CORSO DI DOTTORATO DI RICERCA IN MATEMATICA

XXXV CICLO

A spread of infection model on branching trees

Candidato:
Davide Ciaccia

Docente Guida:
Prof. Alexandre Stauffer

Coordinatore:
Prof. Alessandro Giuliani

Summary

This work contains a discussion on the spread of infection, a model for interacting particles system, in the specific setting of Galton-Watson branching trees.

It is organized in five chapters: in Chapter 1, we recall the basics of interacting particle systems and quickly provide some classical examples. Then we observe more recent evolutions in this subject, in particular by focusing on a class of models known in literature as the spread of infection. We recall one of its basic definitions, examine a couple of specific models, state some known results as well as some open questions, and finally consider more recent works on different types of graphs.

Chapter 2 is dedicated to Galton-Watson tree and it serves as a preparation for our model to be introduced in the next chapter. Here we recall the construction of Galton-Watson trees and examine some properties which we are going to use to prove our results later on.

In Chapter 3 we define a model for the spread of infection on Galton-Watson trees, mention some of its basic properties, consider the survival problem and state two theorems which provide sufficient conditions for its solution in two different scenarios.

Our results will then be proved in Chapter 4, while Chapter 5 is dedicated to some open problems.

Contents

Chapter 1

Interacting particle systems

In this first chapter we present some basic elements in the theory of interacting particle systems. Most of the topics in this introduction are inspired from [4]. This branch of the probability studies was born in the late 1960's, with most of initial works due to Spitzer in the US and Dobrushin in the Soviet Union. Its motivation came from statistical mechanics, more specifically the phase transition phenomenon that often characterizes those models, which is the change of behaviour in the long term evolution of the system as the parameters involved in its definition take different values. Since the beginning, other mathematical models have been adapted to be treated as interacting particle systems and therefore studied with the new tools this theory has developed along the way.

1.1 Introduction

From a pure mathematical perspective, an interacting particle systems can be seen as a further step forward with respect to the theory of Markov processes. We explain this with an example: consider a countable family of particles performing a discrete-time random walk[1] over the set $\{0,1\}$, each one independently of the others. Let $\eta_t(j)$ be the position of the j-th particle at time t. Imagine now to view the entire process η_t as a whole Markov process on the (uncountable) space $\{0,1\}^{\mathbb{N}}$. Whichever initial configuration η_0 is chosen, the distributions η_t at a given time t are a product of measures, with each one of them being singular with respect to the others and with respect to the invariant measure of the process. If some kind of interaction is added, for instance we could add some dependence from the position of the other particles in the transition probabilities, this singularity may be kept, however we no longer have independence-based results and therefore need new techniques.

---

[1] At every time unit, each particle chooses position 0 or 1 with probability $\frac{1}{2}$ each and then moves to it, independently from the other particles and from its present location.

More generally, an interacting particle systems can be summarized as a process involving a finite of infinite quantity of particles (or agents) moving on some space, which is usually taken to be the vertex set V of some finite or infinite graph G. On top of this, some interaction rule is added to the system. Typical questions and results about these processes concern the long term behaviour of their evolution, such as convergence to some equilibrium distribution or the eventual realization of some event. This practical description provided above will be sufficient for our purpose. Indeed a precise construction would require tools related to functional analysis and Markov theory. A formal approach to these procedures can be found in chapters 1, 2 and 3 from [4], whereas we used [25] as a reference for Markov processes. For now, let us move on and consider some examples, starting from their historical formulations.

1.2 An example: the contact process

In this section we shall keep the notation η used before to indicate a generic configuration in the state space. We now begin by providing the classic definition of the *contact process.*

1.2.1 Historical definition and first results

The contact process was introduced by Harris in 1974 and is defined on the state space $\{0,1\}^{\mathbb{Z}^d}$. This means that each element of $\mathbb{Z}^d$ is assigned an attribute η_x equal to 0 or 1. In this setting we consider the integer lattice $\mathbb{Z}^d$ as a graph, therefore we may call its element vertices or sites. An edge exists between two sites x and y if and only if $|x-y|=1$, which mean that either they are equal or they differ by 1 in exactly one coordinate. A possible interpretation for the contact process is as a model for the *spread of infection.* In this sense, a site x with $\eta(x)=1$ is called infected, whereas if $\eta(x)=0$ we say that x is healthy.

The evolution rule is the following: each x in $\mathbb{Z}^d$ with $\eta(x)=1$ waits independently of the others for an exponential clock with rate 1 and then changes its attribute to 0, while sites x with $\eta(x)=0$ change their state to 1 similarly, however their exponential clock rate is equal to

$$\lambda \sum_{y:|y-x|=1} \eta(y)$$

where λ is a positive parameter which is usually called *infection rate.* Notice that the starting configuration μ with all healthy particles is stationary. Moreover, for other initial configurations, it is possible that the distribution η_t converges to some distribution μ' as $t \to \infty$ as indicated in the Markov chains convergence theorem ([25], Theorem 20.1). In this case we say that μ' is an *invariant measure.*

The first studies (until the early 1990s) have analyzed whether there exist other (nontrivial) invariant measures for the process on the integer lattice $\mathbb{Z}^d$.

In [4], Chapter 6 we see that for any fixed dimension $d \geq 1$ there exists a critical value $\lambda_c = \lambda_c(d)$ such that μ is the sole invariant measure for every $\lambda < \lambda_c(d)$; in this case, we say that the infection *dies out* almost surely. On the other hand, it is possible to find at least one additional invariant measure if $\lambda > \lambda_c(d)$, which means that the infection can survive with positive probability for all times, provided that there is at least one infected site at the beginning. This change of global behaviour of the model according to the values of the parameter λ is known as *phase transition.* A later study by Bezuidenhout and Grimmett (see [5]) showed that the infection dies out almost surely at the critical value as well. Finally, there have been numerical attempts to establish some bounds for the critical value, see for example [10] for the one-dimensional process.

1.2.2 Further developments

With the long-term behaviour of the model on the integer lattice being completely understood, focus has then shifted towards the analysis of the contact process on different graphs other than $\mathbb{Z}^d$. Pemantle ([8], 1992) and Stacey ([13], 1996) began by considering homogeneous trees with degree d (each vertex has d neighbours), showing that for every $d \geq 4$ the model can exhibit three possible behaviours, these being separated by two critical values for the parameter λ. More precisely, they proved that there exist two values $\lambda_1 < \lambda_2$ such that the infection dies out almost surely if $\lambda < \lambda_1$, whereas there is positive probability for the infection process starting from a single particle to survive for all times whenever $\lambda > \lambda_1$. However, if $\lambda > \lambda_2$ we have that any fixed vertex v of the tree gets infected infinitely many times with positive probability. This does not occur in the intermediate regime $\lambda_1 < \lambda < \lambda_2$, in which the infection tends to drift away from any finite region, thus infecting each vertex finitely many times almost surely. The former case is known as *strong survival* of the infection, while the latter one is referred to as *weak survival.* The case $d = 3$, which was initially left open, was then solved by Liggett in 1996, who managed to prove that analogue results hold for the model on the binary tree as well (see [12]).

1.2.3 Introducing the contact process on random graphs

In recent years, the growing interest towards more complex (and possibly random) network structures has motivated the analysis of the contact process on even more graphs, such as *Galton-Watson trees* or other types of random graphs. A Galton-Watson tree or *branching tree* is a random graph generated by starting from a single vertex - the *root* and adding a number of additional vertices according to some distribution X. This rule is then repeated countable-many times by taking the sites which were added during the previous step and attaching to each of them independently a number of new vertices, always according to X. Such distribution is usually called *offspring law.* Chapter 2 will be dedicated to Galton-Watson trees, by provide a more formal construction as well as an analysis of some of its properties.

1.2.4 Heavy-tailed, subexponential and light-tailed distributions

Let us consider a random variable Y which takes positive integer values[2] and put $y_k := \mathbb{P}(Y = k)$, $P_k := \mathbb{P}(Y \geq k)$. Since the structure of a branching tree can vary significantly depending on the offspring law, in this section we distinguish among different distribution classes we can find. Definitions 1.1 and 1.2 are taken directly from [24], wheras we refer to [29], Section 1.2 for the definition 1.3.

Definition 1.1. We say that Y is a *heavy-tailed* distribution if

$$\mathbb{E}[e^{\theta Y}] = +\infty$$

for all $\theta > 0$, that is if Y does not admit any positive exponential moment.

Definition 1.2. We say that Y is a *light-tailed* distribution if it is not heavy-tailed.

Definition 1.3. Here we consider a subcategory of heavy-tailed distributions. We say that Y is a *subexponential* distribution if

$$\limsup_{k \to +\infty} \frac{\log{(p_k)}}{k} = 0 \tag{1.1}$$

Notice that starting from condition (1.1) we get

$$\limsup_{k \to +\infty} \frac{\log{(p_k)}}{k} = 0 \iff \limsup_{k \to +\infty} (p_k)^{\frac{1}{k}} = 1 \iff \forall\ \epsilon \in (0,1)\ p_{k_j} > (1-\epsilon)^{k_j} \tag{1.2}$$

for infinitely many indexes k_j.

1.2.5 Contact process on random graphs: some results

In a paper from 2018, Huang and Durrett managed to prove that, in a contact process on a Galton-Watson tree with one infected particle at the beginning, the infection can survive with positive probability. Here is their result:

Theorem 1.4 ([29], Theorem 3)**.** *Consider the contact process on the Galton-Watson tree with some subexponential offspring distribution X such that $\mathbb{E}[X] > 1$, and suppose that only the root of the tree is initially infected. Then for all choices of the parameter λ there is positive probability for the infection to survive for all times.*

Shortly thereafter in 2019, Bhamidi, Nam, Nguyen and Sly showed that the result from Huang and Durett was sharp, in the sense that follows:

---

[2] Random variables which take such values are of interest because of the models we are considering and therefore we shall focus on them, however the definitions will make sense for real-valued distributions as well.

Theorem 1.5 ([30], Theorem 1)**.** *Consider the contact process on the Galton-Watson tree with offspring distribution* X*, and suppose that only the root of the tree is initially infected. If* X *satisfies* $\mathbb{E}[e^{cX}] < +\infty$ *for some* $c > 0$*, then there exists* $\lambda_0 = \lambda_0(X) > 0$ *such that for all* $\lambda < \lambda_0$ *the infection process dies out almost surely.*

In the same paper, the autors also proved that for random graphs with n vertices and degree distribution μ which satisfies some conditions, the contact process exhibits a phase transition, in the sense that there exist two critical thresholds $\lambda_1 < \lambda_2$ such that with high probability the infection survives for a time interval with size at most $n^{1+o(1)}$ whenever the parameter λ is below λ_1, whereas if $\lambda > \lambda_2$ survival occurs for a time window with size at least $e^{\Theta(n)}$ with high probability.

1.3 Spread of infection

In this section we focus on the spread of infection as a family of interacting particle systems, mentioning some additional examples and examining some related results which have been proved so far. As we saw in section 1.2.1, the contact process itself can be interpreted as a model for the spread of infection. This time we shall consider different models which will distinguish between a *working environment* and *agents* or *particles* that interact somehow within it, for example by spreading the infection, whereas the vertices of a graph in the contact process are involved in both actions simultaneously.

More formally, we will have different types of particles (at least two types, in which case they are usually referred to as *healthy* or *infected*) that move on the set V of vertices of some graph through its edges. The criteria that establish how particles move may or may not depend on their type. Given an initial configuration, the process evolves according to some other rule which explains how transitions from one type to another can be accomplished. Let us begin by considering another example known as the *frog model*.

1.3.1 The frog model

In the frog model we have two types of particles, S (*suscettible*) or I (*infected*). Initially, on every vertex of $\mathbb{Z}^d$ we put a number of particles of type S equal to a Poisson random variable with parameter $\mu > 0$, independently per each vertex. We may write Poisson(μ) to denote such distribution. Then, place an additional I particle at the origin. I particles wait for an exponential clock with rate 1, select a neighbour uniformly at random and move to it. This happens independently per each I particle. On the other hand, S particles do not move (equivalently, they jump with rate 0). The interaction between the two types is that every time an I particle encounters a S particle, the latter turns to I type (and consequently start moving according to the rule explained before). This mechanism can be represented in short with the formula

$$S + I \to 2I \tag{1.3}$$

We will refer to this process as frog model *with SI dynamics*. This was initially introduced in 1999 by Telcs and Wormald [14] as a branching random walk where particles branch every time a new vertex is visited for the first time.

As first result they proved the following fact:

Theorem 1.6 ([14], Theorem 5)**.** *Given $d \geq 1$, in the frog model in $\mathbb{Z}^d$ each particle will eventually become infected almost surely.*

With this result accomplished, another natural question involves the speed at which infected particles move. More precisely, define ν_t^I as the number of the infected particles at time t on $\mathbb{Z}^d$. Moreover, given $r > 0$ let B_r be the open ball in $\mathbb{Z}^d$ centred in the origin and with radius r. We say that the infection propagates with positive speed if there exists $c = c(d, \mu) > 0$ such that

$$\nu_t^I \left(\mathbb{Z}^d \setminus B_{ct}\right) \geq 1$$

for all t sufficiently large, almost surely. The following holds:

Theorem 1.7 (Alves, Machado, Popov and Ravishankar [15])**.** *The frog model with SI dynamics has positive speed for all $d \geq 2$ and any $\mu > 0$.*

Actually, in [15] they proved a stronger claim, which we do not present here. Further results have been achieved in dimension $d = 1$ by Kesten, Ramírez and Sidoravicius [23].

1.3.2 SIS and SIR dynamics

Let us now examine two variants of the frog model, although they can be applied to any other model for the spread of infection. This time, the evolution involves a second parameter λ. With *SIS* dynamics we add the possibility for each infected particles to heal after an exponential clock with rate λ. In short, together with the previous rule (1.3) we can write

$$\begin{aligned} S + I &\to 2I \\ I &\xrightarrow{\lambda} S \end{aligned}$$

Observe that the first rule implies that, whenever an I particle turns to S type, any eventual I particles on the same location will convert it immediately to I type again. As a consequence, I particles effectively heal when they are alone on a site.

Similarly, the *SIR* dynamics involves a third type of particle, the *removed* or *dead* particles, denoted by R. The difference with the SIS model is that, after an exponential clock with rate λ, an I particle dies instead of recovering, therefore turning to type R. Since a R particle cannot turn to S or I again, it plays no role in the spread of the infection. This time we have the formulae

$$S + I \to 2I$$
$$I \xrightarrow{\lambda} R$$

Other models could be defined by combining SIS and SIR, however we ignore them as they are not necessary for our purpose.

For the variants described above, we say that the infection *survives* if

$$\mathbb{P}\left(\nu_t^I(\mathbb{Z}^d) \geq 1 \ \forall \ t \geq 0\right) > 0$$

Otherwise, the infection is said to *die out*. Notice that there is always positive probability that the infection *die out*: for example, in the origin there might be only one infected particle at the beginning, and there is positive probability for it to recover (or die) before it can jump or other particles can jump to the origin; this holds for all dimensions d and for every choice of the two parameters μ and λ. A natural question is therefore whether the infection survives with positive probability.

1.3.3 Phase transition in SIR dynamics

For the frog model it was proved that in dimension 1 survival does not occur:

Theorem 1.8 (Alves, Machado and Popov [16], Theorem 1.1)**.** *In the frog model with SIR dynamics, for* $d = 1$ *the infection dies out almost surely for all* $\mu > 0$ *and for all* $\lambda > 0$*.*

On the other hand, in higher dimensions we have that:

Theorem 1.9 (Alves, Machado and Popov [16], Theorems 1.3-1.5)**.** *In the frog model with SIR dynamics, for all* $d \geq 2$ *and all* $\mu > 0$ *there exists* λ_c *such that the infection survives for all* $\lambda < \lambda_c$*, otherwise it dies out.*

The existence of a critical parameter separating two different regimes for the long-term evolution of the model is called *phase transition* (this time, with respect to the parameter λ).

As μ increases, it becomes more difficult for infected particles to heal, as they are less frequently occupying a site without other particles. This leaves the open question of whether the infection can survive in the SIS model when the particle density is large enough, regardless of the recuperation rate.

We also want to mention the works from Alves, Machado and Pupov [16] as well as Lebensztayn and Utra [28], which considered a discrete-time, modified version of the frog model in which at every step particles can die with some positive probability $1 - p$. Here we report some of the main results from Alves, Machado and Pupov:

- In $\mathbb{Z}$, there are sufficient conditions on the initial configuration for the infection to die out almost surely;

- Both on a regular tree with degrees d and in $\mathbb{Z}^d$ with $d \geq 2$, the infection dies out almost surely if the parameter p is small enough;
- If p is close enough to 1, the infection survives with positive probability both on $\mathbb{Z}^d$ (for $d \geq 2$) and on the regular tree with degrees d (for $d \geq 3$).

