leq 3$. When C is hyperelliptic, as in the proof of Theorem 2.19 we conclude that $L \otimes \eta$ has some base points, in contradiction with Lemma 1.9. Therefore, we can assume $1 \leq \mathrm{Cliff}(C) \leq 3$.
We first consider the cases where $\mathrm{Cliff}(C) = 1$ and thus C is either trigonal or a smooth plane quintic.

CASE A: $\mathrm{Cliff}(C) = \mathrm{Cliff}(L \otimes \eta) = 1$, and $\mathrm{Cliff}(L) = 5$. Since $L \otimes \eta$ is either a g^1_3 or a g^2_5, the degree of L is either 3 or 5 and in both cases the equality $\mathrm{Cliff}(L) = 5$ yields a contradiction.

CASE B: $\mathrm{Cliff}(C) = 1$, $\mathrm{Cliff}(L \otimes \eta) = 2$ and $\mathrm{Cliff}(L) = 4$.

In this case C cannot be a smooth plane quintic because otherwise its genus would be 6 in contradiction with the assumptions $d \leq g-1$ and $\mathrm{Cliff}(L) = 4$. Hence, C is trigonal. By Corollary 2.6, a g^r_d with Clifford index 2 satisfies $r \leq 2$, and this implies that $L \otimes \eta$ is either a g^1_4 or a g^2_6. If $L \otimes \eta$ is a g^1_4, then $r(L) = 0$ and $\dim\mathrm{Cliff}_\eta(C) = (0,1)$, in contradiction with Lemma 1.8. If $L \otimes \eta$ is a g^2_6, then L is a g^1_6. Assume $g = 7$, so that the Maroni invariant of C is $m = 1$ by (2.2) and is thus minimal. By Corollary 2.6 we get $L \otimes \eta = \omega_C - 2g^1_3 = 2g^1_3$ and $\omega_C = 4g^1_3$. Since L is base-point free by Lemma 1.9, then $L \in V^1_6$, and so $L = \omega_C - g^1_3 - D_3 = 3g^1_3 - D_3$ for some degree 3 effective divisor D_3. This yields that $\eta = L - (L \otimes \eta) = g^1_3 - D_3$, and so $\mathrm{Cliff}_\eta(C) \leq \mathrm{Cliff}_\eta(g^1_3) = 1$, in contradiction with our hypothesis. Therefore, by Corollary 2.6 we can assume $g \geq 8$ and $L \otimes \eta = 2g^1_3$. Since $L \in V^1_6 \neq \emptyset$, we must have $g \leq 10$. If $g = 8$ (respectively, $g = 9$), the Maroni invariant is $m = 2$ (resp., $m = 3$). In both cases (2.1) yields $e_1 = 4$ and thus $\omega_C \otimes (4g^1_3)^\vee$ is effective, that is, $\omega_C = 4g^1_3 \otimes \mathcal{O}_C(D_{2g-14})$ for some effective divisor D_{2g-14} of degree $2g-14$. Since $L \in V^1_6$, there is an effective divisor E_{10-g} of degree $10-g$ such that $L=\omega_C-(g-6)g^1_3-E_{10-g}$. In particular, for $g=8$ one gets that $\eta=L-(L\otimes\eta)=\omega_C-2g^1_3-E_2-2g^1_3=D_2-E_2$, and thus the contradiction $\mathrm{Cliff}_\eta(C)\le\mathrm{Cliff}_\eta(\mathcal{O}_C(D_2))=1$. If $g=9$, we get $\eta=\omega_C-3g^1_3-E_1-2g^1_3=D_4-(g^1_3+E_1)$ and, having fixed a point p in the support of D_4, we can write $D_4=D_3+p$ and $g^1_3(-p)=E_2$ for some effective divisors D_3,E_2 of degree 3 and 2, respectively, so that $\eta=\mathcal{O}_C(D_3-E_2-E_1)$. We get $\mathrm{Cliff}_\eta(\mathcal{O}_C(D_3))=2=\mathrm{Cliff}(C)$ and thus the contradiction $\mathrm{dimCliff}_\eta(C)=(0,0)$. To exclude the case $g=10$, it suffices to remark that $L\in V^1_6$ implies $L=\omega_C-4g^1_3=\omega_C\otimes L^{-2}$ so that $L^{\otimes 2}$ is special; this contradicts [MS86, Cor. 3].

CASE C: $\mathrm{Cliff}(C)=1$ and $\mathrm{Cliff}(L)=\mathrm{Cliff}(L\otimes\eta)=3$

Both L and $L\otimes\eta$ are g^r_{2r+3} and we can assume $r\ge 2$, as $r=1$ would contradict Lemma 1.12. Because of the inequality $d\le g-1$, the curve C cannot be a smooth plane quintic and is thus trigonal. Again by Corollary 2.6, we have $r\le 3$ and so L and $L\otimes\eta$ are either of type g^2_7, or of type g^3_9 when $g=10$. In the former case, since L and $L\otimes\eta$ are base point free by Lemma 1.9, they cannot lie in U^2_7. Hence, V^2_7 is nonempty (and thus $g\le 9$) and we have $L=\omega_C-(g-6)g^1_3-D_{9-g}$, $L\otimes\eta=\omega_C-(g-6)g^1_3-D'_{9-g}$ for some effective divisors D_{9-g} and D'_{9-g} of degree $9-g$. We obtain $\eta=\mathcal{O}_C(D_{9-g}-D'_{9-g})$, which is a contradiction because for $g=9$ it gives $\eta=\mathcal{O}_C$ and for $g=8$ it implies $\mathrm{Cliff}_\eta(C)\le\mathrm{Cliff}_\eta(\mathcal{O}_C(D_1))=0$.
If L and $L\otimes\eta$ are two distinct g^3_9, Corollary 2.6 implies $g=10$, $L=3g^1_3$, $L\otimes\eta=\omega_C-3g^1_3$ (or viceversa) and the Maroni invariant is not minimal, that is, $m=4$. Hence, $e_1=g-2-m=4$ and $\omega_C(-4g^1_3)$ is effective ; more precisely, if $\pi:C\to\mathbb{P}^1$ is the degree 3 cover, using [MS86, p.172] one computes $\pi_*\omega_C=\mathcal{O}_{\mathbb{P}^1}(4)\oplus\mathcal{O}_{\mathbb{P}^1}(4)\oplus\mathcal{O}_{\mathbb{P}^1}(-2)$ and thus $h^0(\omega_C(-4g^1_3))=2$, that is, $M_6:=\omega_C(-4g^1_3)$ is a g^1_6. We have $\eta=\omega_C-6g^1_3=M_6-2g^1_3$ and $\mathrm{Cliff}_\eta(M_6)=2=\mathrm{Cliff}_\eta(C)$, in contradiction with the assumption that L computes the Prym-canonical Clifford dimension of (C,η).

We will now treat the cases where $\mathrm{Cliff}(C)=2$, that is, C is either tetragonal or a smooth plane sextic (cf. Remark 2.4).

CASE D: $\mathrm{Cliff}(C)=\mathrm{Cliff}(L\otimes\eta)=2$ and $\mathrm{Cliff}(L)=4$.

Then $L\otimes\eta$ is either a g^1_4, or a g^2_6, or a g^3_8 and $g=9$. We exclude the case $L\otimes\eta=g^1_4$, as it would force $r(L)=0$, in contradiction with Lemma 1.8. If $L\otimes\eta=g^2_6$, then $L=g^1_6$.

We may assume that $L\otimes\eta$ is simple (as C is neither trigonal nor bielliptic) and thus defines a birational morphism $f\colon C\to\bar{C}\subseteq\mathbb{P}^2$. Since smooth plane sextics have arithmetic genus 10 and $g\ge d+1=7$, the $\delta-$invariant of $\bar{C}$ satisfies $0\le\delta(\bar{C})\le 3$. Using the same notation as in the proof of Theorem 2.19, we recall the isomorphism (2.5) implying that sections of ω_C are cut out by cubics through the conductor subscheme $E_{\bar{C}}$, which has length $\delta(\bar{C})$. Given a general divisor $D = p_1 + \ldots + p_6 \in |L|$, the Riemann-Roch Theorem yields that $h^0(\omega_C(-D)) = 5 - \delta(\bar{C})$, that is, the divisor $\bar{D} := E_{\bar{C}} + D$ (consisting of $6 + \delta(\bar{C})$ possibly infinitely near points of $\mathbb{P}^2$) imposes one condition less than expected to $|\omega_C|$, and there is a family $\mathcal{D}_{6+\delta(\bar{C})}$ of plane cubics passing through $\bar{D}$ of dimension

$$\dim \mathcal{D}_{6+\delta(\bar{C})} = 4 - \delta(\bar{C}). \tag{2.9}$$

We first consider $0 \leq \delta(\bar{C}) \leq 2$. By Corollary [Har77, Cor. 4.4, ch. V], the divisor $\bar{D} = E_{\bar{C}} + D$ splits as a sum $\bar{D} = \bar{D}_1 + \bar{D}_2$ of two generalized effective divisors $\bar{D}_1, \bar{D}_2 \subset \bar{C}$ such that either $\bar{D}_1$ is contained in a line and has degree ≥ 4, or $\bar{D}_2$ lies on a conic Q and has degree ≥ 7.

If $\bar{D}_2$ has degree ≥ 7 and lies on a conic, then such conic Q would be fixed and thus the family $\mathcal{D}_{6+\delta(\bar{C})}$ would contain only reducible curves of the form $Q \cap l_t$ with $l_t \in |\mathcal{O}_{\mathbb{P}^2}(1)|$ is a line, so that (2.9) yields $\delta(\bar{C}) = 2$. We can write $\omega_C \otimes L^\vee = \omega_C(-D) = \nu^*(\mathcal{O}_{\bar{C}}(1) + q_1 + q_2)$, where q_1, q_2 are the two further intersection points of Q with $\bar{C}$. Hence, the sections of $L = \nu^*(\mathcal{O}_{\bar{C}}(2) - E_{\bar{C}} - q_1 - q_2)$ would be cut out by plane conics γ_t through $E_{\bar{C}} + q_1 + q_2$. Since $g = 8$ and C is not hyperelliptic, then $h^0(C, (L \otimes \eta)^{\otimes 2}) = 6$, that is, all sections of $|(L \otimes \eta)^{\otimes 2}|$ are cut out by plane conics. Since $L^{\otimes 2} = (L \otimes \eta)^{\otimes 2}$, the divisor $\nu^*(2E_{\bar{C}} + 2q_1 + 2q_2) = 2\Delta_C + \nu^{-1}(2q_1 + 2q_2)$ is the pullback to C of the intersection of $\bar{C}$ with a conic; such a conic has to be singular along $E_{\bar{C}}$, which has length 2, and is thus a double line. Hence, $E_{\bar{C}}, q_1, q_2$ are collinear, yielding the contradiction $L = L \otimes \eta$.

