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Use of Minkowski spacetime diagrams in teaching Special Relativity - an instrument to “see” relativistic effects: the spacetime globe

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Ἐν ἀρπχῇ ἦν ὁ λόγος,
καὶ ὁ λόγος ἦν πρὸς τὸν θεόν,
καὶ θεὸς ἦν ὁ λόγος.
οὗτος ἦν ἐν ἀρπχῇ πρὸς τὸν θεόν.
πάντα δι᾽ αὐτοῦ ἐγένετο,
καὶ χωρὶς αὐτοῦ ἐγένετο οὐδὲ ἕν. ὃ γέγονεν
ἐν αὐτῷ ζωὴ ἦν,
καὶ ἡ ζωὴ ἦν τὸ φῶς τῶν ἀνθρώπων·
καὶ τὸ φῶς ἐν τῇ σκοτίᾳ φαίνει,
καὶ ἡ σκοτία αὐτὸ οὐ κατέλαβεν.

Gv 1, 1-5

Abstract

Special Relativity introduces students to Modern Physics, whose importance in the high school is increasing. Nevertheless its teaching and learning is a critical issue. Different solutions have been developed to overcome the encountered difficulties with a particular emphasis on Minkowski's spacetime diagrams.

In this work we describe the *spacetime globe*, a mechanical instrument that allows experiencing Special Relativity hands-on. We show how it is possible to treat all the main phenomena foreseen by Special Relativity with simple laboratory experiences, using the idea of Minkowski's spacetime diagrams. Our approach is based on the idea that Special Relativity can be introduced to students underlying the problem of the point of view, namely that at the root of this theory there is the fundamental issue of understanding how events are seen from different points of view.

In order to understand if it may be an effective didactic tool, we set up a pilot experimentation carried out with five groups of students of the last year of high schools oriented towards scientific studies. We show the result of the pre-test and the post-test submitted to the students from which arises the persistence of the commitment to alternative representations of Classical Mechanics.

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Ognuno, con la propria ineguagliabile e particolare grafia, ha scritto una parte, un capoverso, un rigo o anche una sola parola. E anche se qualche penna ha terminato l'inchiostro o qualche matita è diventata ormai troppo corta per poter scrivere, il suo carattere è rimasto lì dove l'ha impresso.
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Ad Dei maiorem gloriam

Contents

Introduction

SPECIAL RELATIVITY is one of the most beautiful and revolutionary theories: it marks a definitive point in the History of Physics opening to a new era. It unifies the Mechanics with Electromagnetism introducing strong inferences over space and time, thus giving a new vision of the world. Every book of Physics and general Science, being it addressed to students or to general public, contains at least a reference to Special Relativity due to the wide spectrum of phenomena it deals with.

Since many years, high schools are trying to keep up with the new discoveries of Modern Physics, making it necessary to introduce the main topics of Special Relativity within Physics curricula. Students start to face new aspects of reality far from their everyday life, thus encountering difficulties that have been already observed in the undergraduates, mostly tied with their alternative representations in Classical Mechanics. Numerous solutions are being proposed to solve the didactic issue of how effectively teaching Special Relativity in high schools: in literature a common consensus[1] over the use of spacetime diagrams was found but this approach is still not widely diffused. The recent review [7] of Prado et al. clearly points out that "*the literature shows that using spacetime diagrams is an efficient procedure to answer and explaining questions, dilemmas and paradoxes of the theory*" [7].

The present PhD dissertation thesis focuses over a teaching project for high schools on Special Relativity based on the spacetime diagrams. Since literature refers to the concept of reference frame[2] as on of the main source for misunderstandings on Special Relativity (as well as in Classical one), we used a mechanical instrument that allows to visualise the abstract concept of reference frame paying attention to consider it as a "point of view" of an observer, in the same way Dimitriadi and Halkia [13] did.

Chapter 1 ***State of Art*** contains an introduction to some key concepts in Didactic of

[1] [1, 2, 3, 4, 5, 6]

[2] [8, 9, 10, 11, 12]

Physics for the process of learning (conceptual change and capture) and how they are applied to Special Relativity. Then the main features of two different models of student's way of thinking are introduced together with the description of their alternative conceptions in Classical Mechanics and Special Relativity as it emerges from the literature. Finally different approaches to teaching Special Relativity are presented, with particular emphasis on Minkowski's one.