Lebensztayn and Utra considered the same process but on biregular[3] trees. They proved that there exists a critical value for the death parameter p separating the two regimes of almost sure extinction and survival with positive probability.

1.4 A more general model

Now we consider another model for the spread of infection, where we also allow susceptible particles S to move, with the same jump rate 1 as the infected ones. This addition makes the analysis of the model more challenging, as S particles now have the possibility to avoid encountering the I particles. We will refer to this model generically as *spread of infection* (notice that the contact process and the frog model belong to the spread of infection category as well, but since this new model we just defined does not have a specific name, we prefer to keep the expression *spread of infection* for this last one and use *frog model* or *contact process* for the other two).

Here it is possible to consider the SI dynamics, which we recall here in short

$$S + I \to 2I$$

or, alternatively, the SIS or SIR dynamics, given by

$$\begin{aligned} S + I &\to 2I \\ I &\overset{\lambda}{\to} S \end{aligned} \qquad \text{and} \qquad \begin{aligned} S + I &\to 2I \\ I &\overset{\lambda}{\to} R \end{aligned}$$

respectively.

1.4.1 Results for SI, SIS and SIR dynamics

Recall that we say that there is positive speed when there exists a constant $c = c(d, \mu) > 0$ such that

$$\nu_t^I \left(\mathbb{Z}^d \setminus B_{ct}\right) \geq 1$$

for all t sufficiently large, almost surely. For the SI dynamics, we have the following:

Theorem 1.10 (Kesten and Sidoravicius [17], Theorems 1 and 2)**.** *There exists* $c = c(d, \mu)$ *such that* $\nu_t^I \left(\mathbb{Z}^d \setminus B_{ct}\right) \geq 1$ *for all* t *sufficiently large.*

---

[3] A tree is *biregular* with degrees d_1 and d_2 if, when viewed as a bipartite graph, vertices belonging to the same partition have the same degree, which needs to be either of the values d_1 or d_2.

Later on, Kesten and Sidoravicius also proved an even stronger result about the geometry of the set of infected particles, as it evolves in time.

Moving on to the SIS and SIR dynamics, recall from Section 1.3.2 we say that the infection *survives* if

$$\mathbb{P}\left(\nu_t^I(\mathbb{Z}^d) \geq 1 \ \forall \ t \geq 0\right) > 0$$

and otherwise we say that it *dies out.* For the SIS dynamics, the model exhibits a phase transition:

Theorem 1.11 (Kesten and Sidoravicius [18], Theorem 1)**.** *In the SIS dynamics for the spread of infection, for all $d \geq 1$ and for all $\mu > 0$ there exists λ_c such that the infection survives with positive probability if $\lambda < \lambda_c$.*

The proof is based on a slightly modified model, where the infection of an S particle is only possible either when it moves to some I particle or, on the contrary, some I particle moves to its position.

Baldasso and Stauffer ([32], Theorem 1.2) later proved that, for any density $\mu > 0$ and provided that the recovery rate $\lambda > 0$ is small enough, strong survival occurs almost surely when conditioned on weak survival. They also established ([32], Theorem 1.4) that the infection can always survive with positive probability for all recovery rates $\lambda > 0$ if the particle density μ is large enough. Note that the above result does not hold in the model defined by Kesten and Sidoravicius in [18], because it relies on the argument that particles can only heal when they are alone at a site.

Additionally, Dauvergne and Sly [33] extended a result from Kesten and Sidoravicius from [17] by showing that positive speed of propagation occurs for the SIS model even in the case where susceptible and infected particles move at different rates. Furthermore, in [31] they proved that, in the SIR model, the infection survives for all times with positive probability whenever the parameter μ is small enough. They also showed that the infection tends to abandon quickly any finite region it reaches, by leaving a herd immunity regime with mostly removed and few susceptible particles only.

1.5 Spread of infection on other graphs

More recent studies have considered the spread of infection on graph structures other than $\mathbb{Z}^d$. As an example, let us mention a couple of works about the frog model with SI dynamics on regular d-ary trees. We say that a realization of the model is *recurrent* if the root of the tree is visited infinitely many times and it is *transient* otherwise. Hoffman, Johnson and Junge proved ([26], Theorem 1) almost sure recurrence for $d = 2$ and almost sure transience for $d \geq 5$. For the same model, they also proved ([27], Theorem 1.1) that the distance of the farthest infected particle from the root grows linearly in time with high probability, provided that the particle density is large enough.

Motivated by the results recollected in this chapter, in this work we are going to propose a spread of infection model with SIS dynamics on a

Galton-Watson tree and state sufficient conditions for the infection to survive. Before proceeding, it is necessary to establish some properties of Galton-Watson trees that we will make use of: next chapter will be dedicated to this.

Chapter 2

Galton-Watson trees

After having taken a glance at some examples of interacting particle systems, in this chapter we move forward and consider a *Galton-Watson* or *branching* tree. This serves as a preparation for Chapter 3, where such a tree will be the environment on which we establish a model for the spread of infection.

2.1 Introduction

The discussion that has led to the present model known as Galton-Watson process began in the early 1870s by F. Galton, who published a problem about the diffusion of male family names among a given population, and H. W. Watson, who submitted a possible solution (see [1] and [2]). Although the current formulation of the model differs much from the very first problem posed by Galton, it still retains its original interpretation.

In modern terms - see for instance Harris [3] - by Galton-Watson tree we mean a random tree $\mathcal{T}$ originated from the branching process, which can be constructed as follows: choose some random variable X which takes non negative integer values and such that $\mathbb{P}(X = 0) > 0$, then let $\mathcal{T}_0$ be a tree with one vertex $\{r\}$ only. Define inductively the tree $\mathcal{T}_n$ by taking each vertex v with distance $n-1$ from r in $\mathcal{T}_{n-1}$ and adding to it a number of leaves according to the law of X. These are usually referred to as the *children* of v and v on the other hand is called their *parent* vertex. Repeat this for every such v independently. If no vertex with said property can be found in $\mathcal{T}_{n-1}$, simply put $\mathcal{T}_n := \mathcal{T}_{n-1}$. Finally, take

$$\mathcal{T} = \bigcup_{n=0}^{+\infty} \mathcal{T}_n$$

A usual interpretation of $\mathcal{T}$ is *population growth*: starting from one element, each individual is given once in their lifetime the opportunity to create copies of themselves, and it does so according to X, which is typically referred to as the *offspring law.* The resulting final tree $\mathcal{T}$ is simply a genealogical map of said population. An interesting first object of studies in the branching process is the *survival* event

$$\mathcal{S} := \{|\mathcal{T}| = +\infty\}$$

The complement event of $\mathcal{S}$ is called *extinction*. The following holds:

Theorem 2.1. *[Harris [3], Theorem 6.1] Consider a branching process with offspring law X and let $m := \mathbb{E}[X]$. The survival event $\mathcal{S}$ has positive probability if and only if either $m > 1$ or $m = 1$ and $X = 1$ almost surely.*

In other words, no matter how complicated the distribution X can be, its expected value is the only crucial parameter in establishing whether the event $\mathcal{S}$ is possible. Naturally, survival is also guaranteed if we assume that $X \geq 1$ almost surely. The case $m > 1$ is often called *supercritical regime*. We will see that the results concerning our model turn to be trivial if we do not assume that the tree is infinite, therefore we shall always consider this supercritical scenario.

2.1.1 Some notations

Limited portions of the tree

Let Z_0 be the root and, for every $n \geq 1$, let

$$Z_n := \mathcal{T}_n \setminus \mathcal{T}_{n-1}$$

be the *n-th generation*, that is the set of vertices at distance n from the root.

Also, given a vertex v, let $\mathcal{T}(v)$ be the subtree of $\mathcal{T}$ which is obtained by taking v as the root (generation zero), then its offspring as the first generation, and so on. Similarly, let $\mathcal{T}_k(v)$ be the portion of $\mathcal{T}(v)$ which only contains the first k generations of vertices, and let $Z_k(v)$ be the set of vertices in $\mathcal{T}(v)$ at distance k from v.

Cutting out vertices

Another construction that we will make use of is the following: in a tree $\mathcal{T}$, with root v and S a set of some of its children, we denote by $\mathcal{T}^S$ the tree obtained from $\mathcal{T}$ by removing the edges between v and all of its children $w \in S^c$, and then taking the connected component containing v.

Paths and descending paths

A path is a finite or infinite sequence of vertices $P = (v_0, v_1, v_2, ...)$ such that, for every $i \geq 0$, the edge (v_i, v_{i+1}) exists in the tree. We say that a path is *descending* if v_i is the parent of v_{i+1} for all $i \geq 0$.

Given two vertices v and w, let $\{v \to w\}$ denote the event that there exists a descending path from v to w, provided that the definition makes sense given the positions of v and w. Similarly, given a vertex v and a collection A of vertices, we define

$$\{v \to A\} = \bigcup_{w \in A} \{v \to w\} \tag{2.1}$$

Finally, let $\{v \to \infty\}$ be the event that there exists an infinite descending path starting from v.

2.2 The spine decomposition

With the expression *spine decomposition* we mean a different representation of a Galton-Watson tree for the first k generations, provided that it survives up to that point. The following construction is inspired from Lyons, Pemantle and Peres [11], Section 2. Now, starting from the root, for every n choose some ordering among the vertices $w_1, ..., w_{Z_n} \in Z_n$. Since we have survival up to the k-th generation, among the offspring of the root select the first vertex v_1 that belongs to a descending path of length k, starting from the root.

Because of this choice, for every $v \in Z_1$ with $v < v_1$ (according to the ordering), since we have independence, its offspring will not survive up to generation k in the original tree, which means it will die out before its $(k-1)$-th branching step. As for the vertices $v \in Z_1$ with $v > v_1$, in general we do not have any information about their offspring. Proceed then by considering v_1 as the new root; this produces another vertex v_2 at distance 2 from the root with similar properties. Iterate this up to the $(k-1)$-th generation.

We are now left with a subtree $\tilde{\mathcal{T}}_k$ of $\mathcal{T}_k$ in which for every $1 \leq n \leq k-1$ we have appointed a special vertex $v_n \in Z_n$ belonging to a path of length k, every $v \in Z_n$ with $v < v_n$ whose offspring does not reach Z_k and for every $v \in Z_n$ with $v > v_n$ do not need to satisfy any condition. We call the sequence $v_1, ..., v_k$ a *spine*. Note that, conditioning on survival of the tree, the spine can be taken to be an infinite sequence. Also, if we are interested in only the portion of $\tilde{\mathcal{T}}_k$ that effectively goes from the root to Z_k, simply delete for each n all of the vertices $v < v_n$ first. Then, take a look at the vertices $v > v_n$: each of they may or may not belong to a path of length k and therefore they will be kept or deleted accordingly. Go on for all vertices at a given generation and then for all generations until all of the vertices have been analysed. Let us call the final remaining tree $\mathcal{T}'_k$ the *spined* version of $\mathcal{T}_k$. $\mathcal{T}'_k$ satisfies the property that every vertex up to the $(k-1)$-th generation has at least 1 offspring.

Moreover, we want to note that, starting from $\mathcal{T}'_k$, it is possible to construct a *weighted* tree $\mathcal{T}''_k$, with the property that every vertex in $\mathcal{T}''_k$ up to the $(k-1)$-th generation has at least 2 offspring. The procedure is simple: first, contract every path $\mathcal{P} = (w_1, ..., w_l)$ in $\mathcal{T}'_k$ which consists of only edges (w_i, w_{i+1}), with $\deg(w_i) = \deg(w_{i+1}) = 2$, into a single edge (w_1, w_l), then assign it the weight l which is the length of the compressed path. Lastly, assign weight 1 to all of the remaining edges. The advantage of such construction is that inside $\mathcal{T}''_k$ there is a regular binary tree. We shall see that having such hypothesis will make some computations easier. $\mathcal{T}''_k$ will be referred to as the *compressed spined* version of $\mathcal{T}_k$.

2.3 The k-th generation Z_k

When looking at a truncated portion $\mathcal{T}_k$ of the whole tree, it might be useful to consider some properties of Z_k, the farthest level from the root in $\mathcal{T}_k$.

2.3.1 The size of Z_k given $|Z_k| > 0$

Keeping in mind that we shall examine the behaviour of an infection process, in this section we collect other minor results on Galton-Watson trees that will prove to be crucial in the next chapter. Indeed at some point we will need to be able to find a lot of vertices far from the root and, among them, having some with sufficiently high degree. Therefore we introduce the following technical lemma:

Lemma 2.2. *Consider a supercritical Galton-Watson tree with offspring law X. Given $k \geq 0$ there exists a constant $c > 0$, depending only on X, such that*

$$\mathbb{P}\left(|Z_k| < 2^{ck} \mid |Z_k| > 0\right) \leq \exp\left(-ck\right) \tag{2.2}$$

Proof of Lemma 2.2. If $X \geq 2$ almost surely then $m \geq 2$ as well, and there are at least 2^k vertices at distance k from the root almost surely, as in this case $\mathcal{T}$ contains a binary tree.

Let us now focus on the general scenario where $\mathbb{P}(X \leq 1) > 0$. Here we make use of the spine and compressed spine decompositions: firstly, fix a distance h from the root (to be chosen afterwards). From the given tree $\mathcal{T}$ consider the spined version $\mathcal{T}'$ where each vertex in $\mathcal{T}_k$ produces at least 1 child. Because $\mathcal{T}$ has more vertices than $\mathcal{T}'$, it suffices to prove the statement for $\mathcal{T}'$. Let also $\mathcal{T}''$ be the compressed spined version of $\mathcal{T}$. Fix a constant $0 < c < 1$, let $h = ck$ and remark that

$$|Z''_h| \geq 2^h \tag{2.3}$$

Given $w \in Z''_h$, let X_w be the sum of the weights W_e over the edges e in the path (in $\mathcal{T}''$) from the root to w. Recall that the weight W_e indicates that the edge $e \in \mathcal{T}''_k$ corresponds to a contraction of a path in $\mathcal{T}'_k$ which connects the two endpoints of e and with length W_e. Since the $\{W_e\}$ are independent, geometric random variables with parameter p equal to the probability that the offspring law X is equal to 1, we can use a Chernoff bound for X_w (see section A.2) and obtain that for every constant $\epsilon > 0$:

$$\begin{aligned}\mathbb{P}\left(X_w > (1+\epsilon)\frac{h}{p}\right) &= \mathbb{P}\left(W_{e_1} + ... + W_{e_h} > (1+\epsilon)\frac{h}{p}\right)\\ &\leq \exp\left(-\frac{\epsilon^2}{1+\epsilon^2} \cdot \frac{1+\epsilon}{2}h\right)\end{aligned} \tag{2.4}$$

Now this holds for every $w \in Z_h$; let us put

$$b(\epsilon) := \frac{1+\epsilon}{p} \qquad g(\epsilon) := \frac{\epsilon^2}{1+\epsilon^2} \cdot \frac{1+\epsilon}{2} \tag{2.5}$$

Also, let $\mathcal{B}$ be the event

$$\mathcal{B} := \{\exists\ w \in Z''_h : X_w > b(\epsilon)h\} \tag{2.6}$$

We want to show that no one of these X_w can exceed $b(\epsilon)h$ with high probability as h grows. Indeed by making use of the notation with (2.5) and applying the inequalities (2.3) and (2.4), together with the union bound, we can get

$$\begin{aligned}
\mathbb{P}(\mathcal{B}) &= \mathbb{P}\left(\bigcup_{w \in Z''_h} \{X_w > b(\epsilon)h\}\right) \\
&\le \sum_{w \in Z''_h} \mathbb{P}(X_w > b(\epsilon)h) \\
&\underset{\substack{(2.3)\\(2.4)}}{\le} 2^h \exp(-g(\epsilon)h) \\
&= \exp\left[-h\left(g(\epsilon) - \log 2\right)\right] \\
&\le \exp(-h) \left(\underset{h=ck}{=} \exp(-ck)\right)
\end{aligned} \tag{2.7}$$

here in the second last passage we take $\epsilon > 0$ such that $g(\epsilon) \ge 1 + \log 2$, which is possible because $g(0) = 0$ and $\lim\limits_{\epsilon \to +\infty} g(\epsilon) = +\infty$. Since we have chosen a specific value for ϵ, from now on let $b := b(\epsilon)$. Notice that b now only depends on the law of X because of p.