We now assume that $\bar{D}_1$ has degree $d_1 \geq 4$ and is contained in a fixed line l, so that $\omega_C(-D)$ has the further intersection points of l with $\bar{C}$ as base points and its base point free part is cut out by $|\mathcal{I}_{\bar{D}_2/\mathbb{P}^2}(2)|$; equation (2.9) then implies that $\bar{D}_2$ has degree $d_2 = \delta(\bar{C}) + 1$ and thus $d_1 = 5$. We now write $E_{\bar{C}} = E_1 + E_2$ with $\bar{D}_1 = E_1 + D_1$, $\bar{D}_2 = E_2 + D_2$, where all divisors E_i, D_i are effective. Since $\bar{C}$ is singular along E_1 and E_2, the equality $d_1 = 5$ implies that E_1 has degree $e \leq 1$ (because otherwise the intersection number between $\bar{C}$ and l would be > 6, which is absurd). If $e = 1$, then E_2 has degree $\delta(\bar{C}) - 1$, the divisor D_2 has degree 2 and $\omega_C \otimes L^\vee = \omega_C(-D) = \nu^*(\mathcal{O}_{\bar{C}}(2)(-D_2 - E_2)$. We get $L = \nu^*(\mathcal{O}_{\bar{C}}(1) + D_2 - E_1)$ so that $\eta = \nu^*\mathcal{O}_{\bar{C}}(D_2 - E_1) = \mathcal{O}_C(D_2 - \Delta_1)$ where $\Delta_1 \subset C$ is the degree 2 divisor mapping to E_1; we get the contradiction $\mathrm{Cliff}_\eta(C) \leq \mathrm{Cliff}_\eta(D_2) = 1$. If instead $e = 0$, then $E_2 = E_{\bar{C}}$, the divisor D_2 has degree 1 and $\omega_C \otimes L^\vee = \nu^*(\mathcal{O}_{\bar{C}}(2)(-D_2 - E_2 + q)$, where q is a base point and coincides with the further intersection point of $\bar{C}$ with l. We obtain $L = \nu^*(\mathcal{O}_{\bar{C}}(1) + D_2 - q)$ and thus $\eta = \mathcal{O}_C(D_2 - q)$, yielding the contradiction $\mathrm{Cliff}_\eta(C) \leq \mathrm{Cliff}_\eta(D_2) = 0$.

It remains to treat the case $\delta(\bar{C}) = 3$, that is, $g = 7$, where the divisors in $|\omega_C(-D)|$ are cut out by the pencil of cubics through $\bar{D}$, which has degree 9. The curve $\bar{C}$ has only double points as singularities because otherwise it would have a g^1_3. Let $\sigma : S \to \mathbb{P}^2$ be a sequence of blow-ups resolving the singularity of $\bar{C}$, and denote by l the class of the pullback on S of a line so that $L \otimes \eta = \mathcal{O}_C(l)$. We recall that $\omega_S \simeq \sigma^*\mathcal{O}_{\mathbb{P}^2}(-3) + E_1 + E_2 + E_3$, where E_1, E_2, E_3 are effective (possibly reducible) divisors contracted by l such that $E_i^2 = -1$ and $E_i \cdot E_j = 0$ for $i \neq j$. Since $\bar{C}$ has only double points, then $C \in |6l - 2E_1 - 2E_2 - 2E_3|$ and thus $\omega_C = \mathcal{O}_C(3l - E_1 - E_2 - E_3)$. The isomorphism $L^{\otimes 2} = (L \otimes \eta)^{\otimes 2}$ implies that, for every divisor $D \in |L|$, $\mathcal{O}_C(2l - 2D) \simeq \mathcal{O}_C$. We denote by $2D$ the 0-dimensional subscheme of S of length 12 consisting of the 6 points of D along with the tangent direction of C at each of them, and consider the following short exact sequence

$$0 \longrightarrow -4l + 2E_1 + 2E_2 + 2E_3 \longrightarrow 2l \otimes I_{2D/S} \longrightarrow \mathcal{O}_C \longrightarrow 0. \tag{2.10}$$

An element of the linear system $|2l \otimes I_{2D/S}|$ defines a conic totally tangent to $\bar{C}$. Moreover, since a plane sextic with ADE singularities admits at most finitely many totally tangent conics. Indeed, if $q : Y \to \mathbb{P}^2$ is a double cover of $\mathbb{P}^2$ branched along a sextic X_6 with ADE singularities, then Y is a $K3$ surface with ADE singularities and the inverse image on Y of every conic γ totally tangent to X_6 splits as the union of two smooth rational curves. Since smooth rational curves on Y are finitely many, the same holds for the conics γ. Hence, we conclude that $h^0(2l \otimes I_{2D/S}) = 0$ for a general $D \in |L|$. Short exact sequence (2.10) then implies that $h^1(-4l + 2E_1 + 2E_2 + 2E_3) \neq 0$. Because of the equalities

$$h^1(-4l + 2E_1 + 2E_2 + 2E_3) = h^1(l - E_1 - E_2 - E_3) = h^0(l - E_1 - E_2 - E_3),$$

which follow from Serre duality and the Riemann-Roch Theorem, respectively, we conclude that $h^0(l - E_1 - E_2 - E_3) \neq 0$, that is, n_1, n_2, n_3 lie on a line. Therefore, both $L \otimes \eta$ and L are theta characteristics of C. We can write $\omega_C \otimes \eta = \mathcal{O}_C(4l - E_1 - E_2 - E_3 - D)$ for $D \in |L|$; since $h^0((4l - E_1 - E_2 - E_3) \otimes I_{D/S}) \geq 6 = h^0(\omega_C \otimes \eta)$, we conclude that the divisors in $|\omega_C \otimes \eta|$ are cut out by plane quartics passing through n_1, n_2, n_3 and D. We want to show that η can be written as the difference of two effective divisors of degree 3, or equivalently, there exists $D_3 \in \mathrm{Sym}^3(C)$ such that $h^0(\omega_C \otimes \eta(-D_3)) = 4$. Choose a plane cubic $X \in |3l - E_1 - E_2 - E_3 - D|$ and a divisor $D_3 \subset X \cap \bar{C}$ with support disjoint from that of $E_1 + E_2 + E_3 + D$, and consider the following short exact sequence:

$$0 \longrightarrow l \longrightarrow (4l - E_1 - E_2 - E_3) \otimes I_{D+D_3/S} \longrightarrow O_X(4l - E_1 - E_2 - E_3 - D - D_3) \longrightarrow 0.$$

We observe that $\mathcal{O}_X(4l - E_1 - E_2 - E_3 - D - D_3) \simeq \mathcal{O}_X(3l - D - D_3) \simeq O_X$, where the last isomorphism follows from the existence of a pencil of cubics through $D+D_3$. We thus obtain $h^0((4l-E_1-E_2-E_3)\otimes I_{D+D_3/S}) = 4$ and, by restricting to C, $h^0(\omega_C\otimes\eta(-D_3)) = 4$. As a consequence, $\mathcal{O}_C(D_3) \otimes \eta$ is effective and thus the contradiction $\dim\mathrm{Cliff}_\eta(C) = (0, 0)$.

We now treat the case where $g = 9$ and $L \otimes \eta$ is a g^3_8, which implies that L is a g^2_8 and C is tetragonal. Note that $\omega_C \otimes \eta \otimes L^\vee$ is also a g^3_8 and $\omega_C \otimes L^\vee$ is a g^2_8, so that $\omega_C \otimes L^\vee$ also computes the Prim-canonical Clifford dimension of (C, η). By Lemma 1.9 L, $L \otimes \eta$, $\omega_C \otimes L^\vee$, $\omega_C \otimes \eta \otimes L^\vee$ are all base-point free and thus we can use Corollary 2.8 to describe them having fixed a g^1_4, which we call l. Since both $L \otimes \eta$ and $\omega_C \otimes \eta \otimes L^\vee$ are g^3_8, for degree reasons they must be of type (4) in Corollary 2.8, that is, $L \otimes \eta = l + l'$ and $\omega_C \otimes \eta \otimes L^\vee = l + l''$ where l' and l'' are g^1_4 different from l. If we had $L = l + F_4$ with F_4 an effective divisor of degree 4, then $\eta = l' - F_4$ and the equality $\mathrm{Cliff}_\eta(C) = \mathrm{Cliff}_\eta(F_4)$ would contradict the assumption that L computes the Prym-canonical Clifford dimension of C. Therefore, Corollary 2.8 yields either $L = 2l$, or $\omega_C \otimes L^\vee = 2l$. In the former case we get $\eta = l + l' - 2l = l' - l$, while in the latter case we get $\eta = l + l'' - 2l = l'' - l$; in both cases we thus obtain the contradiction $\mathrm{Cliff}_\eta(C) \leq \mathrm{Cliff}_\eta(l) = 1$.

CASE E: $\mathrm{Cliff}(C) = 2$ and $\mathrm{Cliff}(L \otimes \eta) = \mathrm{Cliff}(L) = 3$.

Then both L and $L \otimes \eta$ are g^r_{2r+3}, and Lemma 1.12 implies $r \geq 2$. Since both L and $\omega_C \otimes L^\vee$ compute the Prym-canonical Clifford dimension, then L, $\omega_C \otimes L^\vee$, $L \otimes \eta$, $\omega_C \otimes \eta \otimes L^\vee$ are all base point free by Lemma 1.9.

Assume that C is a smooth plane sextic, so that $g = 10$, and $L, L \otimes \eta$ are two g^2_7, or two g^3_9. To rule out the g^3_9 case, it is enough to apply Lemma 2.9 (smooth plane sextics have gonality 5) excluding the cases (3) and (6) in the lemma because they would imply the existence of a g^1_4. The g^2_7 case can be excluded as follows. Fix a divisor $D = p_1 + \ldots + p_7 \in |L|$; since by Riemann-Roch $h^0(\omega_C - D) = 5$ and in the plane model of C as a smooth sextic ω_C is cut out by plane cubics by the adjunction formula, there exists a 4 dimensional family $\mathcal{D} = |\mathcal{I}_{p_1+\ldots+p_7/\mathbb{P}^2}(3)|$ of cubics passing through $p_1, \ldots p_7$. Then, [Har77, Cor. 4.4, ch. V] implies that either 4 of $p_1, \ldots, p_7$ are collinear, or 7 of them lie on a conic. If the latter case happens, the conic would be fixed, and so $\dim\mathcal{D} = 2 \leq 4$. The former case implies that up to reordering, $p_1 \ldots, p_6$ lie on a line l (as C is a sextic) and $p_7 \in Q$, where Q is a conic. Then, $\dim\mathcal{D} = 4$, and $\omega_C \otimes L^\vee \simeq \mathcal{O}_C(2) - p_7$. This implies $L \simeq \mathcal{O}_C(1) + p_7$, that is, L has a base point at p_7, in contradiction with Lemma 1.9.

Therefore we can assume that C is tetragonal and refer to Corollary 2.8 to classify linear series of type g^r_{2r+3} with $r \geq 2$ on it. Only cases (3) ad (4) in the corollary may occur, as one excludes cases (1) and (2) for degree reasons recalling that L, $L\otimes\eta$ and their residuals are base point free. Case (3) reads like $g^r_{2r+3} = (r-1)g^1_4 + F$ for some effective divisor F. On the other hand, case (4) reads like $g^r_{2r+3} = l + kg^1_4$ with $l = g^{r-2k}_{2r+3-4k}$ of type (3), that is, $l = (r-2k-1)g^1_4 + F$ and thus $g^r_{2r+3} = (r-k-1)g^1_4 + F$. Hence, independently whether we fall in case (3) or (4), we get $L = (r-k-1)g^1_4 + F$ and $L' = (r-k'-1)g^1_4 + F'$ for some effective divisors F, F' of degree $-2r+7+4k \leq 3$ and $-2r+7+4k' \leq 3$, respectively, where the inequalities follow from $r-2k \geq 1$ and $r-2k' \geq 1$; since both $\deg F$ and $\deg F'$ are congruent to $2r+3$ modulo 4, we conclude that $\deg F = \deg F'$ (that is, $k = k'$). Using that $\eta \simeq \mathcal{O}_C(F-F')$ and $h^0(\mathcal{O}_C(F)) = h^0(\mathcal{O}_C(F')) = 1$ since C has no g^1_3, we compute $\mathrm{Cliff}_\eta(\mathcal{O}_C(F)) = \deg F - 1 \leq 2$ in contradiction with the assumption that L computes the Prym-canonical Clifford dimension of C.

CASE F: $\mathrm{Cliff}(C) = \mathrm{Cliff}(L) = \mathrm{Cliff}(L\otimes\eta) = 3$.