Chapter 2 ***The Special Relativity*** contains a review of the main topics of Special Relativity together with a small historical introduction outlining the works preceding Einstein's one that anticipated the contents of his theory. As Arriassecq and Greca [14] pointed out, it is important to contextualised scientific "discoveries" and in particular the Theory of Special Relativity, about which historians do not completely agree on the works already existing before it. This contextualisation aims to remember the reader or to make him aware that the scientific theories are not developed in a *in a conceptual vacuum* ([14]) nor they are an invention of a genius but they are inside the flow of current of works. My dissertation would like to be a "didactic thesis on didactic of Special Relativity", meaning that the reader could learn something else about Special Relativity itself. For this reason in that same chapter I added also a description of this theory from Minkowski's perspective showing how relativistic phenomena imply the existence of an absolute spacetime and how they are implied by it.

Chapter 3 ***The Special Relativity's project*** contains the description of the spacetime globe, the mechanical instrument we used for our project together with the explanation of how to show relativistic phenomena with it. It is a more complete and detailed version of our previous work ([15]). We also showed a virtual version of the instrument through simple animations realised with Python codes with a detailed example in order to fulfil the issues of the Digital Didactic.

In Chapter 4 ***School experimentation*** we described our pilot experimentation implemented in order to understand whether the use of the spacetime globe is an effective didactic tool. It consisted in two lectures carried out with some groups of high-school students of Rome. We explained the topics of Special Relativity using the mechanical instrument and we administered a pre- and a post-test with open and closed questions. Their whole texts are reported in a drive folder (see Appendix C). We reported also some inferences over the questionnaires arising from a statistical analysis of students' answers together with a discussion of their replies to the open questions in terms of misconceptions and common or scientific knowledge.

In Chapter ***Conclusion*** we summarised the conclusions of this work and its future outlooks. Finally in Chapter ***About teaching*** I pointed out some considerations about teaching in relation to students arising from my experience.

Chapter 1

State of Art

SCIENCE educational research is a complex issue: teaching and learning physics is neither a one-way process departing from the teacher and arriving on the student nor the inverse one-way process. The full attention is no more focused on the teacher, the only one "who knows", nor on the student, the one "who builds" his own knowledge.
Knowledge is built in a two-bodies process involving both the teacher and the student in a *teaching-learning process*. It is an adaptive action resolving into a continuous organisation of the teaching apparatus depending on the current context. Teaching depends on how students learn but the learning relays upon how teacher is able to dialogue with students' ideas. Educational research has to examine how students' brain works in the learning process, how students organise the preexisting ideas in the light of new ones and how they face the problems challenging them: it concerns the theory of learning ([16, 17]) and the nature of the students' ideas ([18]).

1.1 Conceptual change and capture

What is learning[1]?
Such an open question can not be solved into an easy description but without focusing on what learning depends on (which is an also wider question), we can state that learning is a process in which a concept is substituted with another one: a *conceptual change*. Here with concept we referred to a single conception (e.g. "space is not absolute") or to a theory (e.g. the theory of Special Relativity) that is a set of subsequent different affirmations both in their mathematical and logical aspects.
As Piaget showed in his studies, since the early age pupils try to give an explanation of natural phenomena starting from their sensory experiences. The first naive experiments

[1] The content of this section refers to the works of Posner et al. [16] and Hewson [17].

children perform allow them to create their own interpretive framework of the reality. It constitutes the substratum the students have to face with during all the learning process; it is the base necessary to activate the construction of knowledge. Indeed the students' prior knowledge has a critical role in learning. When students face a new phenomenon, at first their mind interprets and organises the experience or the concept framing it within an already existent structure: in this way it simplifies, orders and gives meaning to the experience. This first step is known as *assimilation* which has the preexisting concepts as instruments to investigate the new ones: students try to verify if what they already know is sufficient to explain the new concepts. This is a natural part of a learning process but sometime it fails: students understand that they are not able to grasp the full meaning and essence of a new phenomenon or concept because of their insufficient reading key. They need to reorganise or replace their previous commitments. This second step is known as *accommodation.* The whole teaching effort is to allow the transition from assimilation to accommodation to happen in order to favour the conceptual change. However it is clear that there must be certain features of the new concept under which this process is likely to take place. At the very beginning there must be a *dissatisfaction* between the current conceptions and the problems students are trying to solve: that is, students should become aware that their commitments are not able anymore to explain a certain set of phenomena. We can refer to this dissatisfaction as a *cognitive conflict* which paves the way to the accommodation process: it is important this conflict to be created in students' mind otherwise they will not fell the urgency to adopt new concepts. Often *anomalies* in theories can guide the accommodation showing some specific failures that require a new framework of understanding even though sometimes they can not be seen really as anomalies by the students. Thus what might create a conflict in the teacher's mind could not create any dissatisfaction in the students. This is due to their own criteria of judgement which are made up of their previous metaphysical beliefs and epistemological commitments. Researches[2] showed that students also in front of evident (according to the instructor) incongruities between for example relativistic phenomena and Classical Mechanics tried to force their stable certainties to fit with them.