So inequality (2.7) effectively shows that $\mathcal{B}$ occurs with exponentially small probability. Now let us assume that it does not happen: we know that the vertices in Z''_h we have considered before, when viewed as part of $\mathcal{T}'$, find themselves at distance at most bh from the root. Take now

$$c = \frac{1}{b} \tag{2.8}$$

so that

$$bh = bck = k$$

since every vertex in $\mathcal{T}'$ has at least degree 2, here it is possible to find at least $2^h = 2^{ck}$ vertices at distance k from the root. As a consequence, even the original tree $\mathcal{T}$ has at least 2^{ck} vertices at generation k. Finally, observe that from (2.8) we have that c only depends on the law of X. This completes the proof. □

2.3.2 Survival of the tree up to a given generation

The previous lemma was based upon conditioning on the event $\{|Z_k| > 0\}$. However, what could we say if we did not know that such event is verified?

Lemma 2.3. *Let $\mathcal{T}$ be a supercritical Galton-Watson tree with offspring distribution X and let v be its root. Given $\Gamma > 0$, let $\mathbb{P}_\Gamma$ be the probability measure conditioned on the event $\{\deg(v) = \Gamma\}$. Let $w_1, ..., w_{\Gamma-1}$ be the children of the root. Choose some $k > 0$ and recall from section 2.1.1 that $\{w_j \to Z_k\}$ denotes the existence of a descending path from w_j to Z_k. Let ω_j be the indicator of the event $\{w_j \to Z_k\}$. Notice that, since the offspring of the different w_js all have the same probability, we have*

$$\pi_k := \mathbb{P}(w_1 \to Z_k) = \mathbb{P}(w_j \to Z_k)$$

for all j. Define the event

$$\mathcal{Y}_v^k := \left\{\omega_1 + ... + \omega_{\Gamma-1} > \frac{\pi_k}{2}(\Gamma - 1)\right\} \tag{2.9}$$

We have

$$\mathbb{P}_\Gamma\left(\left(\mathcal{Y}_v^k\right)^c\right) \leq \exp\left(-\frac{\pi_k}{8}\Gamma\right) \tag{2.10}$$

Proof of Lemma 2.3. The claim is obvious if the offspring distribution X satisfies $X \geq 1$ almost surely. If $\mathbb{P}(X = 0) > 0$, then the proof is an immediate application of a Chernoff bound for the sum of the i.i.d. Bernoulli variables $\omega_1, ..., \omega_{\Gamma-1}$ as from Theorem A.2. □

2.3.3 Random walks: transience, recurrence

A simple random walk is said *transient* if, for any given vertex w, the process returns to w finite times almost surely. We say that it is *recurrent* if it is not transient. For a supercritical, infinite branching tree it has been proved (see [6], theorem 4.3 and proposition 6.4) that a simple random walk is transient.

Now let v be the root of $\mathcal{T}$ and pick a child w of v. A further consequence of transience is that a random walk starting on w will visit w a finite times almost surely, thus there is positive probability that it never jumps to the root v. We conclude this section with the following observation:

Observation 2.4. In analogy with Lemma 2.3, a similar claim can be stated and proved for events other than $\{w \to Z_k\}$, like for example $\{w \to \infty\}$, the transience or the event that, starting from some child w of the root v, a random walk never visits the root for all positive times. Simply replace the events $\{w_j \to \infty\}$, their indicator functions ω_j and their probability $\pi_k := \mathbb{P}(w_1 \to Z_k)$ accordingly.

2.4 Looking for high-degree vertices

In the section we will examine some results about the distribution of the maximum degree of a possibly large group of vertices. In this case, however, the possibility of establishing valid estimates strongly depends on the properties of the offspring distribution X, therefore we will need to make some assumptions about it.

2.4.1 Defining a new subset of the heavy-tailed distributions

Recall that in section 1.2.4 we defined heavy-tailed, subexponential and light-tailed distribution. Also, the results from Huang and Durrett [29] and Bhamidi, Nam, Nguyen and Sly [30] for the contact process that we introduced in section 1.2.5 distinguished between subexponential and light-tailed distribution. For our model, we require a slightly different alternative to the subexponential distributions. Indeed in section 3.3.2 we will present a first result for Galton-Watson trees with offspring law X which satisfies the following:

$$\forall \theta > 0 \; \exists \; k_0 = k_0(\theta) > 0 \;:\; \mathbb{P}(X \geq k) > \exp(-\theta k) \qquad \forall k \geq k_0 \tag{2.11}$$

Let us observe first that condition (2.11) for a distribution is stronger than belonging to the heavy-tailed class:

Lemma 2.5. *Let R be a random variable that satisfies equation (2.11). Then R does not admit any exponential moments, that is $\mathbb{E}\left[e^{\theta R}\right] = +\infty$ for all $\theta > 0$.*

Proof of Lemma 2.5. Let

$$f_n := \mathbb{P}(R = n), \qquad F_n := \mathbb{P}(R \geq n)$$

and choose some $\theta > 0$. We have

$$\begin{aligned} \mathbb{E}\left[e^{\theta R}\right] &= \sum_{n=1}^{+\infty} e^{\theta n} f_n \\ &= \sum_{n=1}^{+\infty} e^{\theta n} \left(F_n - F_{n+1}\right) \end{aligned}$$

summation by parts yields

$$\begin{aligned} \sum_{n=1}^{+\infty} e^{\theta n} \left(F_n - F_{n+1}\right) &= e^{\theta} F_1 + \sum_{n=2}^{+\infty} \left(e^{\theta n} - e^{\theta(n-1)}\right) F_n \\ &= e^{\theta} F_1 + \left(1 - e^{-\theta}\right) \sum_{n=2}^{+\infty} e^{\theta n} F_n \end{aligned}$$

in the last series, observe that thanks to (2.11) applied with $\frac{\theta}{2}$, for large n we have

$$e^{\theta n} F_n > e^{\frac{\theta}{2} n}$$

and therefore

$$\sum_{n=2}^{+\infty} e^{\theta n} F_n > \sum_{n=2}^{+\infty} e^{\frac{\theta}{2} n} = +\infty$$

thus completing the proof. □

Observe now that the converse is not true, in the sense that it is possible to define a random distribution R which is heavy-tailed but does not satisfy (2.11). Here is an example:

Example 2.6. Let us construct a non-negative random variable R by specifying $F_n := \mathbb{P}(R \geq n)$. Choose some $\delta > 0$ and let $F_0 := 1$, $F_1 := e^{-\delta}$. Now there exists n_1 such that

$$F_1 = e^{-\delta} = e^{-\frac{\delta}{2} n_1} 00$$

therefore, let $F_n \equiv F_1$ for all $1 \leq n \leq n_1$.

At this point we proceed inductively: let us assume that we have defined F_j for all $0 \leq j \leq n_j$ and put

$$F_{n_j+1} := \exp\left(-2^j \delta \left(n_j + 1\right)\right)$$

Then, take n_{j+1} such that

$$\exp\left(-2^j \delta \left(n_j + 1\right)\right) = \exp\left(-\frac{\delta}{2^{j+1}} n_{j+1}\right) \qquad \Longleftrightarrow \qquad n_{j+1} = 2^{2j+1}\left(n_j + 1\right) \tag{2.12}$$

and let

$$F_n \equiv \exp\left(-\frac{\delta}{2^{j+1}} n_{j+1}\right) \quad \forall\ n_j + 1 \leq n \leq n_{j+1}$$

At this point, note that $f_n = \mathbb{P}(R = n)$ satisfies

$$f_0 = 1 - \exp\left(-\delta\right),\ f_{n_j} = \exp\left(-\frac{\delta}{2^j} n_j\right) - \exp\left(-2^j \delta \left(n_j + 1\right)\right),\ f_n = 0 \text{ if } n \neq 0, n_j$$

As a consequence, given $\theta > 0$ we have

$$\begin{aligned}
\mathbb{E}\left[e^{\theta X}\right] &= \sum_{n=0}^{+\infty} e^{\theta n} f_n \\
&= 1 - e^{-\delta} + \sum_{j=1}^{+\infty} \exp(\theta n_j)\left(\exp\left(-\frac{\delta}{2^j} n_j\right) - \exp\left(-2^j \delta \left(n_j + 1\right)\right)\right) \\
&> 1 - e^{-\delta} + \sum_{j=1}^{+\infty} \exp\left(\left(\theta - \frac{\delta}{2^j}\right) n_j\right) = +\infty
\end{aligned}$$

thus R cannot have any finite exponential moments.

Now observe that for every $k \geq 0$ we can choose $j(k)$ such that

$$2^{j(k)-1} \delta n_{j(k)} > \theta k$$

This is possible because (2.12) implies that the sequence $\left\{n_j\right\}_j$ is increasing and therefore, for the same reason, so is the sequence $\left\{2^{j-1}\delta n_j\right\}_j$. As a consequence,

$$2^j \delta \left(n_j + 1\right) > \theta k \implies F_{n_j} = \exp\left(-2^{j-1}\delta n_j\right) < \exp\left(-\theta k\right)$$

for all $j \geq j(k)$. Finally, if j also satisfies $n_j > k$ we have

$$F_{n_j} = \exp\left(-2^{j-1}\delta n_j\right) < \exp\left(-\theta k\right)$$

which means that R does not satify (2.11).

2.4.2 The maximum of i.i.d. random variables

Now that the relevant distributions for our model have been defined, let us go back to the examination of Galton-Watson trees properties. The next step is this short technical lemma about the maximum of a collection of i.i.d. random variables.

Lemma 2.7. *Let $X_1, ..., X_n$ be i.i.d. random variables that satisfy the property (2.11) and fix $\theta > 0$. For $M = M(\theta) > 0$ sufficiently large we have*

$$\mathbb{P}\left(\max\left\{X_1, ..., X_n\right\} < M\right) \leq \exp\left(-n \exp\left(-\theta M\right)\right) \tag{2.13}$$

Proof of Lemma 2.7. We have

$$\begin{aligned}\mathbb{P}\left(\max\left\{X_1, ..., X_n\right\} < M\right) &= \mathbb{P}\left(X_1 < M\right)^n \\ &= \left(1 - \mathbb{P}\left(X_1 \geq M\right)\right)^n \\ &\underset{(2.11)}{<} \left(1 - \exp\left(-\theta M\right)\right)^n \\ &\leq \exp\left(-n \exp\left(-\theta M\right)\right)\end{aligned}$$

where (2.11) requires that $M \geq M(\theta)$. □

2.4.3 The maximum degree among vertices in Z_k

We finish this chapter with one last lemma about the maximum degree we can expect to find among the vertices located some generations away from the root.

Lemma 2.8. *Consider a supercritical Galton-Watson tree $\mathcal{T}$ with offspring distribution X. Let v be the root of $\mathcal{T}$ and, for any given $\Gamma \geq 1$, let $\mathbb{P}_\Gamma$ be the probability measure obtained by conditioning on the event $\{\deg(v) = \Gamma\}$. Choose $\epsilon > 0$.*

(a) There exist two constants $c, \Gamma_0 > 0$, depending only on the law of X, such that the event

$$\mathcal{M}_{v,\epsilon} := \left\{|Z_{\epsilon\Gamma}| \geq 2^{c\epsilon\Gamma}\right\} \tag{2.14}$$

satifies

$$\mathbb{P}_\Gamma\left(\mathcal{M}_{v,\epsilon}^c\right) < \exp(-c\epsilon\Gamma) \tag{2.15}$$

provided that $\Gamma \geq \Gamma_0$.

(b) Assume now that X satisfies the property (2.11) and define the event

$$\mathcal{D}_{v,\epsilon} := \left\{\max_{w \in Z_{\epsilon\Gamma}} \deg(w) \geq 2\Gamma\right\} \tag{2.16}$$

There exist two constants $c' > 0$, depending only on the law of X, and Γ_0', depending on the law of X and on ϵ, such that

$$\mathbb{P}_\Gamma\left(\mathcal{D}_{v,\epsilon}^c | \mathcal{M}_{v,\epsilon}\right) < \exp(-c'\epsilon\Gamma) \tag{2.17}$$

for all that $\Gamma \geq \Gamma_0'$.

In words, provided that Γ is large enough, with very high probability we can find at least $2^{b\epsilon\Gamma}$ vertices in $Z_{\epsilon\Gamma}$ and at least one of them will have a degree which is not smaller than 2Γ. Before proceeding with the proof, we just want to anticipate here that the events (2.14) and (2.16) will play a fundamental role in our main result.

Proof of Lemma 2.8.

(a) If $X \geq 2$ then there are at least $2^{\epsilon\Gamma}$ vertices in $|Z_{\epsilon\Gamma}|$ and thus the claim is verified with $c = 1$, so assume that this is not the case. An application of Lemma 2.3 implies that the event $\mathcal{Y}_v^{\epsilon\Gamma}$ in (2.9) satisfies

$$\mathbb{P}_\Gamma\left(\left(\mathcal{Y}_v^{\epsilon\Gamma}\right)^c\right) \leq \exp\left(-\frac{\pi_{\epsilon\Gamma}}{8}\Gamma\right) \leq \exp\left(-\frac{\pi_\infty}{8}\Gamma\right) \tag{2.18}$$

where recall that, for every $w \in S$, $\pi_{\epsilon\Gamma} = \mathbb{P}(w \to Z_{\epsilon\Gamma})$ and $\pi_\infty = \mathbb{P}(w \to \infty)$; the last passage in (2.18) has the advantage that π_∞ only depends on the offspring distribution. Then, choose Γ large enough so that

$$\frac{\pi_\infty}{2}(\Gamma - 1) > 1 \qquad \Longleftrightarrow \qquad \Gamma > 1 + \frac{2}{\pi_\infty} =: \Gamma_0 \tag{2.19}$$

so that there is at least one good child with good property $\{w \to Z_{\epsilon\Gamma}\}$. Finally, under the event $\mathcal{Y}_v^{\epsilon\Gamma}$ we have $|Z_{\epsilon\Gamma}| > 0$, thus Lemma 2.2 can be applied with k replaced by $\epsilon\Gamma$, so that there exists $\tilde{c} > 0$ such that

$$\mathbb{P}_\Gamma(\mathcal{M}_{v,\epsilon}^c \mid \mathcal{Y}_v^{\epsilon\Gamma}) < \exp(-\tilde{c}\epsilon\Gamma) \tag{2.20}$$

In conclusion, since $\tilde{c}$ only depends on X, we simply combine (2.18) and (2.20) with $c := \frac{\pi_\infty}{8} + \tilde{c}$ to get (2.15).

(b) For this second part, observe first that since the degree D of a vertex is equal to its offspring X in the case of the root, whereas $D = 1 + X$ for all of the other vertices, we can say that properties (2.11) or (**??**) are valid for D if and only if they hold for X. Now we assume that $\mathcal{M}_{v,\epsilon}$ holds and therefore select a group of $2^{\tilde{c}\epsilon\Gamma}$ vertices in $Z_{\epsilon\Gamma}$. At this point we apply Lemma 2.7 to their degrees, with $n = 2^{\tilde{c}\epsilon\Gamma}$ and M replaced by 2Γ. We obtain

$$\mathbb{P}_\Gamma\left(\mathcal{D}_{v,\epsilon}^c | \mathcal{M}_{v,\epsilon}\right) \leq \exp\left(-\exp\left(\Gamma(\epsilon\tilde{c}\log 2 - 2\theta)\right)\right) \tag{2.21}$$

now (2.21) must hold for all $\theta > 0$, provided that $\Gamma \geq \Gamma_0'(\theta)$ is large enough. Therefore we can select $\theta = \frac{\epsilon}{j}$ with j such that

$$\tilde{c}\log 2 - \frac{2}{j} > 0 \qquad \Longleftrightarrow \qquad j > \frac{2}{\tilde{c}\log 2 - 1} \tag{2.22}$$

Now from (2.22) we have that j is a function of $\tilde{c}$ which only depends on X, therefore so does the constant $c' := \left(\tilde{c}\log 2 - \frac{2}{j}\right)$. Now that θ has been chosen, simply put $\Gamma_0' := \Gamma_0'(\theta)$.

Now we prefer to write

$$\exp\left(-\exp\left(c'\epsilon\Gamma\right)\right) \leq \exp\left(-c'\epsilon\Gamma\right) \tag{2.23}$$

because, despite the fact that (2.23) is not a sharp estimate, the analytical expression of the right hand side will be more convient for the future proofs. This concludes Lemma 2.8.

□

Chapter 3

A formal approach to a SIS model

In this chapter we provide a formal definition of a spread of infection model on branching trees, examine a couple more lemmas concerning it, and finally we state our main results.

3.1 Introduction

Consider a supercritical Galton-Watson tree with offspring law X. For every realization of this tree $\mathcal{T}$, let us add particles on its vertices. Given a vertex $v \in \mathcal{T}$ we declare that the number of particles on v at time 0 is distributed as a *Poisson* random variable with mean $\mu \deg(v)$, where $\mu > 0$ is a parameter of our system and $\deg(v)$ is the degree of v. This applies to each vertex v independently. Now that the tree and the particles occupying it are set, we can let the system evolve in time: each particle performs a continuous-time simple random walk on $\mathcal{T}$ independently from the others by waiting for an exponential time (with rate 1), picking a neighbour uniformly at random and then jumping to it.