By Remark 2.4, L and $L\otimes\eta$ are two g^1_5, or two g^2_7, or two g^3_9 and the genus is 10. By Lemma 1.12, we exclude the case where both L and $L\otimes\eta$ are g^1_5. We stress that C cannot have Clifford dimension 3 because in this case the line bundle computing the Clifford index is unique (see [ELMS89, p.174]). Therefore, C is either 5-gonal, or a smooth plane septic.

Assume C is a smooth plane septic. Then, L and $L\otimes\eta$ cannot be two g^3_9 as they occur in genus 10 while C has genus 15. If $L, L\otimes\eta$ are two g^2_7, then they have to be very ample as C has genus 15. The unicity of a g^2_d on a smooth plane curve of degree $d \geq 4$ excludes this case.

We thus conclude that C has Clifford dimension 1 and gonality 5. Since L and $L\otimes\eta$ compute the Clifford index of C, they and their residuals are base-point free and we can apply Lemma 2.9 to classify them.
We first consider the case where $L, L\otimes\eta$ are two g^2_7. We exclude type (1) of the lemma for degree reasons and type (3) because it requires $r-2 \geq 1$. Type (4) reads like $\omega_C - g^2_7 = (g-6)g^1_5$, and comparing the degrees we obtain $g = 7$, against our assumption $g \geq d+1 = 8$. If a g^2_7 is of type (5), then $\omega_C - g^2_7 - (g-7)g^1_5$ is effective and, for degree reasons, this yields $g = 8$. In case (6), $\omega_C - g^2_7 - g^1_5$ is a g^{g-8}_{2g-14} with $g-8 \geq 1$ and this yields the contradiction $\mathrm{Cliff}(g^{g-8}_{2g-14}) = 2 < \mathrm{Cliff}(C) = 3$. Hence, L and $L\otimes\eta$ are of type (2), or of type (5) and $g = 8$.
If both L and $L\otimes\eta$ are of type (2), then $L = g^1_5 + E_2$ and $L\otimes\eta = g^1_5 + E'_2$ for some effective degree 2 divisors E_2, E'_2 satisfying $h^0(\mathcal{O}_C(E_2)) = h^0(\mathcal{O}_C(E'_2)) = 1$; since $\eta = \mathcal{O}_C(E_2 - E'_2)$, this leads to the contradiction $\mathrm{Cliff}_\eta(\mathcal{O}_C(E_2)) = 1$. Similarly, if $g = 8$ and both L and $L\otimes\eta$ are of type (5), then $\omega_C - L = g^1_5 + E_2$ and $\omega_C - L' = g^1_5 + E'_2$ for some effective degree 2 divisors E_2, E'_2 and, as before, one gets the contradiction

$\mathrm{Cliff}_\eta(\mathcal{O}_C(E_2)) = 1$. It remains the possibility where $g = 8$ and one between $L, L \otimes \eta$ is of type (2), while the other is of type (5) with respect to any fixed g^1_5. Without loss of generality, we can assume $L = g^1_5 + E_2$ and $\omega_C(-L \otimes \eta) = g^1_5 + E'_2$ for some effective degree 2 divisors E_2, E'_2. Since every base point free g^2_7 is simple, $L \otimes \eta$ defines a birational morphism $f : C \to \bar{C} \subseteq \mathbb{P}^2$, where $\bar{C}$ has arithmetic genus 15 and thus its δ-invariant is $\delta(\bar{C}) = 7$. We remark that $\bar{C}$ has at most double points as singularities because otherwise $\mathrm{gon}(C) \leq 4$, against our assumptions. In particular, we can fix $g^1_5 = f^*(\mathcal{O}_{\bar{C}}(1) - n)$ where n is a double point of $\bar{C}$. With this choice we have $L = f^*(\mathcal{O}_{\bar{C}}(1) - n) + E_2$ and $L \otimes \eta = f^*(\mathcal{O}_{\bar{C}}(1))$ and thus $\eta = \mathcal{O}_C(E_2) \otimes \nu^*(\mathcal{O}_{\bar{C}}(n))^\vee$, yielding the contradiction $\mathrm{Cliff}_\eta(\mathcal{O}_C(E_2)) = 1$.
It remains to treat the case where $L, L \otimes \eta$ are two g^3_9 and $g = 10$. Cases (3) and (6) of Lemma 2.9 can be ruled out because they would imply the existence of a g^1_4. We exclude all the other possibilities for degree reasons. □

We remark that if C is hyperelliptic and $\eta = \mathcal{O}_C(w_1 + w_2 + w_3 - w_4 - w_5 - w_6)$ (see Section 3.1), then C has a trisecant line, as the divisor $D = w_1 + w_2 + w_3$ is such that $h^0(\mathcal{O}_C(D)) = h^0(\mathcal{O}_C(D) \otimes \eta) = 1$, so by Riemann-Roch, D imposes one condition less than expected to $\omega_C \otimes \eta$.

Chapter 3

The case of hyperelliptic and general curves

As mentioned in the Introduction, we reserve for hyperelliptic curves a separate chapter because, as it will be evident from Proposition 3.2, in this case the Prym-canonical Clifford index keeps track of the geometry of the curve and, in particular, it depends on the nontrivial 2-torsion line bundle η chosen on C. A key point was proved by Verra in [Ver13], that is, if the curve C is hyperelliptic, we can explicitly describe all nontrivial 2-torsion line bundle on C as difference of Weierstrass points. In the first section, we recall this result. Then, we compute the Prym-canonical Clifford index of (C, η) and prove that its Prym-canonical Clifford dimension is always $(0, 0)$. We also determine the rational normal scroll containing a Prym-canonical hyperelliptic curve, and with a computation we show the failure of the normal generation of $\omega_C \otimes \eta$ (this was already known by [LM85, Cor. 3.4], but the argument is different). Finally, as a consequence of the hyperelliptic case, we obtain in the last section that given a general Prym-canonical curve $(C, \eta) \in \mathcal{R}_g$, then $\mathrm{Cliff}_\eta(C) = \mathrm{Cliff}(C) = \lfloor \frac{g-1}{2} \rfloor$.

3.1 Hyperelliptic curves

On a hyperelliptic curve C, let E_t be the family of effective divisors of degree $2t$, where $1 \leq t \leq \lfloor \frac{g+1}{2} \rfloor$, which are supported on $2t$ distinct Weierstrass points. Given $e \in E_t$ and $M \in \mathrm{Pic}(C)$ such that $|M|$ is the only g^1_2 on C, the line bundle $M^{\otimes t}(-e)$ is a square root of $\mathcal{O}_C$, and it is an element of $\mathrm{Pic}^0(C)[2] \setminus \{\mathcal{O}_C\}$. Let

$$\beta_t \colon E_t \to \mathrm{Pic}^0(C)[2] \setminus \{\mathcal{O}_C\}$$

be the map sending $e \in E_t$ to the line bundle $M^{\otimes t}(-e)$, we set $B_t := \beta_t(E_t)$.

Lemma 3.1. *[Ver13, Lemma 4.3]*

1. *The map* β_t *is injective for* $t \leq \lfloor \frac{g}{2} \rfloor$.
2. *For* $t = \frac{g+1}{2}$ *the map* β_t *is* $2:1$ *over its image.*
3. $\mathrm{Pic}^0(C)[2] \setminus \{\mathcal{O}_C\} = \cup_{1 \leq t \leq \lfloor \frac{g+1}{2} \rfloor} B_t$

Note that in $\mathrm{Pic}^0(C)$ there are 2^{2g} 2-torsion line bundles and that the degree 2 cover $C \to \mathbb{P}^1$ has $2g+2$ ramification points, whose set we denote by W. We consider nontrivial 2-torsion line bundles of the form

$$\eta = \mathcal{O}_C(w_1 + \cdots + w_k - w_{k+1} - \cdots - w_{2k}) \tag{3.1}$$

for some $1 \leq k \leq \lfloor \frac{g+1}{2} \rfloor$, where $\{w_1, \ldots, w_k, w_{k+1}, \ldots, w_{2k}\} \subseteq W$ and $w_i \neq w_j$ for $i \neq j$. We remark that writing η as in (3.1) is equivalent to $\eta = M^{\otimes k}(-w_1 - \ldots - w_{2k})$. Due to the fact that η has order 2 and $2w_i \sim 2w_j$, if η' is a line bundle obtained from η by swapping the order of some points in (3.1), then $\eta' \simeq \eta$. Indeed, without loss of generality, we can assume

$$\eta' = \mathcal{O}_C(w_{k+1} + \ldots + w_{k+m} + w_{m+1} + \ldots + w_k - w_1 - \ldots - w_m - w_{k+m+1} - \ldots - w_{2k})$$

and thus obtain

$$\eta - \eta' \simeq \mathcal{O}_C(2(w_1 + \ldots + w_m - w_{k+1} - \ldots - w_{k+m})) \simeq \mathcal{O}_C.$$

Moreover, if $k + k' \leq g$, two different subsets of Weierstrass points

$$P_k := \{p_1, \ldots, p_k, p_{k+1}, \ldots, p_{2k}\} \quad Q_{k'} := \{q_1, \ldots, q_{k'}, q_{k'+1}, \ldots, q_{2k'}\}$$

where $p_i \neq p_j$, $q_i \neq q_j$ for $i \neq j$ provide non-isomorphic 2-torsion line bundles. Indeed, if $k + k' \leq g$ and if we had $\mathcal{O}_C(p_1 + \cdots + p_k - p_{k+1} \cdots - p_{2k}) \simeq \mathcal{O}_C(q_1 + \cdots + q_{k'} - q_{k'+1} - \cdots - q_{2k'})$, then the line bundle

$$\mathcal{O}_C(p_1 + \cdots + p_k + q_{k'+1} + \cdots + q_{2k'}) \simeq \mathcal{O}_C(q_1 + \cdots + q_{k'} + p_{k+1} + \cdots + p_{2k})$$

would be special as $k + k' \leq g$. In particular, up to subtracting possible base points, it should be a multiple of the g^1_2, which is clearly a contradiction because rg^1_2 consists of the preimage of r points of $\mathbb{P}^1$ and the Weierstrass points have always multiplicity 2. If $k + k' = g + 1$, that is, $k = k' = \lfloor \frac{g+1}{2} \rfloor$, $P_k \sqcup Q_k = W$, and by Lemma 3.1 this is the only case in which $\eta \simeq \eta'$. It follows that when g is even all the 2-torsion line bundles $\eta \in \mathrm{Pic}^0(C)$ can be written as (3.1) with $1 \leq k \leq \frac{g}{2}$. Indeed, counting how many η

is of the form (3.1), as k varies in the range $1 \le k \le \frac{g}{2}$, is equivalent to computing the number of subsets $J \subset W$ of even cardinality $2k$ such that $|J| < \frac{1}{2}|W| = g+1$ (we do not consider the subsets $J \subset W$ such that $|J| = g+1$ as for g even, $g+1$ is odd). Since, given a finite set A, the number of subsets of even cardinality is equal to that of odd cardinality, we get that the number of subsets of W of even cardinality is $\frac{1}{2}|\mathcal{P}(W)| = 2^{2g+1}$. If we assume that g is even, than the subsets $J \subset W$ such that $|J|$ is even and $|J| < g+1$ are exactly $\frac{1}{2}2^{2g+1} = 2^{2g}$.

When the genus g is odd, the argument is similar but we have to count also the subsets of W of cardinality exactly $g+1$, which are $\binom{2g+2}{g+1}$. Since the number x of subsets of cardinality strictly less than $g+1$ satisfies

$$2x + \binom{2g+2}{g+1} = 2^{2g+1},$$

the number of η that can be written as in (3.1) with $1 \le k \le \frac{g+1}{2}$ is precisely

$$x + \frac{1}{2}\binom{2g+2}{g+1} = 2^{2g}, \tag{3.2}$$

that is, we obtain every η in this way.