Once the cognitive conflict finally happened, there is a selection of the new proposed conceptions: they must fulfil some parameters to gain acceptance in order to appear to make sense. A new concept must be *intelligible*, namely the students must be able to understand its content sufficiently to permit them to explore its inner implications. Intelligibility requires a concept to be understood in its mathematical aspect: students need to grasp the terms and the symbol used and how they are combined into more complex expressions. This is the case of the Special Relativity which has an easy mathematics and it does not

[2]See Section 1.4 for further details.

create any problems to students managing with its kind of algebra.
But this aspect is not sufficient: intelligibility is the ability of a concept to be coherently represented by the individual. The inner *representation* of a theory may be in the form of images or some kind of networks: it is the frame the information must suit and it addresses the student's attention and researchers.
Another important requirement for a new concept is to be *plausible* that is it has to succeed where the previous ones failed. It has at least to be able of solving the conflict that previously it brought about. The concept has also to be consistent with the other knowledge to prevent internal contradictions. However students can find a concept to be plausible not only for its capability of solving anomalies but also for other reasons: it seems consistent with their current metaphysical beliefs and epistemological conceptions, with other known theories, with past experiences or with their vision of the world.
Finally a concept that is intelligible and plausible, capable of solving anomalies, should be *fruitful* to completely address the process of accommodation, namely it should also guide students in discovering new areas of inquiry.
Sometimes the process of accommodation can take another direction and does not favour a conceptual exchange but a *conceptual capture*: our learning model should take into account that a concept can also be reconciled with other existing concepts. *Reconciliation* is the process in which students give sense to a new concept at the light of their existing knowledge. The new commitment and the previous conceptions are not contradictory but they share some features as they are part of the same set of ideas.
Depending on the status (intelligible, intelligible and plausible or intelligible, plausible and fruitful) of an existing conception and of the new one, a conceptual change or a conceptual capture can happen as well as a completely rejection.
The research of Hewson [17] shows that previous alternative frameworks constitute a robust obstacle towards a deeper learning and understanding of the correct interpretation of natural phenomena in various fields of Physics. He argues that if the status of these alternative conceptions is not accurately debated, the new taught concepts will not be understood: they will be rejected, yet memorised or classified as part of a "science world". It is just another different view of reality, completely detached from the real world.

1.1.1 Focus on Special Relativity

The described process of accommodation seems to be easy and linear. If applied to the case of Special Relativity, it should be outlined as following: students, starting from a dissatisfaction with Classical Physics, start dealing with a new intelligible and plausible theory, which is able to solve all the previous contradictions, to predict new phenomena and to provide applications also to other fields of Physics. This process will allow them to com-

pletely grasp the meaning of Special Relativity and its implications.
However these steps are oversimplified:the basic conceptions of Special Relativity are very complex and it is likely that it should require different time to each one of them to be accommodated. It is not a radical change but a gradual progress during which students accommodate some of the features and the claims of a new single conception and then they gradually modify the other ideas as they have fully understood the meaning and the inferences of the new concepts.
The intelligibility of Special Relativity seems not to be problematic when Einstein's two postulates are concerned. However the representation of the whole theory is quite difficult as it requires to build a coherent description of a world in which the two postulates are true together with their consequences about the concept of space and time.
The inferences on the plausibility of this theory are crucial. Once again instructor must be careful: Special Relativity has some features that historically are said to fit Einstein's requisite for a plausible theory, i. e. parsimony and symmetry, in addition to the fundamental ability of solving the previous anomalies. But students do not share the same Einstein's epistemological beliefs. Special Relativity appears to them as counterintuitive as long as the they will trust in absolute space and time. It is important to find out students' epistemological commitments in order to understand what at first students would consider plausible or not.
Indeed Hewson's research shows how important are the learner's previous metaphysical commitments to Classical Physics. He pointed out that an instruction addressing towards them is effective. In particular he found out how the previous conceptions remained stable also in the first months after instruction on Special Relativity in one undergraduate student: only after many interventions of the instructor his view changed. At the very beginning he tried to reconcile this new theory with his previous knowledge ending up in a product of Einsteinian concepts with Newtonian foundations. The whole theory seems not to represent an issue except for the two mainly counterintuitive phenomena, namely the slow running of a moving clock and the shrinking of the moving road. As a matter of facts, the resistance shown by the student is due to his mechanistic view of the reality where objects have some fixed properties as mass, length and so on. These attributes are independent of the observer's measure, leading to the existence of a preferred observer (the one at rest) who has "more reason" then the moving one. The former will perceive the objects as they really are while the latter will have only a distorted perception of the world: roads appear to shrink, clocks seem running slower. At first, the student integrated two counterintuitive conceptions within his exhausting frame of known phenomena.
This research highlights that students' previous frameworks including metaphysical commitments play a key role in the process of accommodation in learning Special Relativity.