Once the particles are set, let us proceed and consider a SIS infection process, where at time $t = 0$ we add one infected particle to the root. By doing so, we can be sure that there is at least one particle infected to actually start the process, even if the Poisson distribution determining the amount of particles at the origin should be 0. Recall from section 1.3.2 that in the SIS model infected particles (I) immediately spread the infection to suscettible (S) particles on the same vertex. Furthermore, we assume that an infected particle heals (thus turning back to S type) after an exponential time with rate $\lambda > 0$.

3.2 Stationarity

Our work will rely on the fact that the particles are distributed as Poisson because of the following two reasons: firstly, every time we will need to consider only some special subset of particles, because of the *thinning* property (see Proposition B.1) they will be distributed as Poisson too, allowing for quicker computations. Secondly, note how the particle system is in stationarity, as established in the following proposition.

Proposition 3.1. *Let $\mathcal{N}(v,t)$ be the number of particles on a given vertex v at time t. We know that $\mathcal{N}(v,0)$ is a Poisson distribution with parameter $\mu \deg(v)$. We claim that $\mathcal{N}(v,\cdot)$ is stationary, in the sense that, for any given $t > 0$, $\mathcal{N}(v,t)$ is Poisson with parameter $\mu \deg(v)$.*

Proof of Proposition 3.1. Fix a vertex v and a time $t > 0$ and observe that

$$\mathcal{N}(v,t) = \sum_w \mathcal{M}(w,t)$$

where $\mathcal{M}(w,t)$ is the number of particles located in the vertex w at time 0 that satisfy the event $\left\{w \xrightarrow{t} v\right\}$ of having performed a random walk such that they are in v at time t. Now the thinning property from Proposition B.1 implies that $\sum\limits_w \mathcal{M}(w,t)$ is a Poisson distribution, with parameter

$$\sum_w \mu \deg(w) \mathbb{P}(w \xrightarrow{t} v)$$

Now notice that the events $\left\{w \xrightarrow{t} v\right\}$ and $\left\{v \xrightarrow{t} w\right\}$ satisfy

$$\deg(w)\mathbb{P}(w \xrightarrow{t} v) = \deg(v)\mathbb{P}(v \xrightarrow{t} w)$$

and thus we conclude that

$$\begin{aligned} \sum_w \mu \deg(w)\mathbb{P}(w \xrightarrow{t} v) &= \sum_w \mu \deg(v)\mathbb{P}(v \xrightarrow{t} w) \\ &= \mu \deg(v) \sum_w \mathbb{P}(v \xrightarrow{t} w) \\ &= \mu \deg(v) \end{aligned}$$

that is $\mathcal{N}(v,t)$ and $\mathcal{N}(v,0)$ have the same law. □

3.3 The infection problem

The evolution of the model is influenced by - and thus changes according to - these three factors, which play the roles of parameters: the offspring law X, the particle density μ and the healing rate λ.

3.3.1 The survival event

Recall that the infection is said to *survive* if for all times $t > 0$ there is at least one infected particle on the tree. We say that the infection *dies out* if it does not survive. Observe that we have excluded a priori some choice for the parameters involved that would otherwise make the discussion trivial. For example, if $\mu = 0$ then the only particle on the tree is the infected one that we add to the origin, thus it will recover for any $\lambda > 0$ almost surely.

Also, if there are no particles at the root before the infected particle is added and the only infected one heals before jumping to some other vertex, then the infection dies out. Both events happen with positive probabilities for any choice of μ and λ. As a consequence, the actual problem is whether the infection may survive with positive probability.

3.3.2 The case of the finite tree

Also, the infection dies out almost surely if the tree is finite: to see this, observe that there is always positive probability P_0 for the number of initial particles to be smaller than the number of vertices, regardless of the particle density μ. Therefore even if we add another one on the root, the number of total particles still does not exceed the number of vertices. Fix some $T > 0$ and observe that in the time interval $[0, T]$ there is positive probability P_1 for the particles to scatter, so that we have at most one particle per vertex. At this point, for any choice of λ there is positive probability P_2 for all of these particles to heal before any of them manages to jump within the time interval $[0, 2T]$. So the whole process has positive probability $P_0P_1P_2$ to happen in the limited time interval $[0, 2T]$. Therefore, as time goes to infinity, at some point the infection will die almost surely. Now Theorem 2.1 states that a necessary condition for the tree to be infinite is that it is supercritical, thus we require such consequent choice for the offspring distribution X.

3.4 Main results

Here we present our results about the long-time behavious of the spread of infection.

Theorem 3.2. *Consider the SIS model on a Galton-Watson tree with supercritical offspring distribution X that satisfies the property (2.11). For any choice of $\mu > 0$ and $\lambda > 0$, the infection survives with positive probability.*

As anticipated in section 2.3, things change when we remove the property (2.11):

Theorem 3.3. *Consider the SIS model on a Galton-Watson tree with supercritical offspring distribution X which has unbounded support. There exists a constant $\mu_c < +\infty$ such that the infection survives with positive probability for all $\mu > \mu_c$.*

The proof of these theorems can be found in the next chapter.

3.4.1 Phase transition

Both theorems above show that the infection can survive under different hypothesis. However, fixed an offspring law X with the necessary properties, while in Theorem 3.2 we see that the parameters μ and λ do not influence the long term behaviour of the system, in Theorem 3.3 it is required instead that the particle density μ is large enough. As we will see in the final chapter, we believe that the infection dies out when μ is small, although we have not managed to prove it yet. In case this happens, we say that the process undergoes a *phase transition* in the particle density μ.

3.5 Preliminary lemmas

This section contains two lemmas related to our model which we will require to prove our results. We provide their proof in the next section.

3.5.1 Keeping the infection alive

The purpose of this first lemma is to analyse the exchange of particles between an infected vertex v with a high degree and its children in order to show that the infection process can be kept alive for long enough so that a very high number of particles can visit v. This argument is similar to the one that Chatterjee and Durrett used in [20] (in particular, see Section 2) to prove a similar result for the contact process starting with all infected vertices.

Lemma 3.4. *Consider the SIS infection model on a supercritical Galton-Watson tree with offspring law X. Let v be a vertex of X and let $\mathbb{P}_\Gamma$ be the probability distribution conditioned on the event $\{\deg(v) = \Gamma\}$. Let τ be the first time that an infected particle reaches v: we shall only consider the particles that have never visited the parent vertex of v for all times before τ. Let $\tilde{\mathcal{N}}_t$ be the number of such particles that find themselves on v at a given time $t > \tau$; put*

$$\tau' := \inf\left\{t > \tau : \tilde{\mathcal{N}}_t \leq 1\right\}$$

Now take $\mathcal{V}_v$ to be the number of particles that jump on v for their first time within the interval (τ, τ') and consider the event

$$\mathcal{K}_{c,v} := \{\mathcal{V}_v \geq \exp(c\mu\Gamma)\}$$

Then there exist two constants $c, \Gamma_0 > 0$ such that

$$\mathbb{P}_\Gamma\left(K_{c,v}^c\right) \leq 4\exp(-c\mu\Gamma) \tag{3.1}$$

for all $\Gamma > \Gamma_0$. The constants c and Γ_0 both depend on the law of X. Moreover, Γ_0 also depends on μ and is decreasing as a function of it.

3.5.2 Propagation of the infection over generations

Next, we present a second lesser lemma. Its purpose will be to make the proof of our first theorem easier to understand when $X \geq 2$ almost surely, as in the other cases we will require some changes and thus present similar arguments directly in the proof of the corresponding results. The objective here will be trying to understand whether the infection can expand to a specific generation of vertices ahead.

Lemma 3.5. *Consider the SIS infection model on a Galton-Watson tree with offspring law* X *such that* $X \geq 2$ *almost surely. Let* v *be the its root, with* $\deg(v) = \Gamma$*, and define* $\mathbb{P}_\Gamma$ *in analogy with the previous lemma. Assume that there exists a constant* $C > 0$ *such that there are* $\exp(C\Gamma)$ *particles on* v *at time* 0 *and no particles elsewhere. Suppose that each particle, independently from the others, is removed from the tree after an exponential clock with rate* λ*. Fix* $\epsilon > 0$ *and let* $\mathcal{X}_v$ *be the event there are at least* $2^{\epsilon\Gamma}$ *vertices in* $Z_{\epsilon\Gamma}$ *such that each one of them is reached by one of the particles. Then there exists a value of* ϵ *such that*

$$\mathbb{P}_\Gamma(\mathcal{X}_v^c) \leq \exp(-c\Gamma) \tag{3.2}$$

for some constant $c > 0$ *which depends on* ϵ *and on* λ*.*

3.6 Proof of the preliminary lemmas

Let us begin with the proof of Lemma 3.4 first. We are going to divide it in two steps, first under the stronger assumption that $X \geq 2$ almost surely, then in the general case.

Proof of Lemma 3.4, assuming that $X \geq 2$ *almost surely.* The first step is to define the quantity $\mathcal{N}_t$ as follows:

- $\mathcal{N}_0 := 1 + \tilde{\mathcal{N}_\tau}$, where with $\tilde{\mathcal{N}_\tau}$ we indicate the number of particles that have never visited the parent of v for all times before τ and that are on v when an infected particle reaches it from the first time. We add one to such quantity as we want to consider the particle that has brought the infection as well (and which effectively jumps on v in that instant);
- Let $\mathcal{N}_t$ decrease by 1 every time that some particle on v leaves it;
- Let $\mathcal{N}_t$ increase by 1 at time $t > 0$ if there is some particle that has never been in v for all times $s < t$ and moves to v at time t, coming from one of its children.

Now, each child w of v has a number of particles occupying it at a given t distributed as Poisson($\mu \deg(w)$) (see Lemma 3.1). Thus, if we define

$$\alpha_\tau := \mathbb{P}_\Gamma\,(\text{a particle never visits the parent of } v \text{ for all times } t < \tau) \tag{3.3}$$

then the number of particles on w with this property is $\text{Poisson}(\alpha_{\mathcal{T}}\mu \deg(w))$ (see Proposition B.1); for the same reason, if we further put

$$\alpha_{w,t} := \mathbb{P}_{\Gamma}\,(\text{a particle on } w \text{ at time } t > 0 \text{ never visited } v \text{ for all times } s < t) \tag{3.4}$$

then there are $\text{Poisson}(\alpha_{\mathcal{T}}\alpha_{w,t}\mu \deg(w))$ particles on w at time t with said property. Observe that the definition in (3.4) makes sense even in the case $t = +\infty$.

Notice also that since we are assuming $X \geq 2$ a.s., the tree $\mathcal{T}$ contains a binary tree. Thus we deduce that, for a given particle, the function

$$S_t := \text{ distance from the root } r \text{ on the tree } \mathcal{T}$$

is greater than or equal to

$$S_t^B := \text{ distance from the root } r \text{ in the binary tree}$$

with S_t^B behaving like a random walk on $\mathbb{Z}_{\geq 0}$, with probability $\frac{2}{3}$ of increasing by 1 and $\frac{1}{3}$ of decreasing by 1 (except when in position 0, where it jumps almost surely to 1). There is a martingale argument that shows that, for an asymmetric random walk on $\mathbb{Z}$, where the probability of increasing by 1 is greater than the probability of decreasing by 1, it is possible to compute the probability that the process, starting from within an interval (a, b), leaves it from one endpoints instead of the other. By taking the limit $b \to \infty$, we can see that there is positive probability that the random walk never actually reaches a for all times. To see this, look at Section B.2 in the appendix. If we translate this in terms of our tree, we have that

$$\alpha_{w,t} \geq \alpha_{w,\infty} \geq \alpha^B \tag{3.5}$$

where α^B denotes the probability that a random walk on a binary tree that does not start on the root never visits it for all times. The consequence of 3.5 is that the distribution

$$\text{Poisson}(\alpha_{\mathcal{T}}\alpha^B\mu \deg(w))$$

represents a worst-case scenario for the number of particles occupying the children of v which can interact with v itself. Therefore, it will suffice to prove the lemma in such case, therefore let us assume that each child w of v has $\text{Poisson}(\alpha_{\mathcal{T}}\alpha^B\mu \deg(w))$ number of particles. The advantage is that we have eliminated the dependence from time in the parameter of the Poisson distribution.

So how does $\mathcal{N}_t$ evolve in time? We know that each particle will jump independently with rate 1, so that $\mathcal{N}_t$ decreases by 1 with rate $\mathcal{N}_t$. On the other hand, in each child w of v we have particles that jump with rate $\alpha_{\mathcal{T}}\alpha^B\mu \deg(w)$; however, only those that move towards v produce an increase by 1 in $\mathcal{N}_t$, and this happens with probability $\frac{1}{\deg(w)}$. By summing over the $\Gamma - 1$ children we get that $\mathcal{N}_t$ increases by 1 with overall rate

$$r := \alpha_{\mathcal{T}}\alpha^B\mu(\Gamma - 1) \tag{3.6}$$

At this point observe that, even though $\mathcal{N}_t$ is a continuous time process, it actually changes only at the times where some particle jumps according to the rules above. Therefore, to simplify the notations, let us treat $\mathcal{N}_t$ as a discrete process in the first place. Consequently, we can now write

$$\mathcal{N}_{t+1} = \begin{cases} \mathcal{N}_t - 1 & \text{with probability } \frac{N_t}{N_t + r} \\ \\ \mathcal{N}_t + 1 & \text{with probability } \frac{r}{N_t + r} \end{cases} \tag{3.7}$$

and recall that $\mathcal{N}_0 = 1 + \tilde{\mathcal{N}_\tau}$. Now $\tilde{\mathcal{N}_\tau}$ is distributed as a Poisson$(\alpha_\tau \mu \Gamma)$. An application of Theorem A.3 with $\delta = \frac{1}{2}$ implies that

$$\mathbb{P}_\Gamma \left(\tilde{\mathcal{N}_\tau} \le \frac{3}{4} \alpha_\tau \mu \Gamma \right) \le \exp\left(-\frac{9}{128} \alpha_\tau \mu \Gamma \right) \tag{3.8}$$

We will require (3.8) in a few moments. As for relations in (3.7), they suggest that whenever $\mathcal{N}_t < r$, the movement of particles favours an increase of $\mathcal{N}_t$, while on the contrary they tend to let $\mathcal{N}_t$ decrease whenever $\mathcal{N}_t > r$. With the idea that we would like the infection to be alive for a long time, we use the following strategy: assume that $\mathcal{N}_0 > \frac{r}{4}$; this occurs with high probability thanks to (3.8), so long as Γ is large enough so that

$$\frac{3}{4} \alpha_\tau \mu \Gamma > \frac{r}{4} \quad \underset{(3.6)}{\iff} \quad 3\Gamma > \alpha^B (\Gamma - 1)$$

which holds for all positive values of Γ.