Theorem 3.2. *If C is hyperelliptic and η can be written as in* (3.1) *for some $k \le \lfloor \frac{g+1}{2} \rfloor$, then*

$$\mathrm{Cliff}_\eta(C) = k-1 \quad \textit{and} \quad \mathrm{dimCliff}_\eta(C) = (0,0).$$

Proof. Let us consider the line bundle $A = \mathcal{O}_C(w_1 + \ldots + w_k)$ provided by (3.1). We claim that $h^0(A) = h^0(A \otimes \eta) = 1$. Indeed, if we suppose $h^0(A) = r+1$ for some positive r, then up to subtracting its base points, A would be a multiple of the g^1_2, which is clearly a contradiction since the w_i are Weierstrass points. Hence, $\mathrm{Cliff}_\eta(A) = k-1$ and we need to show that $\mathrm{Cliff}_\eta(A) = \mathrm{Cliff}_\eta(C)$.

Firstly, we prove that $\mathrm{dimCliff}_\eta(C) = (0,0)$. Let L be a line bundle computing the Prym-canonical Clifford dimension of (C,η) and, by contradiction, assume $h^0(L) \ge 2$ (recall that by Lemma 1.8 $\mathrm{dimCliff}_\eta(C) \ne (0,r')$ for $r' > 0$). Since L and $L \otimes \eta$ are special, base point free by Lemma 1.9 and of the same degree, then both of them should coincide with $r(L)g^1_2$ which is a contradiction as they are distinct.

If $\mathrm{Cliff}_\eta(C) = \frac{g-1}{2}$ (which implies g odd and $k = \frac{g+1}{2}$ by what we have proved above), we have done. So we may assume $\mathrm{Cliff}_\eta(C) < \frac{g-1}{2}$ (hence, $\deg L < \frac{g+1}{2}$) and prove that $L = A$. Taking $D \in |L|$ and $D' \in |L \otimes \eta|$, it is enough to show that D and D' are sums of Weierstrass points. Since $2D, 2D' \in |L^{\otimes 2}|$ and $\deg L^{\otimes 2} < g+1$, then $L^{\otimes 2}$ is special and is thus a multiple of the g^1_2, as $2D$ and $2D'$ have no points in common. This leads us to the conclusion that D and D' are sums of Weierstrass points, as all divisors of $g^r_{2r} = rg^1_2$ are sum of r fibers of the morphism defined by the g^1_2 and divisors of the form $2(P + \iota(P))$, for some $p \in C$ and ι the hyperelliptic involution, do not appear since we showed that $h^0(L) = h^0(L \otimes \eta) = 1$, so $|D|$ does not contain the g^1_2. Thus,

$$\mathrm{Cliff}_\eta(C) = \mathrm{Cliff}_\eta(A) = k - 1.$$

□

3.1.1 Scroll containing $\varphi_\eta(C)$

We are interested in the rational normal scroll $S \subset \mathbb{P}^{g-2}$ containing the Prym-canonical curve when the curve C is hyperelliptic. We will give a brief introduction; for more details, we refer to [Sch86].

Let $\mathcal{E} = \mathcal{O}(e_1) \oplus \ldots \oplus \mathcal{O}(e_d)$ be a locally free sheaf of rank d on $\mathbb{P}^1$ and let

$$\pi\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$$

denote the corresponding $\mathbb{P}^{d-1}$-bundle.

Definition 3.3. *A rational normal scroll S of type $S(e_1, \ldots, e_d)$ with $e_1 \geq \ldots \geq e_d \geq 0$ and $f = e_1 + \ldots + e_d \geq 2$ is the image of $\mathbb{P}(\mathcal{E})$ in $\mathbb{P}^r = \mathbb{P}H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ through the morphism $j\colon \mathbb{P}(\mathcal{E}) \to S \subset \mathbb{P}^r$ with $r = f + d - 1$.*

The scroll S is a non-degenerate irreducible variety of minimal degree $\deg(X) = f = r - d + 1 = \mathrm{codim}(S) + 1$ in $\mathbb{P}^r$. If all $e_i > 0$ then S is smooth and j is an isomorphism. If some of the $e_i = 0$, then X is singular and j is a resolution of singularities.

The Picard group of $\mathbb{P}(\mathcal{E})$ is generated by the hyperplane class $H = [j^*\mathcal{O}_{\mathbb{P}^r}(1)]$ and the ruling $R = [\pi^*\mathcal{O}_{\mathbb{P}^1}(1)]$, that is, $\mathrm{Pic}\mathbb{P}(\mathcal{E}) = \mathbb{Z}H \oplus \mathbb{Z}R$. Also, when $\mathbb{P}(\mathcal{E})$ has dimension 2, one has

$$H^2 = f, \quad H \cdot R = 1, \quad R^2 = 0.$$

Given a smooth hyperelliptic curve C and the Prym-canonical morphism $\varphi_\eta\colon C \to \mathbb{P}^{g-2}$, we study the rational normal scrolls $S \subset \mathbb{P}^{g-2}$ of degree f containing $\varphi_\eta(C)$. We need $|\omega_C \otimes \eta|$ to be base-point free and simple, and $\varphi_\eta(C)$ to be linearly normal; by Corollary 2.20 this is equivalent to requiring that $\mathrm{Cliff}_\eta(C) \geq 2$. The hyperelliptic pencil $\{D_\lambda\}_{\lambda \in \mathbb{P}^1} = g^1_2$ on C satisfies $h^0(C, \omega_C \otimes \eta(-D_\lambda)) = f = g - 3$. Geometrically, the scroll S is

$$S = \bigcup_{\lambda \in \mathbb{P}^1} \overline{D_\lambda} \subset \mathbb{P}^{g-2}.$$

The type $S(e_1, e_2)$ of the scroll S can be determined as follows (see [Sch86]). First of all, since $S \subset \mathbb{P}^{g-2}$, then $e_1 + e_2 = g - 3$. Consider the following partition of $g - 1$:

$$\begin{aligned}
d_0 &:= h^0(\omega_C \otimes \eta) - h^0(\omega_C \otimes \eta(-D_\lambda)) = 2, \\
d_1 &:= h^0(\omega_C \otimes \eta(-D_\lambda)) - h^0(\omega_C \otimes \eta(-2D_\lambda)), \\
&\vdots \\
d_i &:= h^0(\omega_C \otimes \eta(-iD_\lambda)) - h^0(\omega_C \otimes \eta(-i-1)D_\lambda), \\
&\vdots
\end{aligned}$$

Since D_λ has degree 2, clearly $0 \leq d_i \leq 2 \ \forall i$. For $i \in \{1, 2\}$ the number e_i is defined as

$$e_i := \#\{j \mid d_j \geq i\} - 1.$$

A theorem by Harris and Bertini ensures that with the notation as above, S is a d_0-dimensional rational normal scroll of type $S(e_1, \dots, e_{d_0})$.
To compute d_i, we need a preliminary lemma:

Lemma 3.4. *Let C be a smooth hyperelliptic curve and let $A \in \mathrm{Pic}(C)$. Then, given an effective divisor D moving in the g^1_2, the equality $h^0(C, A(-D)) = h^0(C, A) - 1$ holds if and only if $|A| = n \cdot g^1_2$ plus base points.*

Proof. If $|A| = n \cdot g^1_2$ plus base points, it is obvious.
Conversely, we can assume $|A|$ base-point free, because otherwise we can replace it with its base-point free linear subsystem $|A'| \subset |A|$. Since $h^0(C, A(-D)) = h^0(C, A) - 1$ by hypothesis, then the morphism φ_A defined by A does not separate the points of the divisor D. Since D moves in a linear system of dimension 1, this means that φ_A does not separate all the divisors moving in the g^1_2, that is, φ_A factorizes through $f : C \to \mathbb{P}^1$ defined by the g^1_2. This is equivalent to $|A| = n \cdot g^1_2$. □

Corollary 3.5. *Using the same notation as above, if i is the first index such that $d_i = 1$, then:*

i. $d_l = 2, \ \forall \ 0 \leq l \leq i - 1$;

ii. $d_l = 1, \ \forall \ i \leq l \leq g - 2 - i$.

In particular, the scroll is of type $S(g - 2 - i, i - 1)$.

Proof. Item (i) follows from the definition of the integer i, which also implies $h^0(\omega_C \otimes \eta(-iD_\lambda)) = g - 1 - 2i$. Lemma 3.4 yields that

$$\omega_C \otimes \eta(-iD_\lambda) = (g - 2 - 2i)g^1_2 + p_1 + \ldots + p_{2i+2}$$

where $p_1, \ldots, p_{2i+2}$ are base points. Then,

$$\omega_C \otimes \eta(-lD_\lambda) = (g-2-i-l)g_2^1 + p_1 + \ldots + p_{2i+2}$$

$\forall i \leq l \leq g-2-i$, implying item (ii). □

Since $e_1 \geq 1$ because $|\omega_C \otimes \eta|$ is base-point free, the only case in which S is singular occurs when the Prym-canonical morphism is not an embedding, i.e., when $\eta = \mathcal{O}_C(p+q-x-y)$, with $2(p+q) \sim 2(x+y)$. Indeed, for S to be singular, e_2 has to be zero, meaning that d_0 is the only d_i equal to 2. So, consider

$$d_1 = h^0(\omega_C \otimes \eta(-D_\lambda)) - h^0(\omega_C \otimes \eta(-2D_\lambda)) = 1;$$

from Lemma 3.4 we get that $|\omega_C \otimes \eta(-D_\lambda)|$ is a multiple of the g_2^1 plus base points, in symbols

$$|\omega_C \otimes \eta(-D_\lambda)| = (g-4)g_2^1 + p_1 + p_2 + p_3 + p_4.$$

Since on hyperelliptic curves $\omega_C \sim (g-1)g_2^1$, we get that $\eta \sim 2g_2^1 - (p_1+p_2+p_3+p_4)$. The points p_1, p_2, p_3, p_4 are Weierstrass points, as $4g_2^1 \sim 2(p_1+p_2+p_3+p_4)$, and we can rewrite η as

$$\eta = \mathcal{O}_C(p_1+p_2-p_3-p_4).$$

More generally, the following holds:

Proposition 3.6. *Let* $2 \leq k \leq \lfloor \frac{g+1}{2} \rfloor$ *and* $\eta = \mathcal{O}_C(w_1 + \ldots + w_k - w_{k+1} - \ldots - w_{2k}) \in \mathrm{Pic}^0(C)$ *be a non trivial 2-torsion line bundle. Using the notation as above, if i is the first integer such that $d_i = 1$, then $k = i+1$.*

Proof. A simple argument by induction shows that if $d_{i-1} = 2$, then $h^0(\omega_C \otimes \eta - iD_\lambda) = g-1-2i$, and Lemma 3.4 yields

$$|\omega_C \otimes \eta - iD_\lambda| = (g-2-2i)g_2^1 + p_1 + \ldots + p_{2(i+1)},$$

where $p_1, \ldots, p_{2(i+1)}$ are base points. Since $\omega_C = (g-1)g_2^1$,

$$(g-1-i)g_2^1 \otimes \eta = (g-2-2i)g_2^1 + p_1 + \ldots + p_{2i+2},$$

from which we get

$$\eta = (i+1)g_2^1 - (p_1 + \ldots + p_{2i+2}).$$

This concludes the proof because from $2\eta \sim \mathcal{O}_C$, and thus $2(i+1)g_2^1 \sim 2(p_1+\ldots+p_{2i+2})$, it follows that $p_1, \ldots, p_{2i+2}$ are Weierstrass points, and

$$\eta = \mathcal{O}_C(p_1 + \ldots + p_{i+1} - p_{i+2} - \ldots - p_{2i}),$$

as we proved that any non trivial 2-torsion line bundle in $\mathrm{Pic}^0(C)$ can be written as (3.1). Thus, $k = i+1$. □

We remark that since by Corollary 4.10 the scroll S is of type $S(g-2-i, i-1)$ and $i = k-1$, then $S = S(g-1-k, k-2)$. As an application of our analysis of the scroll S, we can give a new proof of the failure of the normal generation of the Prym-canonical bundle $\omega_C \otimes \eta$ for a hyperelliptic curve C, a result first proved in [LM85, Cor. 3.4] (we recall that a line bundle $L \in \operatorname{Pic}(C)$ with $h^0(C, L) = r+1$ is normally generated if it is very ample and it embeds $C \subseteq \mathbb{P}^r$ such that the hypersurfaces of degree k cut out the complete linear series $|\mathcal{O}_C(k)|$ for any k).