A conceptual exchange can not really happen if there is not such a dissatisfaction leading to a lower reliability of the mechanistic view of reality and of the Classical Mechanics in general.

1.2 Students' thinking

Next to the discussion about what learning is, the other important aspect is how students think[3]. In order to deal with students' alternative interpretative frameworks we have to consider how they interact with new conceptions: our didactic sequences should be organised specifically depending on students' way of thinking. Together with the traditional *misconceptions model* in which students' ideas are considered as a set of fixed elements creating barriers to instruction, a new one has been recently proposed. The *pieces model* assumes students' ideas as context-depending adaptable "knowledge pieces" that learners activate independently or in networks.
Both the models point out some features of the students' ideas Scherr [18] highlighted:

- *determinacy*: students' ideas are not simply correct or incorrect. Often they are referred only to a specific situation but not the correct one. Thus the idea that "the higher the effort, the greater the effect" has an indeterminate state of truth until it is not applied to the valid context (e.g. to the relation force-acceleration and not to force-velocity). Determinacy is the property of an idea to be truth-determinate or truth-indeterminate;

- *coherence*: students' ideas seem often to be contradictory according to the instructor knowledge even if learners do not feel this contrast. Coherence should be related to how the internal system of logics of students creates their networks of ideas but it is biased to the stance of the instructor;

- *context-dependence*: researches found that students' answers to a question are influenced for instance by the setting where a question is asked (at school or in everyday life) and by the situation (during a class or a clinical interview). Thus context-dependence refers to all the surroundings of a question that can influence students' performance;

- *variability*: students' ideas can change or not very quickly during the same situation. Ideas can be fluctuating if they change frequently and they are not stable;

- *malleability*: students' ideas are expected to change after instruction. Malleability refers to the difficulty of the accommodation's process: ideas can be rigid if they are

[3] The content of this section refers to Scherr's work [18].

resistant to the change otherwise pliable.

These features are declined (see Table 1.1) according to the particular modelling of students' thinking towards which instructors and teachers have to behave differently in order to favour the learning process.

	Misconceptions model	Pieces model
Determinacy	True or false	Indeterminate
Coherence	Coherent	Potentially mutually independent
Context-dependence	Context-independent	Potentially context-dependent
Variability	Stable	Potentially fluctuating
Malleability	Rigid	Potentially pliable
Research agenda	Find coherent frameworks	Find useful pieces
Instructional agenda	Elicit, confront, resolve	Refine intuitions
Changes in understanding	Difficult, permanent	Potentially easy, temporary

Table 1.1: Table gathering the features of students' ideas according to the two different models of thinking (from Scherr [18]).

The misconception model is the most traditional description of students' ideas: it outlines the existence of alternative structured logical frameworks students use to solve problems. They are (apparently) truth-determined as students think of their ideas to be correct; it was observed that they are also coherent and rigid, stable and independent of the context a problem refers to. This set of ideas is well organised inside students' mind probably because they had it for a long time and thus we can expect students to have different alternative frameworks and to see them emerging often in various contexts. The instructional sequences should aim to identify these alternative students' interpretations, to make students aware of them, then to show proofs of their inadequacy and finally to help them resolving these contradictions. Because of the solidity of these ideas, the learning is difficult yet permanent.
The pieces model is more dynamical then the previous one: ideas are not gathered into a solid frame and we refer to single pieces of knowledge that students activate independently or in a network, depending on the problems. Ideas do not have a determined state of truth as they depends on the situation and they are context-dependent. Being single pieces, students' ideas are independent of one another, fluctuating and pliable. For this reason we should not expect to have coherence between students' answers, yet we will see contradictions as different situations activate different pieces of knowledge. Instructors can have a prove of this model of thinking looking to the variability of students' answers or in their changing during thought process. Indeed instructors and teachers have to find out

the basic idea that guides students' reasoning which can either help them or not[4]. Once these pieces have been identified, the instruction has to be addressed in order to refine students' intuitions: the elements are not incorrect by themselves but they are only applied to a wrong context, thus requiring a refining. Students should be guided to understand how to incorporate these concepts into their reasoning rather then denying. Instruction is expected to be easier then the misconceptions model but potentially temporary.