Now

$$\frac{r}{4} > 2 \qquad \iff \qquad \Gamma > 1 + \frac{8}{\alpha_\tau \alpha^B \mu} =: \Gamma_0 \tag{3.9}$$

If (3.9) holds then $\mathcal{N}_t > 2$ and thus the infection cannot heal. At this point, note that whenever we have that $\mathcal{N}_t \le \frac{r}{4}$, we can observe that until

$$2 \le \mathcal{N}_t \le \frac{r}{2} \tag{3.10}$$

relation (3.10), combined with (3.7), produces that the increase probability for $\mathcal{N}_{t+1}$ satisfies

$$\frac{r}{\mathcal{N}_t + r} \ge \frac{2}{3} \tag{3.11}$$

Now we observe that, so long as (3.10) is valid, inequality (3.11) shows that $\mathcal{N}_t$ dominates stochastically a random walk W_t on $\mathbb{Z}$ with probability $\frac{2}{3}$ of moving by $+1$, probability $\frac{1}{3}$ of moving by -1 and such that $W_0 = \frac{r}{4}$. The point is that we can show that for this process is very unlikely to reach 2 before $\frac{r}{2}$: to see this, simply apply again the martingale argument from Section B.2, by taking $a = 2$ and $b = \frac{r}{2}$. If t^* is the first time that the random walk leaves the interval I_r and $\beta = \mathbb{P}\left(W_{t^*} = \frac{r}{2} \right)$, we have

$$\beta = \frac{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^{\frac{r}{4}}}{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^{\frac{r}{2}}} \tag{3.12}$$

Note that

$$\begin{aligned}
1-\beta &= \frac{\left(\frac{1}{2}\right)^{\frac{r}{4}}-\left(\frac{1}{2}\right)^{\frac{r}{2}}}{\left(\frac{1}{2}\right)^{2}-\left(\frac{1}{2}\right)^{\frac{r}{2}}} \\
&\leq 4\left(\left(\frac{1}{2}\right)^{\frac{r}{4}}-\left(\frac{1}{2}\right)^{\frac{r}{2}}\right) \\
&\underset{(3.6)}{\leq} 4\exp\left(-\frac{\log 2\alpha_{\mathcal{T}}\alpha^{B}\mu\Gamma}{8}\right) \\
&\underset{c:=\frac{\log 2\alpha_{\mathcal{T}}\alpha^{B}}{16}}{=} 4\exp\left(-2c\mu\Gamma\right) \qquad (3.13)
\end{aligned}$$

So we know that it is very unlikely for the random walk to leave the interval I_r in 2 instead of $\frac{r}{2}$. Therefore until we have $\mathcal{N}_t \geq 2$, we simply keep applying this strategy for every time that $\mathcal{N}_t$ falls into the range (3.10). The question is how long does this process usually go on? If G represents the number of iterations that this strategy needs to be applied for before eventually having only one particle left in v, G is a geometric random variable with probability $\gamma := 1-\beta$ of success and note that for any j we have

$$\begin{aligned}
\mathbb{P}_{\Gamma}(G \leq j) &= 1-\mathbb{P}_{\Gamma}(G > j) \\
&\leq 1-(1-\gamma)^{j} \\
&\leq j\gamma \\
&\underset{(3.13)}{=} 4j\exp\left(-2c\mu\Gamma\right) \\
&\leq 4\exp\left(-c\mu\ \Gamma\right) \qquad (3.14)
\end{aligned}$$

if we take

$$j = \exp\left(c\mu\Gamma\right) \qquad (3.15)$$

This proves a first part (3.1), provided that Γ is large enough as required in (3.9). Also, notice that for every time that the strategy is a success (in the sense that the infection is kept active), we have that $\mathcal{N}_t$ has to grow from $\frac{r}{4}$ up to $\frac{r}{2}$, with

$$\frac{r}{2}-\frac{r}{4}=\frac{r}{4}\underset{(3.9)}{>}2$$

which implies that, for every iteration, we have at least 2 new particles that jump to v. As a consequence, under the event that the number of iterations j satisfies (3.15), there is an exponentially high number (with respect to Γ) of particles that jump to v in the process. This finishes the proof of (3.1) as well, thus concluding the proof in the case $X \geq 2$ almost surely. $\square$

We now proceed with the general case.

Proof of Lemma 3.4, with $\mathbb{P}(X \leq 1) > 0$. We would like to provide the same proof of the previous case, however we cannot make use of relation (3.5) this time, as our tree does not contain a binary tree in general. Therefore, we need a workaround. Given a tree T, let $\mathcal{G}(T)$ be the event that a random walk, starting from a child w of v, never visits v for all positive times. Since X is supercritical, transience implies that there exists $\delta > 0$ and a family $\mathbb{T}$ of trees such that

$$\mathbb{P}_\Gamma(\mathbb{T}) > 0 \tag{3.16}$$

and

$$\mathbb{P}_\Gamma(\mathcal{G}(T) \geq \delta) \tag{3.17}$$

for all trees $T \in \mathbb{T}$. We say that a tree T is *good* if it satisfies (3.16) and (3.17) with δ replaced by $\frac{\delta}{1000}$. Then, we say that a child w of v is *good* if T is good. If P is the probability that w is good, an application of Theorem 2.3 and Observation 2.4 implies that the event $\mathcal{W}$ that at least $\frac{P}{2}\left\lfloor\frac{\Gamma-1}{2}\right\rfloor$ children of v are good satisfies

$$\mathbb{P}_\Gamma(\mathcal{W}^c) \leq \exp\left(-\frac{P}{4}\left\lfloor\frac{\Gamma-1}{2}\right\rfloor\right) \leq \exp\left(-\frac{P}{16}\Gamma\right) \tag{3.18}$$

Under the event $\mathcal{W}$, let $w_1, ..., w_{\left\lfloor\frac{\Gamma-1}{2}\right\rfloor}$ be the good children of v and let S be the collection of them. At this point we simply try to repeat the argument of the lemma in the case $X \geq 2$ almost surely; however, this time we restrict out argument to the tree $\mathcal{T}(v)^S$ (recall from section 2.1.1 that $\mathcal{T}(v)^S$ is obtained by $\mathcal{T}(v)$ by deleting the edges between v and its children that do not belong to S and then taking the connected component containing v. Define the quantities $\mathcal{N}_0$ and $\mathcal{N}_t$, for $t > 0$, just like in the previous case, which we recall here:

- $\mathcal{N}_0 := 1 + \tilde{\mathcal{N}_\tau}$, where with $\tilde{\mathcal{N}_\tau}$ we indicate the number of particles that have never visited the parent of v for all times before τ and that are on v when an infected particle reaches it from the first time. We add one to such quantity as we want to consider the particle that has brought the infection as well (and which effectively jumps on v in that instant);
- Let $\mathcal{N}_t$ decrease by 1 every time that some particle on v leaves it;
- Let $\mathcal{N}_t$ increase by 1 at time $t > 0$ if there is some particle that has never been in v for all times $s < t$ and moves to v at time t, coming from one of its children.

This time the particles we consider for each child w are distributed as Poisson $\left(\alpha_\tau \alpha_{w,t} \mu \left\lfloor\frac{\Gamma-1}{2}\right\rfloor\right)$, where α_τ and $\alpha_{w,t}$ are defined like in (3.3) and (3.4) respectively. In this case, observe that

$$\alpha_{w,t} \geq \alpha_{w,\infty} \geq \alpha := \frac{\delta}{1000} \tag{3.19}$$

thus we restrict the number of particles per each child w to be the (worst) case Poisson $\left(\alpha_\tau \alpha \mu \left\lfloor\frac{\Gamma-1}{2}\right\rfloor\right)$, which is independent of time. Equation (3.19) is a candidate replacement for (3.5) from the case $X \geq 2$ almost surely that we were looking for, therefore now we can simply previous argument to proceed with the rest of the proof. By observing again that particles leave v with rate $\mathcal{N}_t$ and move to w from one of its children with rate

$$r' := \alpha_\tau \alpha \mu \left\lfloor \frac{\Gamma - 1}{2} \right\rfloor \tag{3.20}$$

Let us simplify again the process $\mathcal{N}_t$ by treating it as if time were discrete. In complete analogy with (3.7) we have

$$\mathcal{N}_{t+1} = \begin{cases} \mathcal{N}_t - 1 & \text{with probability } \frac{N_t}{N_t + r'} \\ \\ \mathcal{N}_t + 1 & \text{with probability } \frac{r'}{N_t + r'} \end{cases} \tag{3.21}$$

Since here again $\mathcal{N}_0 = 1 + \tilde{\mathcal{N}_\tau}$, with $\tilde{\mathcal{N}_\tau}$ distributed as Poisson$(\alpha_\tau \mu \Gamma)$, the previous inequality (3.8), namely

$$\mathbb{P}_\Gamma \left(\tilde{\mathcal{N}_\tau} \leq \frac{3}{4} \alpha_\tau \mu \Gamma \right) \leq \exp\left(-\frac{9}{128} \alpha_\tau \mu \Gamma \right) \tag{3.22}$$

still holds. This implies implies that with high probability we have $\mathcal{N}_0 > \frac{r'}{4}$, so long as we require that

$$\frac{3}{4} \alpha_\tau \mu \Gamma > \frac{r'}{4} \quad \underset{(3.20)}{\iff} 3\Gamma > \alpha \left\lfloor \frac{\Gamma - 1}{2} \right\rfloor$$

which is always verified, and

$$\frac{r'}{4} > 2 \iff \left\lfloor \frac{\Gamma - 1}{2} \right\rfloor > \frac{8}{\alpha_\tau \alpha \mu} =: \Gamma_0 \tag{3.23}$$

similarly to (3.9). Finally, if we have

$$2 \leq \mathcal{N}_t \leq \frac{r'}{2} \tag{3.24}$$

we deduce from (3.21) that

$$\frac{r'}{\mathcal{N}_t + r'} \geq \frac{2}{3} \tag{3.25}$$

and at this point we can apply the same martingale argument as in the previous case: whenever we have $\mathcal{N}_t \geq \frac{r'}{4}$ we compare $\mathcal{N}_t$ with a random walk W_t on $\mathbb{Z}$ with probability $\frac{2}{3}$ of increasing by 1, $\frac{1}{3}$ of decreasing by 1, and $W_0 = \frac{r'}{4}$. This is valid so long as condition (3.24) is satisfied. If t^* is the first time that W_t leaves the interval $\left(2, \frac{r'}{2}\right)$ and $\beta' := \mathbb{P}\left(W_{t^*} = \frac{r'}{2}\right)$, we have

$$\beta' = \frac{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^{\frac{r'}{4}}}{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^{\frac{r'}{2}}} \tag{3.26}$$

and

$$\begin{aligned}
1-\beta' \quad &= \quad \frac{\left(\frac{1}{2}\right)^{\frac{r'}{4}} - \left(\frac{1}{2}\right)^{\frac{r'}{2}}}{\left(\frac{1}{2}\right)^{2} - \left(\frac{1}{2}\right)^{\frac{r'}{2}}} \\
&\leq \quad 4\left(\left(\frac{1}{2}\right)^{\frac{r'}{4}} - \left(\frac{1}{2}\right)^{\frac{r'}{2}}\right) \\
&\underset{(3.20)}{\leq} \quad 4\exp\left(-\frac{\log 2\alpha_\tau \alpha\mu\Gamma}{32}\right) \\
&\underset{c':=\frac{\log 2\alpha_\tau\alpha}{64}}{=} \quad 4\exp\left(-2c'\mu\Gamma\right) \qquad (3.27)
\end{aligned}$$

And this shows that it is very unlikely for the random walk to leave the interval $\left(2, \frac{r'}{2}\right)$ in 2 instead of in $\frac{r'}{2}$. This strategy can be applied again for every instance that the condition (3.24) is verified. The conclusion follows directly from the same geometric argument as in the case $X \geq 2$ almost surely: if G represents the number of iterations that this strategy needs to be applied for before eventually having only one particle left in v, G is a geometric random variable with probability $\gamma' := 1-\beta'$ of success and note that for any j we have

$$\begin{aligned}
\mathbb{P}_\Gamma(G \leq j) \quad &= \quad 1 - \mathbb{P}_\Gamma(G > j) \\
&\leq \quad 1 - (1-\gamma')^j \\
&\leq \quad j\gamma' \\
&\underset{(3.13)}{=} \quad 4j\exp\left(-2c'\mu\Gamma\right) \\
&\leq \quad 4\exp\left(-c'\mu\ \Gamma\right) \qquad (3.28)
\end{aligned}$$

if we take

$$j = \exp\left(c'\mu\Gamma\right) \qquad (3.29)$$

This proves a first part (3.1), provided that Γ is large enough as required in (3.23). Also, notice that for every time that the strategy is a success (in the sense that the infection is kept active), we have that $\mathcal{N}_t$ has to grow from $\frac{r'}{4}$ up to $\frac{r'}{2}$, with

$$\frac{r'}{2} - \frac{r'}{4} = \frac{r'}{4} \underset{(3.23)}{>} 2$$

which implies that, for every iteration, we have at least 2 new particles that jump to v. As a consequence, under the event that the number of iterations j satisfies (3.29), there is an exponentially high number (with respect to Γ) of particles that jump to v in the process. This finishes the proof, showing that (3.1) is satisfied with the constant c'. Finally, take $c := \max\left\{\frac{P}{16}, c'\right\}$ so that (3.18) holds as well. $\square$

It remains to prove Lemma 3.5. The strategy here is showing first that it suffices to prove the statement in the case of the binary tree, then proceed by checking that things actually work for such a tree.

Proof of Lemma 3.5. As first step, we claim that the particles from v can visit at least the same amount of vertices that they would do in the special case of a regular binary tree: to see this, let us establish a comparison between two version of the random walks, the first one on a binary tree (B), the second on our general tree (G). For simplicity, let us consider only the particles that at every jump move always to a child and not back to their parent vertex. We now would like to couple the processes (B) and (G), so that whenever (G) moves, so does (B). In the tree (G) we have two cases: the number of possible moves could be either even or odd. If it is odd, we eliminate the last child from our current position, so that we reduce ourselves to the even situation. Now every time the random walk (G) performs a jump to one of the first half of children, simply let the process (B) move to the first child, otherwise let (B) move to the second one. Notice that by this rule we have that two distinct paths in (B) cannot correspond to two identical paths in (G), therefore number of vertices explored by the random walk (G) is at least the same amount explored in (B). This proves the claim. Also note that the random walk on (G) at a given vertex w with degree $\deg(w) = d$ satisfies that the probability of moving to a child of w is given by $\frac{d-1}{d}$, when d is odd, and $\frac{d-2}{d}$, if d is even. Notice that

$$\min\left\{\inf_{d\geq 3, d \text{ even}} \frac{d-2}{d}, \inf_{d\geq 2, d \text{ odd}} \frac{d-1}{d}\right\} = \frac{1}{3}$$

with $\frac{1}{3}$ being the probability that a simple random walk in (B) chooses a specific adjacent vertex at a jump.

So, because of this coupling argument, it suffices to show that the statement is true in the case of the binary tree. Let w be a vertex in $Z_{\epsilon\Gamma}$, pick a particle u and let $\left\{v \underset{u}{\sim} w\right\}$ be the event that u reaches w via the minimal path, without jumping backwards and before being removed, and let $\left\{v \xrightarrow{i} w\right\}$ be the event that some particle manages to jump to v before being deleted. Because the tree is binary, we have

$$\mathbb{P}_\Gamma(v \underset{u}{\sim} w) = (3\,(1+\gamma))^{-\epsilon\Gamma} \tag{3.30}$$

per each particle. Thus the event $\mathcal{X}^c_\Delta$ that some vertex in $Z_{\epsilon\Gamma}$ is not reached by any infected particles satisfies

$$\begin{aligned}
\mathbb{P}_\Gamma(\mathcal{X}^c_\Delta) &= \mathbb{P}_\Gamma\left(\bigcup_{w\in Z_{\epsilon\Gamma}} \left\{v \xrightarrow{i} w\right\}^c\right)\\
&\leq \mathbb{P}_\Gamma\left(\bigcup_{w\in Z_{\epsilon\Gamma}} \bigcap_{\text{Particles } u \text{ on } v} \left\{v \underset{u}{\sim} w\right\}^c\right)\\
&\leq \sum_{w\in Z_{\epsilon\Gamma}} (1-\mathbb{P}_\Gamma(v\sim w))^{\exp(C\Gamma)}
\end{aligned}$$

Now we use that $|Z_{\epsilon\Gamma}| = 2^{\epsilon\Delta}$ in the binary tree, together with (3.30), to obtain

$$\begin{aligned}\sum_{w\in Z_{\epsilon\Gamma}} (1-\mathbb{P}_\Gamma(v\sim w))^{\exp(C\Gamma)} &= 2^{\epsilon\Gamma}\left((1-3(1+\gamma))^{-\epsilon\Gamma}\right)^{\exp(C\Gamma)} \\ &\leq \exp\left(\epsilon\Gamma\log 2 - \exp(C\Gamma)(2(1+\gamma))^{-\epsilon\Gamma}\right) \\ &\leq \exp\left(\epsilon\Gamma\log 2 - \exp\left(\Gamma\left(C-\epsilon\log(2(1+\gamma))\right)\right)\right) \\ &\leq \exp\left(-\Gamma(C+\epsilon\log(1+\gamma))\right) \qquad (3.31)\end{aligned}$$

provided that we have

$$\epsilon < \frac{C}{\log(2(1+\gamma))} \qquad (3.32)$$

and this finishes the proof. □

Chapter 4

Proof of Theorems 3.2 and 3.3

In the first theorem, the proof will consist of an iterative multiscale scheme and will distinguish between the case where we assume that $X \geq 2$ almost surely and then the general case. In each step we show that the infection can spread with very high probability from a vertex with a given degree Δ to another one with degree 2Δ. Finally, we check that this procedure can be repeated infinitely many times with positive probability of succeeding.