Theorem 3.7. *Let (C, η) be a smooth, hyperelliptic Prym curve of genus $g \geq 2$. Then, the Prym-canonical bundle $\omega_C \otimes \eta$ is not normally generated.*

Proof. The statement follows from a straightforward computation, using that C lies on S. In particular, the requirement on the hypersurfaces of degree k on $C \subseteq \mathbb{P}^r$ is equivalent to the condition $H^1(\mathbb{P}^r, \mathcal{I}_C(k)) = 0$ for any k ([ACGH85, Ch. III, D1, p. 140]), and we compute that

$$h^1(\mathbb{P}^{g-2}, \mathcal{I}_{C/\mathbb{P}^{g-2}}(2)) = 3. \tag{3.3}$$

Firstly, we recall that, given $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(g-1-k) \oplus \mathcal{O}_{\mathbb{P}^1}(k-2)$, $\operatorname{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}R$, where

$$\begin{aligned} &H = [j^*\mathcal{O}_S(1)], \quad j\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}^{g-2} \\ &R = [\pi^*\mathcal{O}_{\mathbb{P}^1}(1)], \quad \pi\colon S \to \mathbb{P}^1 \\ &H^2 = \deg(S) = g-3, \quad H \cdot R = 1, \quad R^2 = 0. \end{aligned}$$

To compute the class of $C = aH + bR$ in $\operatorname{Pic}(S)$, for some $a, b \in \mathbb{Z}$, we use that $H \cdot C = \deg(\varphi_\eta(C)) = 2g-2$ and $R \cdot C = 2$, as C is hyperelliptic. It follows that

$$[C] = 2H + 4R.$$

Finally, we are ready to return to our initial goal, that is, to show (3.3). We consider the short exact sequence

$$0 \to \mathcal{I}_{S/\mathbb{P}^{g-2}} \to \mathcal{I}_{C/\mathbb{P}^{g-2}} \to \mathcal{I}_{C/S} \to 0,$$

we tensor it by $\mathcal{O}_{\mathbb{P}^{g-2}}(2)$ and we consider the corresponding exact long sequence with its cohomology. Then,

$$\begin{aligned} h^0(\mathcal{I}_{C/S}(2)) &= h^0(\mathcal{O}_S(2H - C)) \\ &= h^0(\mathcal{O}_S(-4R)) = 0 \end{aligned}$$

and

$$h^1(\mathcal{I}_{C/S}(2)) = h^1(\mathcal{O}_S(2H - C)) = h^1(\mathcal{O}_S(-4R))$$
$$= h^1(\mathcal{O}_{\mathbb{P}^1}(-4)) = h^0(\mathcal{O}_{\mathbb{P}^1}(2)) = 3,$$

as $\mathcal{O}_S(-4R)$ is $\pi^*(\mathcal{O}_{\mathbb{P}^1}(-4))$. Furthermore, since rational normal scrolls are projectively normal (see [ACGH85], p.98), $h^1(\mathcal{I}_{S/\mathbb{P}^{g-2}}(2)) = 0$, but it is easy to see that also $h^2(\mathcal{I}_{S/\mathbb{P}^{g-2}}(2)) = 0$.
It is now immediate that (3.3) holds. □

3.1.2 Syzygies of hyperelliptic curves

Green and Lazarsfeld conjectured that given a very ample line bundle $L \in \mathrm{Pic}(C)$ with $\deg(L) \geq 2g+1+p-2\cdot h^1(L)-\mathrm{Cliff}(C)$, then the property (N_p) holds for L unless the embedding φ_L defined by L embeds C with a $(p+2)$-secant p-plane ([GL86, Conj. 3.4]), and they verified it for $p = 0$ ([GL86, Thm 1]). Motivated by this conjecture, Park proved in [Par10] that the syzygies of a hyperelliptic curve C depend on the scroll S containing C. In order to state his result, we firstly recall some preliminary definitions.

Definition 3.8. *A line bundle* $L \in \mathrm{Pic}(C)$ *is* ***normalized*** *if* $H^0(C, L) \neq 0$ *while* $H^0(C, L(-g^1_2)) \neq 0$.

Definition 3.9. *We define the integer* m *as*

$$m = \max\{t \in \mathbb{Z} \mid H^0(C, \omega_C \otimes \eta(-tg^1_2) \neq 0)\}.$$

The pair $(m, 2g-2-2m)$ *is called the* ***factorization type*** *of* $\omega_C \otimes \eta$.

With the same notation as in Section 0.5,

Definition 3.10. *A minimal free resolution* $R(C)$ *is* ***m-regular*** *if* $\beta_{i,j} = 0$ *for all* $i \geq 0$ *and* $j \geq m$.
The Castelnuovo–Mumford regularity of $R(C)$ *is defined as*

$$\mathrm{Reg}(R(C)) = \min\{m \mid R(C) \text{ is } m- \text{ regular}\}.$$

The property (N_i) mentioned in Section 0.5 can be generalized in the following way.

Definition 3.11. *For* $k \geq 2$ *and* $p \geq 1$*, the property* $(N_{k,p})$ *of a curve* C *embedded through a line bundle* $L \in \mathrm{Pic}(C)$*, is defined by the vanishing* $\beta_{i,j}(C) = 0$ *for* $1 \leq i \leq p$ *and all* $j \geq k$.

For instance, C satisfies the property $N_{k,1}$ if its ideal $\mathcal{I}_C$ is generated in degrees $\leq k$, and property (N_i) for $i \geq 1$ is equivalent to C satisfying properties N_0 (that is, $\omega_C \otimes \eta$ is normally generated) and $N_{2,p}$.
We are now ready to state [Par10, Thm 1.3]:

Theorem 3.12. *Let C be a hyperelliptic curve of genus g and let $L \in \operatorname{Pic}^d(C)$ be a very ample line bundle of degree $d \leq 2g$ and factorization type (m, b). Let ν, τ be integers defined by*

$$\nu := \left\lceil \frac{b-1}{m+b-g-1} \right\rceil, \quad \tau := \left\lceil \frac{2g+1-b}{m} \right\rceil, \quad \text{and} \quad p := \nu(m+b-g-1) - b + 1.$$

Then, the graded Betti numbers of $\varphi_L(C) \subset \mathbb{P}^r$ are as follows:

- $\beta_{i,1}(C) = i\binom{r-1}{i+1}$ *for all* $i \geq 1$;
- $\beta_{i,j}(C) = 0$ *for* $2 \leq j \leq \tau - 1$ *and all* $i \geq 1$;
- $\beta_{i,\tau}(C) > 0$;
- $\beta_{1,\nu}(C) = \ldots = \beta_{p,\nu} = 0$ *and* $\beta_{r,\nu} = 2g + 1 - d$;
- $\beta_{i,\nu}(C) > 0$ *for all* $p + 1 \leq i \leq r$;
- $\beta_{i,j}(C) = 0$ *for all* $i \geq 1$ *and* $j \geq \nu + 1$.

Therefore, the Castelnuovo–Mumford regularity of $\varphi_L(C)$ is equal to $\nu + 1$. Also, $\varphi_L(C)$ satisfies property $N_{\nu,p}$ while it fails to satisfy property $N_{\nu,p+1}$.

As a consequence, we get the following result:

Corollary 3.13. *Let C be a hyperelliptic curve of genus g and let $\eta = \mathcal{O}_C(w_1 + \ldots + w_k - w_{k+1} - \ldots - w_{2k})$ with $w_1, \ldots, w_{2k}$ Weierstrass points and $3 \leq k \leq \lfloor \frac{g+1}{2} \rfloor$. Set*

$$\nu = \begin{cases} 5 & \text{for } k = 3 \\ 4 & \text{for } k = 4 \\ 3 & \text{for } k \geq 5 \end{cases} \qquad \text{and} \quad p = \nu(k-2) - 2k + 1.$$

Then:

i. the Castelnuovo–Mumford regularity of $\varphi_\eta(C)$ is $\nu + 1$;

ii. $\varphi_\eta(C)$ satisfies $N_{\nu,p}$ but fails property $N_{\nu,p+1}$.

Proof. Given a smooth hyperelliptic Prym curve (C, η) of genus g and φ_η the Prym-canonical embedding, $\varphi_\eta(C)$ is contained in the rational normal scroll $S = (g-1-k, k-2)$ with $2 \leq k \leq \lfloor \frac{g+1}{2} \rfloor$ (see Proposition 3.6 and Corollary 4.10), or equivalently, $S = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(g-2k+1))$. This is equivalent to saying that the factorization type of $\omega_C \otimes \eta$ is $(m, b) = (g-k-1, 2k)$ where $1 \leq k \leq \lfloor \frac{g+1}{2} \rfloor$. We compute the integers ν, τ, p under the assumption that $\omega_C \otimes \eta$ is very ample, or equivalently, $k \geq 3$:

$$\nu = \begin{cases} 5 & \text{for} \quad k = 3 \\ 4 & \text{for} \quad k = 4 \\ 3 & \text{for} \quad k \geq 5 \end{cases}$$

$$p = \nu(k-2) - 2k + 1.$$

□

Since $\tau = \lfloor \frac{2g+1-2k}{g-k-1} \rfloor = \lfloor 2 + \frac{3}{g-k-1} \rfloor$, Theorem 3.12 yields that it is possible to compute explicitly the graded Betti numbers.

3.2 General curves

Let $\mathcal{R}_g$ be the moduli space of Prym curves and consider a general curve $(C, \eta) \in \mathcal{R}_g$. Firstly, we show that the function $\mathrm{Cliff}_\eta(C)$ is a lower semicontinuous function in families of Prym curves. Then, the computation of the Prym-canonical Clifford index for a general curve will follow directly from the case of hyperelliptic curves. A brief introduction with background is given below, we refer to [ACG11, ch. XXI]. The basic existence theorem of the relative Picard variety is the following:

Theorem 3.14. *[ACG11, Thm 2.1, Ch. XXI, Sec. 2] Let d be an integer. Let $p \colon \mathcal{C} \to S$ be a family of smooth curves of genus $g > 1$ parametrized by a scheme S. Suppose that p admits a section. Then there exists a scheme over S*

$$\mathbf{Pic}^d(p) \to S$$

and a line bundle $\mathcal{L}_d = \mathcal{L}_d(p)$ over $\mathcal{C} \times_S \mathbf{Pic}^d(p)$ which restricts to a degree d line bundle on each fiber of p and which satisfies the following universal property. For every morphism $f \colon T \to S$ and every line bundle $\mathcal{L}$ on $\mathcal{C} \times_S T$, restricting to a degree d line bundle on each fiber of $q \colon \mathcal{C} \times_S T \to T$, there exists a unique lifting $\varphi \colon T \to \mathbf{Pic}^d(p)$ of f such that

$$\mathcal{L} = (\mathrm{id} \times \varphi)^* \mathcal{L}_d \otimes q^*(\mathcal{Q})$$

for some line bundle $\mathcal{Q}$ on T. The line bundle $\mathcal{L}_d$ is called a Poincaré bundle of degree d.