1.3 Alternative conceptions in Classical Mechanics

The roots of Special Relativity lay deeply in some features of Classical Mechanics. The researches on Einsteinian Relativity show also that students' problems are related to concepts of Mechanics ([19]) which are not thoroughly treated at the time of classical instruction, first of all the concept of reference frame and event ([17, 20, 8, 9, 21]).
Many authors discussed about the influences the previous knowledge has on the learning process of Special Relativity: for instance Hewson [17] stressed the importance of the Newtonian commitments into the explanation of relativistic phenomena as well as Selçuk [22] pointed out that some misconceptions would arise from a failure in the accommodation of the concepts of Galilean Relativity. While investigating problems about the speed of light, Villani and Pacca [23] found that university students use pre-relativistic concepts very similar to Saltiel's "spontaneous" kinematic ([20] and later in the text). Moreover they specified that it is far from being real to assume that students learning Special Relativity have already fully grasped the meaning of Galilean Relativity. Indeed de Hosson et al. [9] stressed the deep students' misunderstanding of the structure of Classical Mechanics which is an obstacle to the comprehension of Special Relativity. Thus students' alternative representations of Classical Mechanics should be taken seriously into account while trying to outline their thinking on Special Relativity as the former involves concepts that are far from being simple, even if easily relating to everyday life experience. Students learn that motion is frame-dependent in contrast with the usual dichotomous distinction between "rest" and "motion" which is commonly always thought as well defined.
The major problem literature highlights is the concept of *reference frame*, a set of observers at rest with respect to each other, an operative concept which seems not to be hold by students ([9]). In their review on Special Relativity, Alstein et al. [21] began from analysing the literature's result on the use and understanding of reference frames and on other classical concepts that will be discuss later in the text.
Different authors as Villani and Pacca [23], Özcan et al. [10], Scherr et al. [8] and de Hosson et al. [9] agree that it is necessary to start from the concept of reference frame for a

[4]As for example the idea that "the higher the effort, the grater the effect" which is correct if applied to the relation force-acceleration and not to force-velocity.

full understanding of Special Relativity in order to demolish the idolatrous stone of the absolute frame, rebuilding the correct representation.
Special Relativity as well as Classical one needs a full correct understanding of the meaning of reference frame ([8, 21]). For instance some of the alternative conceptions about loss of simultaneity are actually alternative conceptions about reference frames.
Panse et al. [11] outlined seven different alternative conceptions about reference frames:

I *reference frames are concrete objects*: it seems that students are likely to associate the idea of a reference frame together with a physical bodies. They refer to them as if they are concrete objects: this seems to be only a convenient way of thinking, perhaps related to the next ones. For instance a ship can have such a reference frame that suffers friction in water too;

II *reference frames have physical extension*: students share the idea that reference frames have some bounders, delimited by the physical extent of the body the are fixed to, and then each object defines a different reference frame regardless of the relative state of motion. There are three major aspects: the physical extension of the frame, its boundary and a finite length of the coordinate axes related to the dimension of the object. For instance a ball thrown out of moving car is going outside the reference frame of the car or two different observers at rest with respect to each other constitute two different reference frames;

III *reference frames have appendices*: students are likely to consider as part of a larger reference frame small moving bodies inside it regardless of their speed with respect to it. For instance a man walking on the deck of a ship as well as a shoot bullet are part of the ship's reference frame;

IV *particular phenomena happen in particular frames*: this less prominent conception has phenomena happening in particular reference frame, either the one in which it takes place or the one which the phenomena is viewed from. A stone dropped inside a train takes place in its reference frame while a stone dropped outside in the ground's one. In the same way, there is the belief that some events could happen only in one particular reference frame nor in two different: for instance a ball trowing out of a moving train is a phenomenon for the reference frame of the ball itself, not for the train or platform's one;

V *real and apparent motions*: students have the tendency to choose the most easy and natural reference frame, even when it is not the most favourable to study a particular motion. This is not a real problem except if they use it to express judgements about the realness or the apparentness of that motion. For instance a child in a train leaving a station is really moving with respect to the platform while this one is not moving with respect to the child. A third "neutral" observer would be necessary to express a true statement whether it is real or apparent - an heritage of Newton's idea of absolute space, as also Dimitriadi and Halkia [13] pointed out;

VI *physical description from viewing*: when students are describing a motion with respect to a particular frame, they are likely to refer to its visual appearance or to the moving object as seen from that frame. Students are certain of a motion in a particular frame because they can see that motion without using the law of transformation of velocities from one frame to another. It was also found that they tend to confuse measuring with viewing. This conception can be related to the connotation of the *observer* leading students to consider physical viewing as the main issue in relative description of phenomena;

VII *pseudorelativism*: students have a natural idea of relativity of motion that does not include the idea of a reference frame, in the way that the description of motion is determined by different observers. If this idea is combined with the previous alternative conceptions, different descriptions of phenomena (as trajectories of moving objects) within the same reference frame emerge depending on how they are viewed.