Proof of Theorem 3.2, assuming that $X \geq 2$ almost surely. Take a vertex v_0 such that $\deg(v_0) := \Delta_0$ and assume that an infected particle jumps to v_0 at some time $t_0 < +\infty$. Lemma (3.4) applied to the tree $\mathcal{T}(v_0)$ with $\Gamma = \Delta_0$ implies that

$$\mathbb{P}\left(\mathcal{K}^c_{c_1,v_0}\right) \leq 4\exp(-c_1\mu\Delta_0) =: F_{1,1} \tag{4.1}$$

provided that Δ_0 is large enough. Here the constant $c_1 > 0$ is independent of μ and Δ_0. At this point the second part of Lemma 2.8 (recall that the first part is trivial if $X \geq 2$ almost surely), applied to the tree $\mathcal{T}(v_0)$, $\Gamma = \Delta_0$ and some $\epsilon > 0$ to be chosen later, implies that

$$\mathbb{P}\left(\mathcal{D}^c_{v_0,\epsilon}\right) < \exp(-c_2\epsilon\Delta_0) =: F_{1,2} \tag{4.2}$$

provided that Δ_0 is large enough. Here the constant c_2 depends only on the law of X. Thus, assuming that $\mathcal{D}_{v_0,\epsilon}$ is verified, there exists a vertex $v_1 \in Z_{\epsilon\Delta_0}$ with $\deg(v_1) \geq 2\Delta_0$. Finally, assuming that the event $\mathcal{K}_{c_1,v_0}$ from Lemma (3.4) hold, we have at least $\exp(c\Delta_0)$ particles on v_0. Within $\mathcal{T}_{v_0}$, we can couple the evolution of each one of those particles to the process described in Lemma 3.5 to get that there exists a choice for the previous ϵ such that

$$\mathbb{P}(\mathcal{X}^c_{v_0}) \leq \exp(-c_3\Delta_0) =: F_{1,3} \tag{4.3}$$

with c_3 a function of ϵ, μ, λ and X. Now thanks to (4.1), (4.2) and (4.3) we have shown that the infection can spread from v_0 to v_1 with probability

$$P_1 := (1 - F_{1,1})\,(1 - F_{1,2})\,(1 - F_{1,3})$$

At this point, we could apply this argument again to the vertex v_1 obtain that the infection reaches another vertex v_2, with degree $\deg(v_2) \geq 2\deg(v_1) \geq 4\Delta_0$, with probability

$$P_2 := (1 - F_{2,1})\,(1 - F_{2,2})\,(1 - F_{2,3})$$

Inductively, for every $j \geq 0$ the infection spreads from a vertex v_j with $\deg(v_j) \geq 2^j\Delta_0$ to another vertex v_{j+1} with $\deg(v_{j+1}) \geq 2^{j+1}\Delta_0$ with probability

$$P_j := (1 - F_{j,1})\,(1 - F_{j,2})\,(1 - F_{j,3})$$

where

$$F_{j,1} \leq 4\exp\left(-c_1\mu 2^j\Delta_0\right), \quad F_{j,2} \leq \exp(-c_2\epsilon 2^j\Delta_0), \quad F_{j,3} \leq \exp(-c_3 2^j\Delta_0)$$

Observe that the quantities $F_{j,i}$ above have the form

$$\phi\exp\left(-\psi 2^j\right)$$

for some positive constants ϕ and ψ. We would like to show that

$$\phi\exp\left(-\psi 2^j\right) \leq \frac{1}{12}\frac{1}{2^j} \tag{4.4}$$

for some $j \geq j_0$. Condition (4.4) rewrites

$$\frac{2^j}{j} \geq \frac{\log(24\phi)}{\psi}$$

which clearly holds for all j greater than some j_0. At this point, for the application of Lemmas 3.4 and 2.8 in the first two steps of the first iteration it was necessary to have Δ_0 large enough, let us say larger than some $\tilde{\Delta}$. At this point, choose Δ_0 such that

$$\Delta_0 \geq 2^{j_0}\tilde{\Delta} \tag{4.5}$$

so that all required conditions on Δ_0 are satisfied. Also, define

$$\phi = 4 \qquad \psi = \max\{c_1\mu\Delta_0, c_2\epsilon\Delta_0, c_3\Delta_0\}$$

so that

$$\max_{i=1,2,3} F_{j,i} \leq \phi\exp\left(-\psi 2^j\right) \underset{(4.4)}{\leq} \frac{1}{12}\frac{1}{2^j} \tag{4.6}$$

for all $j \geq j_0$. Finally, let $\mathcal{F}_j$ be the event that the j-th iteration of this procedure fails, and $\mathcal{F}$ the event that the whole iterative scheme, starting from the step

j_0, fails at some point. We have

$$\begin{aligned} \mathbb{P}(\mathcal{F}) &= \mathbb{P}\left(\bigcup_{j\geq j_0} \mathcal{F}_j\right) \\ &\leq \sum_{j\geq j_0} \mathbb{P}(\mathcal{F}_j) \\ &\underset{(4.6)}{\leq} \sum_{j\geq j_0} \frac{1}{4}\frac{1}{2^j} \\ &\leq \frac{1}{2} \end{aligned} \tag{4.7}$$

(4.7) shows that the iterative procedure works with positive probability. Thus, for the infection to survive for all times, it suffices that it manages to reach some vertex with degree at least Δ_0, large enough as indicated in (4.5). Observe that this happens with positive probability, as for instance the root has positive probability of having degree at least Δ_0. This concludes the proof. $\square$

Proof of Theorem 3.2, with $\mathbb{P}(X \leq 1) > 0$. Take now a generic offspring law X with $\mathbb{E}[X] > 1$ and such that it satisfies the (2.11) property. Take again a vertex v_0, with $\deg(v_0) = \Delta_0$ to be determined, and assume that some infected particle moves to v_0 at some time $t_0 < +\infty$. Let S be a set which consists of $\left\lfloor \frac{\Delta_0 - 1}{2} \right\rfloor$ children of v_0. Recall from section 2.1.1 that $\mathcal{T}(v_0)^S$ is the subtree of $\mathcal{T}(v_0)$ where we delete the children of v_0 which do not belong in S and take the connected component containing v_0. This time, because of independence, we will need to split the iterative argument between $\mathcal{T}(v_0)^S$ and $\mathcal{T}(v_0)^{S^c}$

Firstly we proceed like in the previous case and apply Lemma (3.4) to the tree $\mathcal{T}(v_0)^S$, with $\Gamma = \left\lfloor \frac{\Delta_0 - 1}{2} \right\rfloor$: there exists $c_1 > 0$ such that, for large enough Δ_0 we have

$$\mathbb{P}\left(\mathcal{K}^c_{c_1,v_0}\right) \leq 4\exp(-c_1\mu\Delta_0) =: F_{1,1} \tag{4.8}$$

where c_1 does not depend on μ and Δ_0.

At this point we need a different construction. Since X is supercritical, pick a constant $d \geq 0$ and define the truncated distribution X_d associated to X by

$$X_d := X | X \leq d \tag{4.9}$$

that is

$$\mathbb{P}(X_d = x) = \begin{cases} \frac{\mathbb{P}(X=x)}{\mathbb{P}(X\leq d)} & \text{if } x \leq d \\ 0 & \text{otherwise} \end{cases}$$

Since the X_d converge in probability to X as $d \to +\infty$, we can choose d large enough so that

$$\mathbb{E}[X_d] > 1 \tag{4.10}$$

Take now $\epsilon > 0$, to be chosen afterwards. From now on, we will consider only the tree $\mathcal{T}_{\epsilon\Delta_0}(v_0)^{S^c}$. Then, let $T := \hat{\mathcal{T}}_{\epsilon\Delta_0}(v_0)^{S^c}$ be the truncated version of

$\mathcal{T}_{\epsilon\Delta_0}(v_0)^{S^c}$ where we eliminate the offspring of all vertices that have more than $d-1$ children. Because of condition 4.10, T is supercritical, therefore we can apply Lemma 2.8 to it, with $\Gamma = \left\lfloor \frac{\Delta_0 - 1}{2} \right\rfloor$, to get that for large enough Δ_0 we have

$$\max\left\{\mathbb{P}\left(\mathcal{M}^c_{v_0,\epsilon}\right), \mathbb{P}\left(\mathcal{D}^c_{v_0,\epsilon} \mid \mathcal{M}_{v_0,\epsilon}\right)\right\} < \exp(-c_2\epsilon\Delta_0) =: F_{1,2} \tag{4.11}$$

for some $c_2 > 0$ which depends only on the law of X. Assuming that $\mathcal{M}_{v_0,\epsilon}$ and $\mathcal{D}_{v_0,\epsilon}$ hold, there exists some vertex $v_1 \in Z_{\epsilon\Delta_0}(v_0) \cap T$ such that $\deg(v_1) \geq 2\deg(v_0) = 2\Delta_0$.

Take now $w \in Z_{\epsilon\Delta_0}(v_0) \cap T$. For a particle u, let $\left\{v_0 \underset{u}{\sim} w\right\}$ be the event that u reaches w from v_0 via the minimal path, without jumping backwards and before it attempts to heal, and let $\left\{v_0 \xrightarrow{i} w\right\}$ for the event that the infection can spread from v to w. We have

$$\mathbb{P}(v_0 \underset{u}{\sim} w) \leq \left((d-1)\,(1+\gamma)\right)^{-\epsilon\Delta_0} \tag{4.12}$$

where we used the fact that every vertex in $T \setminus \{Z_{\epsilon\Delta_0}(v_0) \cap T\}$ has no more than $d-1$ children. Now the event $\mathcal{X}^{*c}_{\Delta}$ that some $w \in Z_{\epsilon\Delta_0}(v_0) \cap T$ is not reached by any infected particles satisfies

$$\begin{aligned}
\mathbb{P}(\mathcal{X}^{*c}_{\Delta}) &= \mathbb{P}\left(\bigcup_{w \in Z_{\epsilon\Delta_0}(v_0) \cap T} \left\{v_0 \xrightarrow{i} w\right\}^c\right) \\
&\leq \mathbb{P}\left(\bigcup_{w_0 \in Z_{\epsilon\Delta_0}(v_0) \cap T} \; \bigcap_{\text{Particles } u \text{ on } v} \left\{v_0 \underset{u}{\sim} w\right\}^c\right) \\
&\leq \sum_{w \in Z_{\epsilon\Delta_0}(v_0) \cap T} \left(1 - \mathbb{P}(v_0 \underset{u}{\sim} w)\right)^{\#\{\text{Particles } u \text{ on } v\}}
\end{aligned}$$

Now under the event $\mathcal{K}_{v_0}$ we know that

$$\#\left\{\text{Particles } u \text{ on } v\right\} \geq \exp(C\mu\Delta_0) \tag{4.13}$$

with $C > 0$ independent from μ and Δ_0. Also, conditioning on $\mathcal{M}_{v_0,\epsilon}$, we have that there exists $b > 0$, depending only on the distribution X, such that

$$|Z_{\epsilon\Delta_0}(v_0) \cap T| \geq 2^{b\epsilon\Delta_0} \tag{4.14}$$

At this point we combine (4.13) and (4.14) to write

$$\begin{aligned}
&\sum_{w \in Z_{\epsilon\Delta_0}(v_0) \cap T} \left(1 - \mathbb{P}(v \sim w)\right)^{\#\{\text{Particles } u \text{ on } v\}} \\
&\leq 2^{b\epsilon\Delta} \left((1 - (d-1)(1+\gamma))^{-\epsilon\Delta}\right)^{e^{C\mu\Delta_0}} \\
&\leq \exp\left(b\epsilon\Delta_0 \log 2 - \exp\left(C\mu\Delta_0((d-1)(1+\gamma))^{-\epsilon\Delta_0}\right)\right) \\
&\leq \exp\left(b\epsilon\Delta_0 \log 2 - \exp\left(\Delta_0\left(C\mu - \epsilon\log((d-1)(1+\gamma))\right)\right)\right)
\end{aligned}$$

Finally, choose ϵ such that

$$\epsilon < \frac{C\mu}{\log((d-1)(1+\gamma))}$$

Now if Δ_0 is large enough we have

$$\begin{aligned}
&\exp\left(b\epsilon\Delta_0 \log 2 - \exp\left(\Delta_0\left(C\mu - \epsilon\log((d-1)(1+\gamma))\right)\right)\right) \\
\leq\ &\exp\left(-\frac{1}{2}\exp\left(\Delta_0(C\mu - \epsilon\log((d-1)(1+\gamma)))\right)\right) \\
\leq\ &\exp\left(c_3\Delta_0\right) =: F_{1,3} \qquad (4.15)
\end{aligned}$$

where $c_3 := \frac{1}{2}\left(C\mu - \epsilon\log((d-1)(1+\gamma))\right)$ is a function of ϵ, μ, γ and X but it does not depend on Δ_0. So, by combining (4.8), (4.11) and (4.15), we have shown that the infection spreads from v_0 to a vertex v_1 with $\deg(v_1) \geq 2\Delta_0$ with probability

$$P_1 := (1-F_{1,1})(1-F_{1,2})(1-F_{1,3})$$

for large enough Δ_0. At this point the iterative argument starts again: we can apply the three steps again to v_1 to obtain that the infection spreads to a vertex v_2 with $\deg(v_2) \geq 2\deg(v_1) \geq 4\Delta_0$ with probability

$$P_2 := (1-F_{2,1})(1-F_{2,2})(1-F_{2,3})$$

Inductively, for every $j \geq 0$ the infection spreads from a vertex v_j with $\deg(v_j \geq 2^j\Delta_0$ to another vertex v_{j+1} with $\deg(v_{j+1}) \geq 2^{j+1}\Delta_0$ with probability

$$P_j := \left(1-F_{j,1}\right)\left(1-F_{j,2}\right)\left(1-F_{j,3}\right) \tag{4.16}$$

with

$$F_{j,1} \leq 4\exp\left(-c_1\mu 2^j\Delta_0\right), \quad F_{j,2} \leq \exp(-c_2\epsilon 2^j\Delta_0), \quad F_{j,3} \leq \exp(-c_3 2^j\Delta_0) \tag{4.17}$$

Here again the failure probabilities $F_{j,i}$ have the form

$$\phi\exp\left(-\psi 2^j\right)$$

for some positive constants ϕ and ψ. In analogy with the previous case, equation

$$\phi\exp\left(-\psi 2^j\right) \leq \frac{1}{12}\frac{1}{2^j} \tag{4.18}$$

rewrites

$$\frac{2^j}{j} \geq \frac{\log(24\phi)}{\psi} \tag{4.19}$$

which is verified for all j greater than some j_0. Now it remains to choose Δ_0: application of Lemmas 3.4 and 2.8, together with the computations in (4.15), required Δ_0 large enough, let us say greater than some $\tilde{\Delta}$. Take then

$$\Delta > 2^{j_0}\tilde{\Delta} \tag{4.20}$$

so that (4.19) is definitely verified. At this point simply define

$$\phi = 4 \qquad \psi = \max\left\{c_1\mu\Delta_0, c_2\epsilon\Delta_0, c_3\Delta_0\right\}$$

so that

$$\max_{i=1,2,3} F_{j,i} \leq \phi \exp\left(-\psi 2^j\right) \underset{(4.18)}{\leq} \frac{1}{12}\frac{1}{2^j}$$

for all $j \geq j_0$. In conclusion, let $\mathcal{F}_j$ be the event that the j-th iteration of this scheme fails, and $\mathcal{F}$ the event that the whole iteration process, starting from the step j_0, fails at some point. A similar calculation to the one in (4.7) shows that

$$\mathbb{P}(\mathcal{F}) \leq \frac{1}{2} \tag{4.21}$$

As a consequence, the iterative procedure works with positive probability. Thus, for the infection to survive, it suffices that it spreads to some vertex with degree at least Δ_0 as from (4.20). By observing that the root, for instance, can have degree larger than this Δ_0 with positive probability, the theorem is proved. □

Now let us proceed with the proof of the second theorem. In this case, the argument we applied the previous proof does not work, because without the property (2.11) we are unable to find vertices with higher degree to spread the infection to. However, we will show that it is possible to keep the infection alive by observing how particles move among vertices that have a similar, high enough degree.

Proof of Theorem 3.3. For this second theorem, it is possible to proceed via a percolation argument as follows: firstly, take Δ' such that:

$$\mathbb{P}(X \geq \Delta') = c(\Delta') > 0 \tag{4.22}$$

which is possible because X has unbounded support. Also, there exists $\Delta'' > 0$, only depending on the offspring law X, such that Lemma 3.4 can be applied to any vertex w with $\deg(w) \geq \Delta''$, for all $\mu \geq 1$: it suffices to define Δ'' as the Γ_0 in the statement of the lemma, with $\mu = 1$ and then use the monotonicity in μ. Finally, let

$$\Delta := \max\left\{\Delta', \Delta''\right\} \tag{4.23}$$

At this point, assume that the infection has reached a vertex v with degree Δ and let S be a collection of $\left\lfloor \frac{\Delta-1}{2} \right\rfloor$ vertices among its children. Apply Lemma 3.4 to the tree $\mathcal{T}(v)^S$ to obtain that

$$\max_{v \in \mathcal{T}(v)^S} \left\{\mathbb{P}\left(\mathcal{K}^c_{c_1,v}\right)\right\} \geq 1 - \exp\left(-c_1\mu\Delta\right) =: P_1 \tag{4.24}$$

For some $c_1 > 0$ independent of μ. Observe that P_1 tends to 1 as $\mu \to +\infty$. Also, the is positive probability P_0 for the infection to spread to a vertex with degree Δ: for example, the root may have degree Δ with positive probability.