The scheme $\mathbf{Pic}^d(p)$ is the relative Picard variety. Given (C, p_1) a stable 1-pointed curve of genus g and $\nu \geq 3$ fixed, the ν-fold log canonical embedding of C in $\mathbb{P}^r$ is defined by the linear system $|\omega_C \otimes \mathcal{O}_C(p_1)^{\otimes \mu}|$, where $r = (2\mu - 1)(g - 1) + \mu n - 1$ (see [ACG11, Ch. XI, Sec. 5]). Under the additional assumption that $p \colon \mathcal{C} \to S$ is a family of μ-log canonically embedded, smooth, 1-pointed, genus g curve parametrized by a smooth connected variety S, we consider the relative symmetric product

$$\mathcal{C}_d \to S,$$

which may be viewed as the Hilbert scheme

$$\mathcal{C}_d = \mathrm{Hilb}^d_{\mathcal{C}/S}.$$

The idea of the proof of Theorem 3.14 is to define the scheme $\mathbf{Pic}^d(p)$ as the quotient $\mathcal{C}_d$ modulo an equivalence relation R (the existence of such an effective quotient is ensured by the "Scholie" Theorem due to Grothendieck, see [Gro95, Sec. XX]). For this purpose, we introduce the notion of relative universal effective divisor of degree d

$$\begin{array}{ccc} \mathcal{D} \subset \mathcal{C} \times \mathcal{C}_d & \xrightarrow{\pi} & \mathcal{C}_d \\ \big\downarrow & & \\ S & & \end{array},$$

which is defined as the universal family over $\mathrm{Hilb}^d_{\mathcal{C}/S}$. The adjective "universal" recalls that, as the universal divisor of degree d over a fixed curve (see [ACGH85, Lemma 2.1, Ch. IV]), the universal property holds for the divisor $\mathcal{D}$. At the same time, by "relative" one means that $\mathcal{D}$ is a Cartier divisor in $\mathcal{C} \times S$ which does not contain fibers of the projection onto S. In [ACG11, Ch. XXI] it is possible to read also the construction of $\mathbf{Pic}^d(p)$ in the analytic category. Before stating the next result, we recall that a family of Prym curves over a base scheme S consists of a triple $(\chi \xrightarrow{f} S, \epsilon, \beta)$, where $f \colon \chi \to S$ is a flat family of quasi-stable curves, $\epsilon \in \mathrm{Pic}(\chi)$ is a line bundle and $\beta \colon \epsilon^{\otimes 2} \to \mathcal{O}\chi$ is a sheaf homomorphism, such that for every point $s \in S$ the restriction $(X_s, \epsilon_{X_s}, \beta_{X_s} \colon \epsilon_{X_s}^{\otimes 2} \to \mathcal{O}_{X_s})$ is stable Prym curve of genus g.

Proposition 3.15. *The function*

$$(C, \eta) \mapsto \mathrm{Cliff}_\eta(C)$$

is lower semicontinuous in families of Prym curves.

Proof. Given a smooth family of Prym curves $p\colon \mathcal{C} \to S$, let $s \in S$ be a generic point and (C_s, η_s) a generic curve, where $C_s = p^{-1}(s)$. For some effective divisor D_s on C_s, consider a line bundle $L_s = \mathcal{O}_{C_s}(D_s) \in \mathrm{Pic}(C_s)$ such that $\mathrm{Cliff}_\eta(C_s) = \mathrm{Cliff}_\eta(L_s)$. Then, for some effective divisor E_s, $\eta_s = \mathcal{O}_{C_s}(D_s - E_s) \in \mathrm{Im}(\phi_s)$, where $\phi_s\colon C_{s,d} \times C_{s,d} \to \mathrm{Pic}^0(C_s)$ is the difference map induced on C_s. We have the following diagram:

$$\begin{array}{ccc} \mathcal{C}_d \times \mathcal{C}_d & \xrightarrow{\phi} & \mathbf{Pic}^0(p) \\ & \searrow^{\pi_d} & \downarrow p_d \\ & & S \end{array} \tag{3.4}$$

where ϕ is the relative difference map and $\pi_d\colon \mathcal{C}_d \times \mathcal{C}_d \to S$ is induced by p. Also, by definition of family of Prym curves, there exists a section $s \mapsto \eta_s \in \mathbf{Pic}^0(p)$. Let $0 \in S$ be a special point and $C_0 = p^{-1}(0)$. By the smoothness of the family p, one can extend the divisors D_s, E_s to 1-dimensional families of effective divisors contained in the relative universal effective divisor $\mathcal{D}$ of degree $\deg(D_s)$ such that at $0 \in S$, and one gets $\eta_0 = \mathcal{O}_{C_0}(D_0 - E_0) \in \mathrm{Pic}^0(C_0)$, that is, $\eta_0 \in \mathrm{Im}\phi_0$. Let us denote $L_0 = \mathcal{O}_{C_0}(D_0)$ and $L_0 \otimes \eta_0 = \mathcal{O}_{C_0}(E_0)$, then

$$\begin{aligned} \mathrm{Cliff}_{\eta_0}(L_0) &= \deg(L_0) - h^0(L_0) - h^0(L_0 \otimes \eta_0) + 1 \leq \\ &\leq \deg(L_s) - h^0(L_s) - h^0(L_s \otimes \eta_s) + 1 = \mathrm{Cliff}_{\eta_s}(L_s) \end{aligned}$$

because $h^0(L_0) \geq h^0(L_s)$ and $h^0(L_o \otimes \eta_0) \geq h^0(L_s \otimes \eta_s)$ by upper semicontinuity. This concludes the proof. □

Corollary 3.16. *Let $(C, \eta) \in \mathcal{R}_g$ be a general curve. Then,*

$$\mathrm{Cliff}_\eta(C) = \left\lfloor \frac{g-1}{2} \right\rfloor.$$

Proof. By Proposition 3.2, when $k = \lfloor \frac{g+1}{2} \rfloor$ and C is hyperelliptic, then $\mathrm{Cliff}_\eta(C) = \lfloor \frac{g+1}{2} \rfloor$. Let $p\colon \mathcal{C} \to S$ be a smooth family of Prym curves, $s \in S$ a generic point and $0 \in S$ a special point such that $C_0 = p^{-1}(0)$ is hyperelliptic and η_0 is such that $k = \lfloor \frac{g+1}{2} \rfloor$. Then, Proposition 3.15 implies that

$$\left\lfloor \frac{g-1}{2} \right\rfloor = \mathrm{Cliff}_{\eta_0}(C_0) \leq \mathrm{Cliff}_{\eta_s}(C_s).$$

Conversely, by Remark 1.6,

$$\mathrm{Cliff}_{\eta_s}(C_s) \leq \left\lfloor \frac{g-1}{2} \right\rfloor,$$

and this yields the conclusion. □

We remark that the above corollary, together with the inequality (1.5), implies that given a general curve $(C, \eta) \in \mathcal{R}_g$, then $V_e^{e-f}(\omega_C \otimes \eta) = \emptyset$ for $e - f < \lfloor \frac{g-1}{2} \rfloor$.

Chapter 4

Prym-canonical Clifford dimension of curves on Nikulin surfaces

In all the cases we considered (Examples 1.6.1, 1.6.2, 1.6.3, bielliptic curves in Theorem 2.13, hyperelliptic curves in Theorem 3.2, general curves in Corollary 3.16, curves with Prym-canonical Clifford index equal to 1 and 2 in Theorem 2.19 and 2.21), the Prym-canonical Clifford dimension of the curve is $(0,0)$. By analogy with the case of curves with Clifford dimension > 1 (see Section 0.4), we define the following:

Definition 4.1. *A curve C is **Prym-exceptional** if* $\mathrm{dimCliff}_\eta(C) \gneq (0,0)$.

We prove that it cannot be $(0,r')$ with $r' \geq 1$ (Lemma 1.8) or $(1,1)$ (Lemma 1.10). Therefore, a Prym-exceptional curve $(C,\eta) \in \mathcal{R}_g$ satisfies $\mathrm{dimCliff}_\eta(C) \gneq (1,1)$. Inspired by the example constructed by Eisenbud, Lange, Martens and Schreyer in [ELMS89], we consider curves on a special class of $K3$ surfaces. We recall that:

Definition 4.2. *A **Nikulin surface** is a $K3$ surface S endowed with a nontrivial double cover $\pi\colon \tilde{S} \to S$ with a branch divisor $N := N_1 + \ldots + N_8$ consisting of eight disjoint smooth (-2)-curves on S.*

Since by construction, the divisor N is divisible by 2 in $\mathrm{Pic}(S)$, there exists $M \in \mathrm{Pic}(S)$ such that $2M \sim \sum_{i=1}^{9} N_i$.

Definition 4.3. *The **Nikulin lattice** $\mathcal{N}$ is the rank 8 sublattice of $\mathrm{Pic}(S)$ generated by $N_1, \ldots, N_8$ and M.*

Given a smooth curve $C \subset S$ of genus g orthogonal to the Nikulin lattice, a polarized Nikulin surface $(S, \mathcal{O}_S(C))$ is *standard* if the embedding of the rank 9 lattice $\Lambda := \mathbb{Z}[C] \oplus_\perp \mathcal{N} \subset \mathrm{Pic}(S)$ is primitive, and *non-standard* otherwise. Blowing down the

(-1)-curves $E_i := \pi^{-1}(N_i) \subset \tilde{S}$, one obtains a minimal $K3$ surface $\sigma\colon \tilde{S} \to Y$, together with an involution $\iota \in \mathrm{Aut}(Y)$ having eight fixed points corresponding to the images $\sigma(E_i)$ of the exceptional divisors; the involuion ι is called a *Nikulin involution.* Moreover, one has the following standard diagram

$$\begin{array}{ccc} Y & \xleftarrow{\ \sigma\ } & \tilde{S} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \bar{Y} & \longleftarrow & S \end{array}$$

where $\bar{Y} = Y/\iota$ is the eight-nodal quotient of Y.
Nikulin surfaces are often used in the literature to consider Prym curves on $K3$ surfaces. Given S a standard Nikulin surface, $H = \mathcal{O}_S(C - M)$ is a polarization of genus $g-2$, and so it defines a morphism $S \to \mathbb{P}^{g-2}$, which is an embedding if S is general, e.g., $\mathrm{Pic}(S) \cong \Lambda$. There exists a natural candidate for the Prym-canonical bundle of $C \subset S$. Indeed, $\mathcal{O}_C(M)$ has degree zero, as $C \cdot N_i = 0$ $\forall i = 1, \ldots, 8$, and since $\mathcal{O}_C(N_1 + \ldots + N_8)_{|C} \simeq \mathcal{O}_C$ and $N_1 + \ldots + N_8 \sim 2M$, the line bundle $\eta = \mathcal{O}_C(M)$ is a non-trivial two torsion line bundle on C, so that $H_{|C} = \mathcal{O}_C(C - M) \simeq \omega_C \otimes \eta$.

We describe below two attempts to construct Prym-exceptional curves on a Nikulin surface. Our computations show, however, that the curves under consideration fail to satisfy the required condition.