As a result, Panse et al. [11] pointed out that undergraduates seem to consider reference frames as decorative elements without using them to physically formulate and explore the meaning of the principle of relativity. Synthesising, students have difficulties to determine *what makes a reference frame* ([24]). In the same way Saltiel and Malgrange [20] found that students never think in terms of reference frame. Students seem to "geometrise" the space imaging motions happening in an unique geometrical space which is "frozen", independent of observers. Students give birth to a *spontaneous kinematic*: motion becomes a property of the moving body rather then of the observer as well as their trajectory that can be rigidly transferred from one reference frame to another, regardless of observers and time. More explicitly velocities as well as forces exist independently of reference frames with the consequence that motion and rest are defined intrinsically. These ideas are settled in the rooted conception that motion happens only in the frame where there are forces which cause it revealing the existence of a "proper", real motion with respect to a non physical one which is an optical illusion.
The result of the research of Ramadas et al. [12] confirmed that the Saltiel and Malgrange's geometrical view of motion affects also the main features of Galilean Relativity: students prefer to trust to their intuitive kinematic ideas of time and distance and to the theory of "physical drag" for velocity composition. Time interval between fixed events is implicitly thought to be invariant but students do not perceive the contradiction when a calcula-

tion violates the absoluteness: time invariance is not considered as an axiom of Galilean Relativity. As far as distances are concerned, despite of the simultaneity of two events, students consider the relative distance to be invariant because of their intuitive concept of frozen space, often to the detriment of time invariance. Distances covered on moving platform are often seen as equal for observers on the same reference frame and on the ground's one, despite of the platform's speed. Similarly ([20]), when starting and arriving points are identified by some fixed targets on the moving platform, the distance covered does not change according to different observers as the one travelled on the platform is more "intrinsic", more real. Finally students succeed in correctly transforming velocity between reference frames, not using at all the transformation relations but a mechanism of a physical drag as if the moving reference frame is carrying the moving observer in it. An evidence of this reasoning is that on a moving platform moving forward is "easier" then backward. Moreover students do not consider the principle of Galilean Relativity as a powerful tool to determine answers to different problems but only another among the laws to be memorised and its violation is rarely recognised.

Finally we have to take into account students' understanding of time and simultaneity which was deeply investigated by Scherr et al. [8] in relativistic context, revealing some alternative concepts that should be already clear from Classical Mechanics. In particular both Scherr et al. [8] and de Hosson et al. [9] found that students are likely to consider the time of an event as the one at which a light signal reaches an observer, showing a missing right perception of the meaning of event as already Hewson [17] shown. The consequence, strictly connected to the idea of reference frame, is that different observers standing in different positions but in the same reference frame will disagree on the time coordinate of an event. At the same time observers in the same position are considered belonging to the same reference frame, regardless of their state of motion. Similarly, de Hosson et al. [9] found that reception of light signals by different observers in the same point of space but in relative motion creates some problems to students, revealing difficulties linked to the concepts of events and reference frames.

The aim of this short review of the main issues concerning the relation between Galilean and Special Relativity was to underline that a misunderstanding of Classical Mechanics can still be dominant in students trying to solve relativistic problems; it constitutes an additional source of alternative concepts that strongly influence their answers.

1.4 Alternative conceptions in Special Relativity

In the previous Section we have shown that an incorrect interpretation of the concept of reference frames plays a key role in understanding Special Relativity. It is straightforward that if students believe in the existence of a preferred reference frame or of a real and apparent motion, they will miss the full meaning of Einstein's theory. Even if Special Relativity predicts new phenomena which are relativistic ones, often students' misconceptions are ascribable to their incorrect understanding of Classical Mechanics as we will see. We now report what the literature identifies as the most frequent alternative frameworks in Special Relativity summarised some years ago by Aslanides [25] and more recently by Alstein et al. [21] and Prado et al. [7]. We will not distinguish between the different level of instructions: as Dimitriadi and Halkia [13] quoted, most of the researches have been carried out with university students (undergraduates or graduates) but an higher instruction is not a synonymous of a deeper understanding as Scherr et al. [8] highlighted.

1.4.1 Postulates of Special Relativity

About the two postulates of Special Relativity we have to distinguish between their individual understanding and their role within the theory as a whole. Indeed Posner et al. [16] wrote that taken individually the two postulates are not problematic but the comprehension within the theory as a whole is problematic.