Next, we want to establish an argument which is similar to the iteration method in the previous theorem. Since X is supercritical, take X_d to be the truncated version of X like in (4.9): then, choose d large enough so that

$$\mathbb{E}[X_d] > 1$$

Fix now $\epsilon > 0$. From now on, let us proceed and consider only $\mathcal{T}_{\epsilon\mu\Delta}(v)^{S^c}$. Let $T := \hat{\mathcal{T}}_{\epsilon\mu\Delta}(v)^{S^c}$ be the truncated version of $\mathcal{T}_{\epsilon\mu\Delta}(v)^{S^c}$ where we cut the offspring of all vertices with degree higher than d. T is supercritical because of the choice of d, thus we can apply the first part of Lemma 2.8 to get that we can find a set of vertices $Z \subseteq Z_{\epsilon\mu\Delta} \cap T$, with $|Z| = 2^{b\epsilon\mu\Delta}$, for some $b > 0$. This occurs with probability

$$P_2 \geq 1 - \exp(-c_2\epsilon\mu\Delta) \tag{4.25}$$

with c_2 independent of μ. Observe that, similarly to (4.24), P_2 tends to 1 as $\mu \to +\infty$. Fix now $w \in Z$. Given a particle u, let $\left\{v \underset{u}{\sim} w\right\}$ be the event that u reaches w from v via the minimal path, without jumping backwards and before it attempts to heal, and let $\left\{v \underset{i}{\to} w\right\}$ be the event that the infection can spread from v to w. By assumption the degrees in $\mathcal{T}_{\epsilon\mu\Delta}(v)^{S^c} \setminus Z$ are bounded by d. Moreover, under the event $\mathcal{K}_v$ from Lemma 3.4, there are at least $\exp(C\mu\Delta)$ distinct particles that jump to it from one of the children in S, with C independent of μ. We have

$$\mathbb{P}(v \underset{u}{\sim} w) \leq ((d-1)(1+\gamma))^{-\epsilon\mu\Delta} \tag{4.26}$$

Then, for the event $\mathcal{X}_v^c$ that there exists a vertex in Z which is not reached by any infected particles we can write:

$$\begin{aligned}
\mathbb{P}(\mathcal{X}_\Delta^{*c}) &= \mathbb{P}\left(\bigcup_{w\in Z} \{v \to w\}^c\right) \\
&\leq \mathbb{P}\left(\bigcup_{w\in Z} \bigcap_{\text{Particles } u \text{ on } v} \left\{v \underset{u}{\sim} w\right\}^c\right) \\
&\leq \sum_{w\in Z} (1 - \mathbb{P}(v \sim w))^{\#\{\text{Particles } u \text{ on } v\}}
\end{aligned}$$

Now we use that $|Z| = 2^{b\epsilon\mu\Delta}$ together with (4.26) to obtain

$$\begin{aligned}
&\sum_{w\in Z} (1 - \mathbb{P}(v \sim w))^{\#\{\text{Particles } u \text{ on } v\}} \\
&= 2^{b\epsilon\mu\Delta}\left((1 - (d-1)(1+\gamma))^{-\epsilon\mu\Delta}\right)^{\#\{\text{Particles } u \text{ on } v\}} \\
&\leq \exp\left(b\epsilon\mu\Delta\log 2 - (\#\{\text{Particles } u \text{ on } v\})((d-1)(1+\gamma))^{-\epsilon\mu\Delta}\right) \\
&\leq \exp\left(b\epsilon\mu\Delta\log 2 - \exp\left(\Delta\mu\left(C - \epsilon\log((d-1)(1+\gamma))\right)\right)\right) =: h(\Delta,\mu)
\end{aligned}$$

At this point, choose ϵ such that

$$\epsilon < \frac{C}{\log((d-1)(1+\gamma))} \tag{4.27}$$

and observe that, this time, it does not depend on μ. We have shown that all of the vertices in Z get infected with probability

$$P_3 \geq 1 - h(\Delta, \mu) \tag{4.28}$$

Here again we have that P_3 tends to 1 as $\mu \to +\infty$.

Now we are not able to show that in Z we can find a vertex of a degree higher than 2Δ. However, notice that (4.22) implies that the number $\mathcal{E}$ of vertices with degree at least Δ satisfies

$$\mathbb{E}[\mathcal{E}] = 2^{b\mu\epsilon\Delta} c(\Delta) \tag{4.29}$$

Thus, by combining (4.29) with the argument before, considering that the good events we would like to happen have probabilities defined in (4.24), (4.25), (4.28), we can write that the number $\tilde{\mathcal{E}}$ of vertices in Z, with degree at least Δ and that are reached by some infected particle coming from v, is such that

$$\mathbb{E}[\tilde{\mathcal{E}}] \geq 2^{b\mu\epsilon\Delta} c(\Delta) P_0 P_1 P_2 P_3 \tag{4.30}$$

Finally, observe that $\mathbb{E}[\tilde{\mathcal{E}}] \to +\infty$ as $\mu \to +\infty$, so there exists $1 < \mu_c < +\infty$ such that $\mathbb{E}[\tilde{\mathcal{E}}] > 1$ for all $\mu > \mu_c$. The idea is that, if we find some vertex $w \in Z$ with $\deg(w) \geq \Delta$, then we can simply repeat this argument where we take a set S of $\deg(w) - \left\lfloor \frac{\Delta-1}{2} \right\rfloor$ children of w to apply Lemma 3.4 to, and then use the rest $\left\lfloor \frac{\Delta-1}{2} \right\rfloor$ children in S^c to try to reach new vertices with degree at least Δ.

At this point, the spread of infection among vertices with degree at least Δ can be seen as a branching process, where we have calculated the expected value $\mathbb{E}[\tilde{\mathcal{E}}]$ of its offspring distribution. Assuming that $\mu > \mu_c$, the supercriticality condition $\mathbb{E}[\tilde{\mathcal{E}}] > 1$ shows that there is positive probability for such branching tree to be infinite. In terms of our infection process, the infection is kept alive for all times among vertices whose degree is at least Δ with positive probability. This concludes the proof. $\square$

Chapter 5

Open questions and future developments

In this last chapter we list some of the possible questions and topics that might be worth investigating later on.

5.1 A complete discussion of the phase transition regime

Theorem 3.3 shows that, whenever the offspring distribution is light-tailed and has unbounded support, the infection can survive for sufficiently high density of particles. For the moment, if we remove the second hypothesis on X, all we have is that our proof does not work.

As for the other side of the problem, that is if the parameter μ is too small, it is reasonable to think that the infection dies out almost surely, mirroring the behaviour of the contact process as seen in Theorem 1.5. In order to understand this, a good strategy could be based on a better analysis and control of the behaviour of the infection in some precise, finite region of the tree, let us say between a given vertex v which receives an infected particle and k generations $\mathcal{T}_k(v)$ after it somewhere. Then, it is possible that an argument similar to the one in section 3.3.2 might be adapted, though we have not managed to do it so far.

5.2 Sufficient conditions for strong survival

Recall from section 1.2.2 that our results provide sufficient conditions for the *weak survival*, however they do not address the *strong survival* question, which requires that set of times in which the root is occupied by at least an infected particle is unbounded. Such a fact would create a distinction between three different phases like in the standard contact process. For that model, however, this question seems to be open on graphs different from $\mathbb{Z}^d$ or regular trees. It may be possible to deduce some bounds for the duration of the infection like the ones from [30] for the contact process that we mentioned in section 1.2.5, however they have not yet proved to be useful in ensuring strong survival.

5.3 The SIR model

Recall from section 1.3.2 that the SIR model, infected particles die after an exponential clock with rate λ instead of recovering. Generally speaking, this makes the infection difficult to be kept alive at a given vertex even when the particle density is high: to see this, imagine a site v with N infected particles occupying it. In the SIS model we can say that there will always be infected particles on v so long as there are at least two of them. This idea is what the entire Lemma 3.4 is based on. This property will then allow for many of the N particles to move around and spread the infection elsewhere. On the other hand, in the SIR model we have that said N particles will last up to a random time equal to the maximum of N i.i.d. exponential variables with rate λ, so not only does the argument used in Lemma 3.4 fail, but the overall time the infection is maintained at v will usually be shorter.

Observe however that Lemma 3.4 is actually the only part of our entire proof that ceases to be valid in the SIR model: notice that the recovery rate λ appears only in Lemma 3.5 and in the final step of proof of Theorem 3.2 (without the assumption that $X \geq 2$ almost surely) and Theorem 3.3. In all these cases we want particles to reach many vertices and bring the infection to them, thus we simply asked that they did it before attempting to heal for the first time. Now we simply have to say that they bring the infection before they die, and everything else works the same way as for the SIS model.

5.4 Extending the results to random graphs

Another possible topic for future studies is the extension of the results on the trees to more general random graphs. This is useful as there are many situations where a tree is not the most suitable graph structure that represents a real-life network the model takes inspiration from. Galton-Watson trees are usually a good starting point to analyse more complicated graphs and extend to them some previous results already proved for the tree. There are examples in literature where transitions from trees to general graphs have already been established, see for example [22]. It could also be possible to establish some estimates for the duration of the infection like in the works from [30] about the contact process, which we mentioned in the previous section 5.2.

Appendix A

Chernoff bound

The Chernoff bound is an estimate used to show that a certain random variable X is concentrated near its expected value, provided that for such variable has a finite exponential moment $\mathbb{E}[e^{\theta X}]$ for some $\theta \geq 0$. Indeed given $\epsilon > 0$ we have

$$\begin{aligned}\mathbb{P}(X \geq (1+\epsilon)\mathbb{E}[X]) &= \mathbb{P}(e^{\theta X} \geq e^{\theta(1+\epsilon)E[X]}) \\ &\leq \frac{\mathbb{E}[e^{\theta X}]}{e^{\theta(1+\epsilon)\mathbb{E}[X]}} \\ &= \mathbb{E}[e^{\theta X}]e^{-\theta(1+\epsilon)\mathbb{E}[X]}\end{aligned}$$

where in the second passage we applied the Markov inequality to the non-negative random variable $e^{\theta X}$. Now we see that having an upper bound for $\mathbb{E}[e^{\theta} X]$ allows for an upper bound for $\mathbb{P}(X \geq (1+\epsilon)\mathbb{E}[X])$. Furthermore, since the previous estimate is valid for all $\theta > 0$ we have

$$\mathbb{P}(X \geq (1+\epsilon)\mathbb{E}[X]) \leq \inf_{\theta > 0} \mathbb{E}[e^{\theta X}]e^{-\theta(1+\epsilon)\mathbb{E}[X]}$$

Now we proceed by applying the argument above to some assigned distributions.

A.1 Chernoff bound for a sum of Bernoulli variables

Theorem A.1. *Let $X_1, ..., X_n$ be indipendent variables in $\{0, 1\}$. Let*

$$X = \sum_{i=1}^{n} X_i$$

and $p = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_i]$. Then we have

$$\mathbb{P}(X \geq (1+\epsilon)np) \leq \left(\frac{e^{\epsilon}}{(1+\epsilon)^{1+\epsilon}}\right)^{np} \tag{A.1}$$

Another useful bound is

$$\mathbb{P}(X \geq (1+\epsilon)np) \leq \exp\left(-\frac{\epsilon^2 np}{2} + \frac{\epsilon^3 np}{6}\right) \leq \exp\left(-\frac{\epsilon^2 np}{2+\epsilon}\right) \tag{A.2}$$

Proof. Given $\theta > 0$ we have

$$\begin{aligned}
\mathbb{P}(X \geq (1+\epsilon)np) &= \mathbb{P}(e^{\theta X} \geq e^{\theta(1+\epsilon)np}) \\
&\leq e^{-\theta(1+\epsilon)np}\ \mathbb{E}[e^{\theta X}] \\
&= e^{-\theta(1+\epsilon)np}\ \prod_{i=1}^{n} \mathbb{E}[e^{\theta X_i}]
\end{aligned}$$

where we used the Markov inequality and the independence of the X_i variables. Since for each i we have $\mathbb{E}[e^{\theta X_i}] = p_i(e^\theta - 1) + 1$ for some $p_i \in (0,1)$, we can write

$$\begin{aligned}
\mathbb{P}(X \geq (1+\epsilon)np) &\leq e^{-\theta(1+\epsilon)np} \prod_{i=1}^{n} \big(p_i(e^\theta - 1) + 1\big) \\
&\leq e^{-\theta(1+\epsilon)np} \left(\sum_{i=1}^{n} \frac{p_i(e^\theta - 1) + 1}{n}\right)^n \\
&= e^{-\theta(1+\epsilon)np} \big(p(e^\theta - 1) + 1\big)^n \\
&\leq \exp\big(-\theta(1+\epsilon)np + np(e^\theta - 1)\big) =: f(\theta)
\end{aligned}$$

where in the second inequality we used the relation between the arithmetic and the geometric mean which follows directly from Jensen's inequality:

$$\log\left(\sum_{i=1}^{n} \frac{y_i}{n}\right) \geq \sum_{i=1}^{n} \frac{\log y_i}{n} = \log\left(\prod_{i=1}^{n} y_i^{\frac{1}{n}}\right)$$

which holds for any $y_1, y_2, ..., y_n > 0$. Observe that $f(\theta) > 0$ for $\theta \in (0, +\infty)$, $f(0) = 1$, $\lim_{\theta \to +\infty} f(\theta) = +\infty$. Therefore the value of θ that minimizes f must satisfy

$$f'(\theta) = f(\theta)np(e^\theta - (1+\epsilon)) = 0$$

that is $e^\theta = 1 + \epsilon$. Finally we get

$$\mathbb{P}(X \geq (1+\epsilon)np) \leq \exp(-(1+\epsilon)\log(1+\epsilon)np + np\epsilon) = \left(\frac{e^\epsilon}{(1+\epsilon)^{1+\epsilon}}\right)^{np}$$

and this proves equation (A.1). Equation (A.2) follows directly from (A.1): indeed $\log(1+\epsilon) = \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} - \frac{\epsilon^4}{4} + o(\epsilon^5)$ implies

$$(1+\epsilon)\log(1+\epsilon) - \epsilon \geq \frac{\epsilon^2}{2} - \frac{\epsilon^3}{6}$$

and thus (A.2) is proved. □

Theorem A.2. *Let $X_1, ..., X_n$ be independent variables in $\{0, 1\}$. Let*

$$X = \sum_{i=1}^{n} X_i$$

and $p = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[X_i]$. Then we have

$$\mathbb{P}(X \leq (1-\epsilon)np) \leq \exp\left(-\frac{\epsilon^2 np}{2}\right) \tag{A.3}$$

Proof. For every $\theta > 0$ we apply the same argument as in the previous theorem, thus obtaining

$$\begin{aligned}
\mathbb{P}(X \leq (1-\epsilon)np) &= \mathbb{P}(e^{-\theta X} \geq e^{-\theta(1-\epsilon)np}) \\
&\leq e^{\theta(1-\epsilon)np} \ \mathbb{E}[e^{-\theta X}] \\
&= e^{\theta(1+\epsilon)np} \ \prod_{i=1}^{n} \mathbb{E}[e^{-\theta X_i}] \\
&= e^{\theta(1-\epsilon)np} \left(-p(1-e^{-\theta}) + 1\right)^n \\
&\leq \exp\left(\theta(1-\epsilon)np - (1-e^{-\theta})np\right)
\end{aligned}$$

Here again we choose the value of θ given by $e^{-\theta} = 1 - \epsilon$ that minimizes $g(\theta) = \exp\left(\theta(1-\epsilon)np - (1-e^{-\theta})np\right)$. Finally we get

$$\mathbb{P}(X \leq (1-\epsilon)np) \leq \exp\left(-np((1-\epsilon)\log(1-\epsilon) + \epsilon)\right) \leq \exp\left(-\frac{\epsilon^2 np}{2}\right)$$

and so (A.3) is proved. □

A.2 Lower bound for the sum of geometric i.i.d. variables via Chernoff

The Chernoff bound for the sum of Bernoulli variables allows for another pair of useful estimates, such as for a sum of geometric variables. First, take G to be a geometric distribution with probability of success p. A sampling of G, let us say $\{G = n\}$, can be represented as a sequence of $n-1$ zeros followed by a final 1, meaning respectively the first failures and the final success. Thus, a sum of geometric i.i.d. variables $G_1, ..., G_k$ is simply a long string of k such sequences. For every integer m, the event $\{G_1 + ... + G_k > m\}$ simply means that in the first m bits of the string we can see less than k ones. Recalling that each bit of the string is the outcome of some Bernoulli variable X_j with parameter p, more generally we can write:

$$\begin{aligned}
\mathbb{P}\left(G_1 + ... + G_k > (1+\epsilon)\frac{k}{p}\right) &= \mathbb{P}\left(X_1 + ... + X_{(1+\epsilon)\frac{k}{p}} < \frac{1}{1+\epsilon} p(1+\epsilon)\frac{k}{p}\right) \\
&\underset{(A.3)}{\leq} \exp\left(-\frac{\epsilon^2}{1+\epsilon^2}(1+\epsilon)\frac{k}{2}\right)
\end{aligned}$$

Note that Theorem A.1 also allows for an upper bound of the sum $G_1 + ... + G_k$ via a similar argument.