4.1 Prym curves on Nikulin surfaces of Picard rank 10

We consider a lattice $\mathcal{N} \oplus_{\perp} (\mathbb{Z}[D] \oplus \mathbb{Z}[\Gamma])$, where $\mathcal{N}$ is the Nikulin lattice, and D and Γ are divisors such that $D^2 = 2r - 2$ with $r \geq 2$, $D.\Gamma = 1$, $\Gamma^2 = -2$. The intersection matrix has rank $\rho = 10$ and signature $(1, \rho - 1)$, and the lattice $\mathcal{N} \oplus_{\perp} (\mathbb{Z}[D] \oplus \mathbb{Z}[\Gamma])$ is even; Morrison Theorem yields that it occurs as the Picard group of some algebraic $K3$ surface S ([Mor84, Thm 2.9 (i)]). Since $D^2 > 0$, applying the Picard-Lefschetz reflections and replacing D with $-D$ if necessary, we may assume D nef.

Lemma 4.4. *Let S be a Nikulin surface with Picard lattice* $\mathrm{Pic}(S) = \mathcal{N} \oplus_{\perp} (\mathbb{Z}[D] \oplus \mathbb{Z}[\Gamma])$ *as above. If $R = \sum_{i=1}^{8} a_i N_i + bD + c\Gamma$ is an irreducible curve such that $R^2 = -2$ and $R \cdot D = 0$, then $R = \sum_i a_i N_i$.*
Moreover, Γ and N_i are irreducible for any $i = 1, \ldots, 8$.

Proof. Indeed, $R.D = 0$ yields that $(bD + c\Gamma).D = 0$, and since D is big and nef, the Hodge Index Theorem implies that $(bD + c\Gamma)^2 < 0$ (as $(bD + c\Gamma)^2 = 0$ would imply that $bD + c\Gamma$ is a multiple of D, in contradiction with $D^2 > 0$). Since R is a (-2)-curve,

$$-2\sum_{i=1}^{8}(a_i)^2 + (aD + b\Gamma)^2 = -2,$$

and so $a_i = 0$ for any $i = 1, \ldots, 8$, as $a_i \neq 0$ for some $i = 1, \ldots, 8$, would imply $-2\sum_{i=1}^{8}(a_i)^2 \leq -3$. However, $R.D = 0$ with $R = aD + b\Gamma$ contradicts [ELMS89, Lem. 4.2], where the authors show that the divisor D on a $K3$ surface with Picard group $\mathbb{Z}[D] \oplus \mathbb{Z}[\Gamma]$ is very ample.
This yields also that Γ is irreducible. Indeed, since $\Gamma.D = 1$ and any effective nonzero divisor having zero intersection with D is of the form $\sum_i a_i N_i$, we may write $\Gamma = \Gamma_0 + \sum_j \delta_j N_j$, for an irreducible and effective divisor Γ_0 such that $\Gamma_0^2 = -2$ and $\Gamma_0.D = 1$. Then, $\Gamma - \sum_j \delta_j N_j = \Gamma_0$ is effective, and $-2 = -2 - 2\sum_j(\delta_j)^2$ leads to the conclusion that any δ_j is zero. Using the same argument, one can conclude that also N_i is an irreducible curve for any $i = 1, \ldots, 8$. □

We recall the following result of Knutsen:

Proposition 4.5. *[Knu09, Prop. 4.1] Let L be a line bundle on a $K3$ surface S such that $L \sim 2D + \Gamma$ with D and Γ smooth curves satisfying $D^2 \geq 2$, $\Gamma^2 = -2$ and $\Gamma.D = 1$. Assume furthermore that there is no line B on S satisfying $0 \leq B^2 \leq D^2 - 1$ and $0 < B.L - B^2 \leq D^2$.*
Then $|L|$ is base point free and all smooth curves in $|L|$ are exceptional, of genus $g = 2D^2 + 2 \geq 6$, Clifford index $c = D^2 - 1 = \frac{g-4}{2}$ and Clifford dimension $r = \frac{1}{2}D^2 + 1$. Moreover, for any smooth curve $C \in |L|$ the Clifford index is computed only by $\mathcal{O}_C(D)$.

We will now prove the following:

Proposition 4.6. *Let S be a Nikulin surface such that $\mathrm{Pic}(S) = \mathcal{N} \oplus_\perp (\mathbb{Z}[D] \oplus \mathbb{Z}[\Gamma])$ with D and Γ as above, and consider the line bundle $L \sim 2D + \Gamma \in \mathrm{Pic}(S)$. Then, the linear system $|L|$ contains exceptional curves in the classical sense.*

Proof. First of all, we show that L is nef. Assume that L is not nef. Then, there exists an irreducible curve C such that $C \cdot L < 0$, which reads like $2C \cdot D + C \cdot \Gamma \geq C \cdot \Gamma$, where the last inequality occurs because D is nef. Then, $C \cdot \Gamma < 0$, and so $C = \Gamma$, as C and Γ are irreducible. It follows that $C \cdot L = \Gamma \cdot L = 0$, a contradiction.
At this point, we want to apply Proposition 4.5, and we assume that a divisor $B =$

$\sum_i \alpha'_i N_i + \beta' D + \delta'\Gamma$ such that $0 \leq B^2 \leq D^2 - 1$ and $0 < B.L - B^2 \leq D^2$ exists. Since

$$(L-B)^2 = L^2 + B^2 - 2L.B = L^2 - 2(B.L - B^2) - B^2$$
$$\overset{(i)}{\geq} L^2 - 2D^2 - D^2 + 1 = D^2 + 3 \geq 5,$$

and

$$(L-B).L = L^2 - (B.L) \overset{(ii)}{\geq} L^2 - (D^2 + B^2)$$
$$\overset{(iii)}{\geq} L^2 - 2D^2 + 1 = 2D^2 + 3 \geq 7,$$

where the inequalities $(i), (ii), (iii)$ follow from our assumptions on B, then both B and $L - B$ are effective, and thus,

$$B.L = \beta'(2D^2 + 1) \geq 0 \tag{4.1}$$

and

$$(L-B).L = (2 - \alpha')(2D^2 + 1) \geq 0. \tag{4.2}$$

Inequality (4.1) yields $\alpha' \geq 0$, but $\alpha' = 0$ would implies $B.L = 0$, and by the Hodge Index Theorem, $B^2 = 0$ and $B \sim kL$ for some integer k, and so the contradiction $L^2 = 0$. From inequality (4.2) one obtains $0 < \alpha' \leq 2$, and we claim that $\alpha' = 1$ and $B.L = 2D^2 + 1$. Indeed, $\alpha' = 2$ leads to the contradiction $(L - B).L = 0$. Then, our requirements on B read like $D^2 + 1 \leq B^2 \leq D^2 - 1$.
Hence, by Proposition 4.5 and [SD74, Prop. 2.6 (i)], the generic member of $|2D + \Gamma|$ is a smooth, irreducible, exceptional curve (in the sense of [ELMS89]), and the line bundle computing its Clifford index is $\mathcal{O}_C(D)$. The Clifford index of C is then $2r - 3$. (see [ELMS89, Thm 4.3], [ELMS89, Thm 3.6]). □

At this point, we may ask whether $\mathcal{O}_C(D)$ with $h^0(C, \mathcal{O}_C(D)) = r + 1 \geq 2$ and $\deg(D) = C.D = 4r - 3$ computes the Prym-canonical Clifford dimension of (C, η) with $\eta = \mathcal{O}_C(M)$. For the proof of the next result, we will make use of the following result due to Knutsen and Lopez.

Theorem 4.7. *[KL07] Let X be a $K3$ surface and let L be a line bundle on X such that $L > 0$ and $L^2 \geq 0$. Then $H^1(L) \neq 0$ if and only if one of the two following occurs:*

1. *$L \sim nE$ for $E > 0$ nef and primitive with $E^2 = 0$, $n \geq 2$ and $h^1(L) = n - 1$;*
2. *there is a divisor $\Delta > 0$ such that $\Delta^2 = -2$ and $\Delta.L \leq -2$.*

Lemma 4.8. *Let S be a Nikulin surface with* $\mathrm{Pic}(S) = \mathcal{N} \oplus_\perp (\mathbb{Z}[D] \oplus \mathbb{Z}[\Gamma])$ *with D and Γ as above, and consider $C \in |2D + \Gamma|$. Then,*

$$\mathrm{Cliff}_\eta(\mathcal{O}_C(D)) = 2r - 2.$$

Proof. The short exact sequence

$$0 \to \mathcal{O}_S(D - M - C) \to \mathcal{O}_S(D - M) \to \mathcal{O}_C(D - M) \to 0$$

yields that

$$h^0(\mathcal{O}_C(D \otimes \eta)) = h^0(\mathcal{O}_C(D - M)) \geq h^0(\mathcal{O}_S(D - M)) \geq \chi(\mathcal{O}_S(D - M)) = r - 1 \quad (4.3)$$

because $h^0(\mathcal{O}_S(D - M - C)) = 0$ as $\mathcal{O}_S(D - M - C) = \mathcal{O}_S(-(D + \Gamma + M))$ is not effective. We now show that the two inequalities in (4.3) are equalities. We need to show that

$$h^1(\mathcal{O}_S(D - M)) = 0 = h^1(\mathcal{O}_S(D - M - C)). \quad (4.4)$$

We assume that there exists an effective divisor

$$\Delta = \sum_i \alpha_i N_i + \beta D + \gamma \Gamma \in \mathrm{Pic}(S) \quad (4.5)$$

for $\alpha_i \in \frac{1}{2}\mathbb{Z}$ and some integers β, γ, such that $\Delta^2 = -2$, and $\Delta(D - M) \leq -2$. Then,

$$(D - M - \Delta)^2 = (D - M)^2 - 2(D - M)\Delta + \Delta^2 \geq (D - M)^2 + 2, \quad (4.6)$$

and since Hodge Index Theorem implies that

$$(D - M - \Delta)^2.D^2 \leq [(D - M - \Delta).D]^2,$$

by inequality (4.6) one gets that $[(D - M)^2 + 2].D^2 \leq (D - M - \Delta)^2.D^2$, and so

$$(D^2 - 2).D^2 \leq [(D - M - \Delta).D]^2 = (D^2 - \Delta.D)^2 \overset{(iv)}{\leq} (D^2 - 1)^2, \quad (4.7)$$

where for the inequality (iv) we used $\Delta.D \geq 1$, as D is nef. Note that (4.7) yields that $0 \leq \Delta.D \leq 2$, and thus $D^2 = 2r - 2 \leq 2$, that is, $r \leq 2$. However, when $r = 2$, Theorem 4.7 gives no information, and we will treat this case separately.
If $\Delta.D = 0$, then $\gamma = -2(r - 1)\beta$, and $\Delta.M \geq 2$ yields that $\sum_i \alpha_i \leq -2$. Since Δ is a (-2)-curve,

$$-\sum_i \alpha_i^2 + (r - 1)\beta^2 - 4(r - 1)^2\beta^2 - 2\beta^2(r - 1) = -1,$$

which can be rewritten as

$$-\sum_i \alpha_i^2 = (r-1)(4r-3)\beta^2 - 1, \tag{4.8}$$

which is absurd. Indeed, if $r = 1$, then $\sum \alpha_i^2 = 1$, that is, there exists an integer $1 \leq j \leq 8$ such that $\alpha_j = 1$ and $\alpha_i = 0$ for any $i \neq j$, in contradiction with $\sum \alpha_i \leq -2$. If $r \geq 2$, the absurdity arises because $-\sum_i \alpha_i^2 < 0$, while the right hand side of (4.8) is ≥ 1.
The case $\Delta.D = 1$ cannot occur. Indeed, it implies $\gamma = 1 - 2(r-1)\beta$ and $\sum_i \alpha_i \leq -3$, and so $\Delta^2 = -2$ can be rewritten as

$$-\sum \alpha_i^2 - (4(r-1)^2 + (r-1))\beta^2 + (4(r-1)+1)\beta - 1 = -1,$$

that is equivalent to

$$-\sum_i \alpha_i^2 = (4r-3)\beta[(r-1)\beta - 1]$$

which is a contradiction, whether β is positive or negative.
If $\Delta.D = 2$, then $r = 2$, and $(D-M)^2 = -2$; inequality in (4.6) gives no information. Assume that $h^1(\mathcal{O}_S(D-M) \geq 1$, then Riemann-Roch yields that