First postulate - principle of relativity

The first postulate is nothing more then the extension of the Galilean principle of relativity to the optics and electrodynamic phenomena. Thus one can infer that if students do not use the principle of Classical Relativity as an operational tool (Section 1.3), its use in Special Relativity could not be different. Indeed Gousopoulos et al. [26] found that students failed into perceiving the equivalence among motionless and uniform motion which is a more fundamental classical issue rather then a relativistic one. The following works are not explicitly referring to Special Relativity but contain some important students' conceptions about the Galilean principle of relativity sometimes within relativistic situations. These learning difficulties have been addressed however by the Alstein et al.'s work as part of the frame of misconceptions about Special Relativity.
The work of Pietrocola and Zylbersztajn [27] goes in this direction: when asked if high or low speeds influence some phenomena (mechanics ones as well as electrodynamics ones), students do not use the principle of relativity but some arguments not always correct related to the concept of inertia. Indeed they use the not-existence of pseudo-forces to explain the equivalence of laws between different reference frames ([21]).
In the same way Bandyopadhyay [28] found that students do not use the principle of relativity in solving the problems, preferring a more procedural strategy. They understand it in a kinematic and not dynamical way, obtaining that reference frames are equivalent as there is a kinematic reciprocity given by the relative speed. This statement is however

true also for non-inertial frames that share a kinematic quantity, the relative acceleration. Bandyopadhyay also noticed that students do not fully grasp the relation between the invariance of laws and the transformation of physical quantities and that, as a consequence, the former implies the latter and not the Galilean transformations as a student suggested. This problematic was also pointed out by Özcan et al. [10].

Second postulate - light's speed

The review of Alstein et al. reported that the second postulate in literature received less attention then the first. According to Aslanides [25] and Dimitriadi and Halkia [13] students find it easier at least to be understood and to be applied while conversely Kamphorst et al. [24] underlined the difficulty students face dealing with it. Maybe the textual understanding of the statement is easier as at first it does not involve particular and complex reasoning.
However this is not sufficient: we can not infer that students fully hold its meaning as its consequences are not yet completely accepted. As Posner et al. [16] suggested, *the more one accepts Newtonian mechanics the harder it will be to imagine a world in which the postulate about constancy of the speed of light is true.* Indeed, as we will see from the subsequent misconceptions, especially about simultaneity, the Newtonian heritage still influences students' reasoning in relativistic context. Dimitriadi and Halkia found that students think of light as having an intrinsic own speed which is constant (c) but differently measured by various observers. This reveals that Galilean velocity addition is still adopted and that motion is view as intrinsic, in the way Saltiel and Malgrange [20] found. Similarly, Gousopoulos et al. [26] found that students report correctly the principle of invariance of light speed but still use Galilean addition of velocities.
Also Kamphorst et al. drew attention to the fact that students carry on using pre-relativistic model even dealing with light speed as for example using Galilean velocity addition and interpreting c as the maximum speed one can obtain. In the same way, Prado et al. [7] found that there is *still confusion between the invariance of the speed of light and the feature of being a limiting velocity* and that the limited value of the speed of light is ascribed to the insufficiency of technology ([7, 13]).

1.4.2 Simultaneity

The understanding of the concept of simultaneity has been widely investigated by Scherr and the outcomes of her researches have been published in different works ([8, 29, 18, 30]). As a general result it emerges that students fail to spontaneously apply the relativity of simultaneity ([8]). In particular Scherr underlined two different students' attitudes:

- *simultaneity is observer-dependent*: this misconception is rooted into students' ideas of reference frames as constituted by the observer himself. Scherr found two different ways of reasoning. Many students tend to identify the time of an event with the time at which the light signal from the event reaches the observer (as also shown in [9, 13]): therefore simultaneity depends on the time order of the received light signals. As also de Hosson et al. [9] pointed out, observers in the same reference frame could not agree on the simultaneity of two events. Instead other students believe into the sensory experiences of the observers: if an observer does not have any direct evidence of an event, it has not happened. Actually they fail to recognise that observers can have access to additive information from other observers in their own reference frame.

- *simultaneity is absolute*: this result is quite predictable and it was also found by other researches ([22, 13, 26]). Students strongly believe that simultaneity is absolute and thus that two events happen at the same time in each reference frames. Moreover the fact that observers are intelligent and correct for the signal travel time seems not to help students ([13]) for this is considered as the demonstration of their belief: as the observers make corrections then simultaneity is absolute. Any other appearance of the contrary is an illusion due to the different receptions of the light signals. In addition Scherr found also that the signal travel time is thought to be influenced by the desynchronization term in the Lorentz's transformation ($-v\,dx/c^2$) while its reception by the relative motion which affects the timing of events in the moving reference frame only in this way.