A.3 Chernoff bound for a simple random walk on $\mathbb{Z}$

As next example, consider a random walk on $\mathbb{Z}$: a particle starts from the origin. At each step the particle jumps to the right with probability $p \geq \frac{1}{2}$ and jumps to the left with probability $1 - p$. Let $X_i \in -1, 1$ be i.i.d. random variables which indicate the i-th jump of the particle, that is p is the probability that $X_i = 1$ for all i. Let $S_0 = 0$ and

$$S_t = S_{t-1} + X_t = \sum_{i=1}^{t} X_i$$

be the position of the particle after t steps. Let

$$\rho = \mathbb{E}\left[X_i\right] = p - (1 - p) = 2p - 1 \geq 0$$

be the bias of the random walk. For every t we have

$$\mathbb{E}[S_t] = t\rho$$

We want to derive a Chernoff bound for S_t by using what we have proved for the sum of Bernoulli variables: since $Y_i = \frac{1+X_i}{2}$ is a Bernoulli variable with parameter p, let

$$Y = \sum_{i=1}^{t} Y_i = \sum_{i=1}^{t} \frac{1 + X_i}{2} = \frac{t + S_t}{2}$$

and

$$\mu = \mathbb{E}[Y] = \frac{t + \mathbb{E}[S_t]}{2} = \frac{t + t\rho}{2} = \frac{t(1 + \rho)}{2}$$

By combining Theorems A.1 and A.2 we deduce that for every $\delta \in (0, 1)$ we have

$$\mathbb{P}(|Y - \mu| \geq \delta\mu) \leq 2 \exp\left(-\frac{\delta^2\mu}{3}\right)$$

and so we can write

$$\begin{aligned}
\mathbb{P}(|S_t - t\rho|) \geq \delta t\rho) &= \mathbb{P}\left(\frac{|S_t - t\rho|}{2} \geq \frac{\delta t\rho}{2}\right) \\
&= \mathbb{P}\left(|Y-\mu| \geq \frac{\delta t\rho}{2\mu}\mu\right) \\
&= \mathbb{P}\left(|Y-\mu| \geq \left(\frac{\delta\rho}{1+\rho}\right)\mu\right) \\
&\underset{\text{Theorems } A.1+A.2}{\leq} \exp\left(-\frac{\delta^2\rho^2\mu}{3(1+\rho)^2}\right) \\
&= \exp\left(-\frac{\delta^2\rho^2 t}{6(1+\rho)}\right)
\end{aligned}$$

where the Theorems A.1 and A.2 are applied with the constant $\epsilon = \frac{\delta\rho}{1+\rho} \in (0,1)$. Next we shall see a concentration result for a Poisson variable.

A.4 Concentration for a Poisson variable via Chernoff

Theorem A.3. *Let X be a Poisson variable with mean λ. For every $\delta \in (0,1)$ we have*

$$\mathbb{P}\left(\left|X-\lambda\right| \geq \frac{\delta\lambda}{2}\right) \leq \exp\left(-\frac{\delta^2}{16}\lambda\right) \tag{A.4}$$

Proof. Indeed for every $\delta, \theta > 0$ we have

$$\begin{aligned}
\mathbb{P}\left(X \geq \lambda\left(1+\frac{\delta}{2}\right)\right) &= \mathbb{P}\left(e^{\theta X} \geq e^{\theta\lambda\left(1+\frac{\delta}{2}\right)}\right) \\
&\leq \mathbb{E}\left[e^{\theta X}\right] e^{-\theta\lambda\left(1+\frac{\delta}{2}\right)} \\
&= \exp\left(\lambda\left(e^{\theta} - 1 - \theta\left(1+\frac{\delta}{2}\right)\right)\right)
\end{aligned}$$

Now observe that

$$\begin{aligned}
e^{\theta} - 1 - \theta &= \frac{\theta^2}{2}\left(1 + \sum_{j=1}^{+\infty}\theta^j \frac{2!}{(j+2)!}\right) \\
&\leq \frac{\theta^2}{2}\left(1 + \sum_{j=1}^{+\infty}\left(\frac{\theta}{3}\right)^j\right) \\
&\leq \frac{\theta^2}{2}\frac{1}{1-\frac{\theta}{3}} \\
&\leq \frac{3}{4}\theta^2
\end{aligned}$$

provided that $\theta < 3$. In order to get the last inequality we restrict further to $0 < \theta < 2$. Going back to the proof we conclude that

$$\begin{aligned}\mathbb{P}\left(X \geq \lambda\left(1+\frac{\delta}{2}\right)\right) &\leq \exp\left(\lambda\left(\frac{3\theta^2}{4}-\frac{\theta\delta}{2}\right)\right)\\ &\leq \exp\left(-\frac{\delta^2}{16}\lambda\right)\end{aligned}$$

by taking $\theta = \frac{\delta}{2}$.
Similarly, for every $\delta, \theta > 0$ we can write

$$\begin{aligned}\mathbb{P}\left(X \leq \lambda\left(1-\frac{\delta}{2}\right)\right) &= \mathbb{P}\left(e^{-\theta X} \geq e^{-\theta\lambda\left(1+\frac{\delta}{2}\right)}\right)\\ &\leq \mathbb{E}\left[e^{-\theta X}\right] e^{\theta\lambda\left(1+\frac{\delta}{2}\right)}\\ &= \exp\left(\lambda\left(e^{-\theta}-1+\theta\left(1+\frac{\delta}{2}\right)\right)\right)\\ &\leq \exp\left(\lambda\left(\frac{\theta^2}{2}-\frac{\theta\delta}{2}\right)\right)\\ &= \exp\left(-\frac{\delta^2}{8}\lambda\right)\end{aligned}$$

by taking $\theta = \frac{\delta}{2}$. Here in the second last passage we used that $e^{-\theta} - 1 + \theta \leq \frac{\theta^2}{2}$.
The theorem follows by combining the two inequalities. $\square$

Appendix B

Other tools

B.1 Thinning property for a Poisson distribution

Proposition B.1. *Assume that* $\mathcal{N}$ *is a Poisson*(λ) *distribution, indicating the quantity of some random objects. Assume that each of the said objects can satisfy some event* E *independently from all of the others. Assume also that* E *and* $\mathcal{N}$ *are independent. Then the number* $\mathcal{M}$ *of objects that satisfy the event* E *is also a Poisson distribution, with parameter* $p\lambda$*, where* p *is the probability that an object satisfies the event* E*.*

Proof of Proposition B.1. Let P_k be the event that k objects satisfy the event E. We have

$$\begin{aligned}\mathbb{P}(\mathcal{M}=m) &= \mathbb{P}(P_m)\\ &= \sum_{n\geq m}\mathbb{P}(\{\mathcal{N}=n\}\cap P_m)\\ &= \sum_{n\geq m}\mathbb{P}(\mathcal{N}=n)\mathbb{P}(P_m)\\ &= \sum_{n\geq m}\exp(-\lambda)\frac{\lambda^n}{n!}\mathbb{P}(P_m) \qquad \text{(B.1)}\end{aligned}$$

Now by definition of P_m we have to choose m particle that satisfy event E, thus

$$\mathbb{P}(P_m) = \binom{n}{m}p^m(1-p)^{n-m} \qquad \text{(B.2)}$$

by putting equation (B.2) into (B.1) and rearranging the terms we get

$$\begin{aligned}
\sum_{n\geq m} \exp(-\lambda)\frac{\lambda^n}{n!}\mathbb{P}(P_m) &= \exp(-\lambda)\sum_{n\geq m}\frac{\lambda^n}{n!}\frac{n!}{m!(n-m)!}p^m(1-p)^{n-m} \\
&= \frac{\exp(-\lambda)p^m}{m!}\sum_{n\geq m}\frac{\lambda^n(1-p)^{n-m}}{(n-m)!} \\
&= \frac{\exp(-\lambda)(p\lambda)^m}{m!}\sum_{n\geq m}\frac{((1-p)\lambda)^{n-m}}{(n-m)!} \\
&\underset{j:=n-m}{=} \frac{\exp(-\lambda)(p\lambda)^m}{m!}\sum_{j\geq 0}\frac{((1-p)\lambda)^{j}}{j!} \\
&= \frac{\exp(-p\lambda)(p\lambda)^m}{m!}
\end{aligned}$$

which is the law of a Poisson($p\lambda$) distribution as we wanted. □

B.2 A martingale argument for a biased random walk on $\mathbb{Z}$

This section of the appendix is based on the theory of martingales. We will not provide an introduction to it, but simply refer to [7], Chapter 10.

Let us consider a random walk $W_t = \sum_{t\geq 0} X_i$ on $\mathbb{Z}$, where $W_0 = X_0 = w_0 \in \mathbb{Z}$. For simplicity we assume that W_t is a discrete time process, since a similar argument works for the continuous version as well. The increments X_i, $i \geq 1$ are equal to 1 with probability $p > \frac{1}{2}$ and -1 with probability $q = 1 - p$. The first thing we are going to prove is that

$$Q_t := \left(\frac{q}{p}\right)^{W_t}$$

is a martingale w.r.t. the canonical filtration

$$\mathcal{F}_t := \sigma(W_0, W_1, ..., W_t)$$

Indeed we have

$$\begin{aligned}
\mathbb{E}[Q_{t+1}|\mathcal{F}_t] &= \mathbb{E}\left[\left(\frac{q}{p}\right)^{W_{t+1}}\Big|\mathcal{F}_t\right] \\
&= \left(\frac{q}{p}\right)^{W_t}\mathbb{E}\left[\left(\frac{q}{p}\right)^{X_{t+1}}\Big|\mathcal{F}_t\right] \\
&= Q_t\left[\left(\frac{q}{p}\right)p + \left(\frac{p}{q}\right)q\right] \\
&= Q_t
\end{aligned}$$

As a next step, given $a < w_0 < b \in \mathbb{Z}$ with $W_0 = w_0$ consider

$$\tau := \inf\{t \geq 0 : W_t < a \text{ or } W_t > b\}$$

now τ is clearly a stopping time w.r.t. $(\mathcal{F}_t)$. We would like to show that $\alpha := \mathbb{P}(W_\tau = b)$ satisfies

$$\alpha = \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^{w_0}}{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b} \tag{B.3}$$

This follows from Doob's optional stopping theorem, which is applicable here since $Q_{t\wedge\tau}$ is bounded:

$$\left(\frac{q}{p}\right)^b \alpha + \left(\frac{q}{p}\right)^a (1-\alpha) = \mathbb{E}[Q_\tau] = \mathbb{E}[Q_0] = \left(\frac{q}{p}\right)^{w_0}$$

thus we immediately get the value of α as in equation (B.3).

Bibliography

[1] Francis Galton (1873), *Problem 4001*, Educational Times, Vol. 25, Number 143.

[2] Henry William Watson (1873), *Problem 4001*, Educational Times, Vol. 26, Number 148.

[3] Theodore E. Harris (1963), *The theory of branching processes*, Grundlehren der mathematischen Wissenschaften, Vol. 119.

[4] Thomas M. Liggett, *Interacting Particle Systems*, Grundlehren der mathematischen Wissenschaften, 1985, Vol. 276

[5] Carol Bezuidenhout and Geoffrey Grimmett, *The critical contact process dies out*, The Annals of Probability, 1990, Vol. 18, 1462-1482.

[6] Russell Lyons, *Random walks and percolation on trees*, The Annals of Probability, 1990, Vol. 18, 931-958

[7] David Williams, *Probability with martingales*, Cambridge University Press, 1991.

[8] Robin Pemantle, *The contact process on trees*, The Annals of Probability, 1992, Vol. 20, 2089-2116.

[9] Russell Lyons, Robin Pemantle and Yuval Peres, *Ergodic Theory on Galton-Watson Trees: Speed of Random Walk and Dimension of Harmonic Measure*, Ergodic Theory Dynamical Systems, 1994, Vol. 15.

[10] Thomas M. Liggett, *Improved upper bounds for the contact process critical value*, The Annals of Probability, 1995, Vol. 23, 697-723.

[11] Russell Lyons, Robin Pemantle and Yuval Peres, *Conceptual proofs of* $L \log L$ *criteria for mean behavior of branching processes*, The Annals of Probability, 1995, Vol. 23 (3), 1125-1138.

[12] Thomas M. Liggett, *Multiple transition points for the contact process on the binary tree*, The Annals of Probability, 1996, Vol. 24, 1675-1710.

[13] A. M. Stacey, *The existence of an Intermediate Phase for the Contact Process on Trees*, The Annals of Probability, 1996, Vol. 24, 1711-1726.

[14] András Telcs and Nicholas C. Wormald, *Branching and tree indexed random walks on fractals*, Journal of Applied Probability, 1999, Vol. 36 (4), 999-1011.

[15] Oswaldo Alves, Fabio Machado, Serguei Popov and Krishnamurthi Ravishankar, *The shape theorem for the frog model with random initial configuration*, The Annals of Applied Probability, 2002, Vol. 12, 533-546.

[16] Oswaldo Alves, Fabio Machado and Serguei Popov), *Phase transition for the frog model*, Electronic Journal of Probability, 2002, 7-21.

[17] Harry Kesten and Vladas Sidoravicius, *The spread of a rumor or infection in a moving population*, The Annals of Probability, 2005, Vol. 33 (6): 2402-2462.

[18] Harry Kesten and Vladas Sidoravicius, *A phase transition in a model for the spread of infection*, Illinois Journal of Mathematics, 2006, Vol. 50, 547-634.

[19] Harry Kesten and Vladas Sidoravicius, *A shape theorem for the spread of an infection*, Annals of Mathematics, 2008, Vol. 167 (3): 701-766.

[20] Shirshendu Chatterjee and Rick Durrett, *Contact processes on random graphs with power law degree distributions have critical value 0*, The Annals of Probability, 2009, Vol. 37, 2332-2356.

[21] Craig A. Tracy, Harold Widom, *Asymptotics in ASEP with Step Initial Condition*, Communications in Mathematical Physics, 2009, Vol. 290, 129-154

[22] Shankar Bhamidi, Remco van der Hofstad and Gerard Hooghiemstra, *First passage percolation on random graphs with finite mean degrees*, The Annals of Applied Probability, 2010, Vol. 20, No. 5, 1907-1965.

[23] Harry Kesten, Alejandro F. Ramírez and Vladas Sidoravicius, *Asymptotic shape and propagation of fronts for growth models in dynamic random environment*, Probability in Complex Physical Systems, Springer Berlin Heidelberg, 2012, 195-223.

[24] Sergey Foss, Dmitry Korshunov and Stan Zachary, *An Introduction to Heavy-Tailed and Subexponential Distributions*, Springer, 2013.

[25] David A. Levin, Yuval Peres and Elizabeth L. Wilmer, *Markov Chains and Mixing Times*, Second Edition, PDF version available on the author's website http://yuvalperes.com/, 2017.

[26] Christopher Hoffman, Tobias Johnson and Matthew Junge, *Recurrence and transience for the frog model on trees*, The Annals of Probability, 2017, Vol. 45, No. 5, 2826–2854.

[27] Christopher Hoffman, Tobias Johnson and Matthew Junge, *Infection spread for the frog model on trees*, Electronic Journal of Probability, 2019, No. 112, 1-29.

[28] Elcio Lebensztayn and Jamie Utria, *Phase transition for the frog model on biregular trees*, 2020. arXiv:1811.05495v2.

[29] Xiangying Huang and Rick Durrett, *The Contact Process on Random Graphs and Galton-Watson Trees*, ALEA-Latin American Journal of Probability and Mathematical Statistics, 2020, Vol. 17, 159-182.

[30] Shankar Bhamidi, Danny Nam, Oanh Nguyen and Allan Sly, *Survival and extinction of epidemics on random graphs with general degree*, The Annals of Probability, 2021, Vol. 49, 244-286.

[31] Duncan Dauvergne and Allan Sly, *The sir model in a moving population: propagation of infection and herd immunity*, arXiv preprint arXiv:2209.06037, 2022.

[32] Rangel Baldasso and Alexandre Stauffer, *Local and global survival for infections with recovery*, Stochastic Processes and their Applications, 2023, Volume 160: 161-173.

[33] Duncan Dauvergne and Allan Sly, *Spread of infections in a heterogeneous moving population*, Probability Theory and Related Fields, 2023, 1-59.