$$h^0(\mathcal{O}_S(D-M)) = \chi(\mathcal{O}_S(D-M)) + h^1(\mathcal{O}_S(D-M)) \geq 2,$$

meaning that the linear system $|\mathcal{O}_S(D-M)|$ can be written as

$$|\mathcal{O}_S(D-M)| = |B| + F, \tag{4.9}$$

where $B \in \mathrm{Pic}(S)$ is such that $h^0(S, \mathcal{O}_S(B)) \geq 2$ and $|B|$ has no fixed components (and thus by [SD74, Cor. 3.2] is base-point free), and $F \neq \emptyset$ has $h^0(\mathcal{O}_S(F)) = 1$. Together with (4.9), this yields that

$$2 = (D-M).D = (B+F).D = B.D + F.D,$$

where the first equality comes by the assumption $r = 2$. Since $B^2 \geq 0$ and D is irreducible, then $B \cdot D \geq 2$, and thus $F.D = 0$. By Lemma 4.4, then $F = \sum_i \xi_i N_i$ with $\xi_i \geq 0$ $\forall i$. Then,

$$B = D - M - F = D - \sum_i \left(\xi_i + \frac{1}{2}\right) N_i,$$

and since $h^0(S, \mathcal{O}_S(B)) \geq 2$,

$$0 \leq B^2 = D^2 - 2\sum_i \left(\gamma_i + \frac{1}{2}\right)^2 = 2\Big[1 - \sum_i \left(\xi_i + \frac{1}{2}\right)^2\Big]. \tag{4.10}$$

The inequality in (4.10) implies that $\forall i\ \xi_i = 0$, namely $F = 0$, which is a contradiction.

We now prove the second vanishing in (4.3), namely, $h^1(S, \mathcal{O}_S(D - M - C)) = 0$. We remark that $D - M - C = -(D + M + \Gamma)$, and thus by Serre duality,

$$H^1(S, \mathcal{O}_S(D - M - C)) = H^1(S, \mathcal{O}_S(D + M + \Gamma)),$$

which can be written as $D + M + \Gamma = (D - M) + \Gamma + N_1 + \ldots + N_8$. We consider the following short exact sequence holds

$$0 \to \mathcal{O}_S(D-M) \to \mathcal{O}_S(D-M+\Gamma+N_1+\ldots+N_8) \to \mathcal{O}_{\Gamma+N_1+\ldots+N_8}(D-M+\Gamma+N_1+\ldots+N_8) \to 0;$$

Since $\Gamma^2 = N_i^2 = -2$, $\Gamma \cdot N_i = N_i \cdot N_j = 0$, $N_i \cdot M = -1$, $\Gamma \cdot M = 0$, $D \cdot N_i = 0$, $D \cdot \Gamma = 1$, the right-hand side term is isomorphic to $\bigoplus_9 \mathcal{O}_{\mathbb{P}^1}(-1)$, which together with $h^1(\mathcal{O}_S(D - M)) = 0$ just proved, finally implies $h^1(\mathcal{O}_S(D + M + \Gamma)) = 0$.

Hence, the Prym-canonical Clifford index of the divisor D of degree $D \cdot C = 4r - 3$ is:

$$\mathrm{Cliff}_\eta(\mathcal{O}_C(D)) = 4r - 3 - (r + 1) - (r - 1) + 1 = 2r - 2.$$

□

We also remark that $\mathrm{Cliff}(C) = \mathrm{Cliff}(\mathcal{O}_C(D)) = 2r - 3$.
It is natural to ask whether $\mathcal{O}_C(D)$ computes the Prym-canonical Clifford dimension of C. This would imply that (C, η) is Prym-exceptional. However, since the difference map $\phi_d \colon C^d \times C^d \to \mathrm{Pic}^0(C)$ is surjective for $d \geq \lfloor \frac{g+1}{2} \rfloor = 2r - 1$, there are effective divisors F, F' of degree $2r - 1$ such that $\eta = \mathcal{O}_C(F - F')$. Thus $\mathrm{Cliff}_\eta(\mathcal{O}_C(F)) \leq \deg(F) - 1 = 2r - 2$, and $\mathcal{O}_C(D)$ does not compute $\mathrm{dimCliff}_\eta(C)$.
In conclusion, given a $K3$ surface S with $\mathrm{Pic}(S) = \mathcal{N} \oplus_\perp (\mathbb{Z}[D] \oplus \mathbb{Z}[\Gamma])$, a curve $C \sim 2D + \Gamma$ on S is exceptional, and $\mathcal{O}_C(D)$ computes its Clifford dimension; however, $\mathcal{O}_C(D)$ does not compute $\mathrm{dimCliff}_\eta(C)$.

4.2 Prym curves on Nikulin surfaces of Picard rank 9

We provide an example of Prym curve (C, η) with Prym-canonical Clifford index less than $\lfloor \frac{g-1}{2} \rfloor$. In particular, we take a Nikulin surface S with Picard group $\mathrm{Pic}(S) = \mathcal{N} \oplus_\perp \mathbb{Z}(D)$, with D such that $D^2 = 2r - 2$, and a curve $C \in |2D|$. Then, the divisor D has degree $4r - 4$, and the genus of C is $g(C) = 4r - 3$, so that a general $(C, \eta) \in \mathcal{R}_{g(C)}$ has Prym-canonical Clifford index $\mathrm{Cliff}_\eta(C) = \lfloor \frac{g-1}{2} \rfloor = 2r - 2$. We want to compute $\mathrm{Cliff}_\eta(\mathcal{O}_C(D))$.

Proposition 4.9. *Let S be a Nikulin surface with* $\mathrm{Pic}(S) = \mathcal{N} \oplus_\perp (\mathbb{Z}[D])$ *with D such that $D^2 = 2r-2$, and consider $C \in |2D+\Gamma|$. Then,*

$$\mathrm{Cliff}_\eta(\mathcal{O}_C(D)) = 2r-3.$$

Proof. Since D is nef, Theorem 4.7 implies $h^1(\mathcal{O}_S(D)) = 0$, and thus $h^0(\mathcal{O}_C(D)) = \chi(\mathcal{O}_S(D)) = r+1$. To compute $h^0(\mathcal{O}_C(D)\otimes\eta) = h^0(\mathcal{O}_C(D-M))$, we consider the long exact sequence in cohomology associated with

$$0 \to \mathcal{O}_S(D-M-C) \to \mathcal{O}_S(D-M) \to \mathcal{O}_C(D-M) \to 0.$$

One gets that $h^0(\mathcal{O}_S(D-M-C)) = h^0(\mathcal{O}_S(-D-M)) = 0$, and by arguments similar to those used to prove the equalities in (4.4), one shows also the vanishing of $h^1(\mathcal{O}_S(D-M)) = 0$ and $h^1(\mathcal{O}_S(-D-M)) = h^1(\mathcal{O}_S(D+M)) = 0$. It follows that $h^0(\mathcal{O}_C(D-M)) = r-1$, and so

$$\mathrm{Cliff}_\eta(\mathcal{O}_C(D)) = 4r-4-r-1-r+1+1 = 2r-3.$$

□

We ask again whether $\mathcal{O}_C(D)$ computes the Prym-canonical Clifford dimension of C, thus implying that (C,η) is Prym-exceptional. However, the following computations strongly suggest that there exists a line bundle $L \in \mathrm{Pic}(C)$ such that $h^0(L) = h^0(L\otimes\eta) = 1$, and $\mathrm{Cliff}_\eta(L) = \deg(L)-1 \leq \mathrm{Cliff}_\eta(\mathcal{O}_C(D)) = 2r-3$. This implies that $\mathcal{O}_C(D)$ does not compute $\mathrm{dimCliff}_\eta(C)$.
Let us assume that such a line bundle $L \in \mathrm{Pic}(C)$ of degree $2r-2$ exists, and consider the only effective divisor $\xi \in |L|$. Riemann–Roch yields that

$$h^0(\omega_C \otimes \eta(-\xi)) = g(C) - \deg(\xi) = 2r-1.$$

Since $\omega_C\otimes\eta = \mathcal{O}_C(2D-M)$, one has the following short exact sequence

$$0 \to \mathcal{O}_S(-M) \to \mathcal{O}_S(2D-M)\otimes\mathcal{I}_{\xi/S} \to \omega_C\otimes\eta(-\xi) \to 0,$$

which implies $h^0(\mathcal{O}_S(2D-M)\otimes\mathcal{I}_{\xi/S}) = h^0(\omega_C\otimes\eta(-\xi)) = 2r-1$.
At this point, we consider a curve $C' \in |2D-M|$ of degree $4r-4$ and with genus $g(C') = 4r-5$. The following short exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(2D-M)\otimes\mathcal{I}_{\xi/S} \to \omega_{C'}(-\xi) \to 0$$

yields that $h^0(\omega_{C'}(-\xi)) = h^0(\mathcal{O}_S(2D-M)\otimes\mathcal{I}_{\xi/S}) - 1 = 2r-2$. Then, by Riemann–Roch, one gets that $h^0(\mathcal{O}_{C'}(\xi)) = h^0(\omega_{C'}(-\xi)) + \deg(\xi) - g(C') + 1 = 2$, that is, $\mathcal{O}_{C'}(\xi)$ moves

in a g^1_{2r-2} on $C' \in |2D-M|$. Since $\mathrm{Cliff}(C') \leq \mathrm{Cliff}(\mathcal{O}_{C'}(D)) = 2r-4$, and C' is not Prym-exceptional, then C' has linear series of type g^1_{2r-2} (we expect that there exists a 0-dimensional family of g^1_{2r-2} on C', and that the Lazarsfeld-Mukai bundle of a g^1_{2r-2} is $\mathcal{O}_S(D) \oplus \mathcal{O}_S(D-M)$).
Denoting by $\mathcal{H}ilb^{2r-2}(S)$ the Hilbert scheme of 0-dimensional subscheme of S of lenght $2r-2$, we now consider the incidence variety

$$I = \{(C', \xi, C) \ \in |2D-M| \times \mathcal{H}ilb^{2r-2}(S) \times |2D+\Gamma| \mid \xi \subset C \cap C',\ \mathcal{O}_{C'}(\xi) = g^1_{2r-2}\},$$

and the two projections $\pi_1 \colon I_1 \to \mathcal{H}ilb^{2r-2}(S)$, where

$$I_1 = \{(C', \xi) \ \in |2D-M| \times \mathcal{H}ilb^{2r-2}(S) \mid \mathcal{O}_{C'}(\xi) = g^1_{2r-2}\}$$

has dimension $4r-4$, and $\pi_2 \colon I_2 \to \mathcal{H}ilb^{2r-2}(S)$, where

$$I_2 = \{(C, \xi) \ \in |2D+\Gamma| \times \mathcal{H}ilb^{2r-2}(S) \mid \xi \subset C\}.$$

One can check that the dimension of the fibers of π_1 coincide with $h^0(\mathcal{O}_S(2D-M) \otimes \mathcal{I}_{\xi/S}) = 2r-2$, implying that dim $\mathrm{Im}(\pi_1) = 2r-2$, and the fibers of π_2 have dimension $h^0(\mathcal{O}_S(2D) \otimes \mathcal{I}_{\xi/S}) = h^0(\omega_C(-\xi)) = g(C) - \deg(\xi) = 2r-1$. Then, $\dim I_2 = 2r-2+2r-1 = \dim|2D|$, leading to the surjectivity of $q \colon I_2 \to |2D|$ (if q has finite-dimensional fibers, as we believe).

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