Scherr's analysis showed that, even after instruction, students tried to combine what they thought to have learnt with their existent conceptual framework according to which simultaneity could be absolute but yet relative, depending on the reception of the time interval signals. In particular she proposed ([18]) that students held three different beliefs: events are simultaneous if observers receive a light signal from them at the same instant of time, simultaneity is absolute and each observer constitutes a different reference frame. Thus, summarising ([21]), simultaneity does not depend on the relative motion between the observers and the events. As a consequence, students fail to apply the correct procedure of measuring time and of ordering events time using synchronised clocks.

1.4.3 Causality

Strictly tied with the argument of simultaneity, there is the issue of time ordering, namely causality. As Scherr et al. [29] highlighted, students' strong belief in absolute simultaneity prevent them from considering real implications of the cause-effect relationship. Scherr proposed a scenario in which a tape player is in the middle of a moving train with respect to the ground. Two light wavefronts are sent towards the tape player from the two side of the train, simultaneously in the reference frame of the train. If the signals reach the tape player at the same time, it remains silent whereas if the one coming from the front of the train arrives firstly, it plays otherwise if the one coming from the rear arrives firstly, it again stays silent. It is straightforward to see that a missed understanding of simultaneity gives rise to two different realities: in the moving reference frame an observer will listen to the music whilst in the ground's one an observer will not hear any sound. Students actually try to overcome this incompatibility depicting two different situations:

- students think that in the train's reference frame the tape player sounds while in the ground's one not. Even if they strongly debate about the impossibility that in the two reference frames different events happen, they conclude *that special relativity implies that events occurring in one frame do not necessarily occur in all frames* ([29]);
- students invent "alternative realities" borrowing ideas from Quantum Mechanics in which the tape player both sounds and does not sound.

However students seem not to understand that they have to choose between the violation of causality (the same event does not happen in all the reference frames) and the abandonment of absolute simultaneity. They rather prefer to create a personal scenario in which causality is irrelevant in their analysis and absolute simultaneity is preserved as in a pre-relativistic world. Scherr argued that this due to the fact that students do not reach such a level of deep understanding to consider violation of causality.

1.4.4 Time dilation

A very common misconception about time dilation was found by Gousopoulos et al. [26] in almost all the students, namely the perception of time as a frame-independent, thus an absolute physical quantity: *Time is what it is, it can't change.*
Gibson [31] investigated the use of time dilation phenomenon and observed that students get confused using the formula as they do not understand which variable refers to a reference frame. Moreover they often use the formula just because a time is given in a problem. Selçuk [22] found that students think that time flows differently for different observers belonging to different reference frames but the time dilation phenomenon is asymmetric since it effects only the observer in motion. In the same way, according to Özcan et al. [10], time dilation can not occur in the reference frame at rest. Aslanides [25] suggested that this may be linked to students' inclination to consider as "real" the physics phenomena happening on the ground or on the Earth's reference frame ([20, 21, 22, 13, 12, 31]), thus implying the existence of an absolute reference frame ([11]). Moreover he introduced

a new misconception strongly connected with this one that he called "asymmetric time dilation": if time runs slowly for clocks “absolutely moving”, therefore in comparison it must goes faster for clocks “absolutely at rest”.
Then the root of this misconception lays in the misunderstanding of the relativity of motion which again turns in students’ wrong ideas about reference frames.

1.4.5 Length contraction

Similarly to what was already found for the time dilation phenomenon, Gibson [31] found that students get confused using the length contraction formula and they employ it when a length is involved in a problem. Additionally Gibson [31] and Gousopoulos et al. [26], as already Posner et al. [16] did, observed that students tend to consider this phenomenon as apparent: they think that an object according to the moving reference frame appears to be shorter but it has always the same length. This feature is more marked with respect to the time dilation phenomenon as also Dimitriadi and Halkia [13] highlighted: as time is more an abstract concept, it is easier to accept its relativity. Conversely, they are still committed to a mechanistic view of the world as already Hewson [17] pointed out according to which objects have fixed properties as for instance the length: as a consequence the result of a measurement on the Earth is the only really “true”. The same result was obtained by Gousopoulos et al. [26] and Selçuk [22] who both added that, as for time dilation, students believe that the contraction happen only in the “moving” reference frame and sometimes also in all the directions, not only along the one of motion.
Aslanides [25] considered also the “asymmetric length contraction” according to which if a “moving” observer gets a length contracted, then it must be lengthened according to the observer “at rest”.

1.4.6 Velocity addition

Both Aslanides [25] and Alstein et al. [21] reported that there are not documented learning difficulties about the relativistic addition of velocities. Dimitriadi and Halkia [13] found that students have some problems into understanding that the maximum speed one can obtain is the one of light and that this limit is not a matter of technology rather an *intrinsic property of nature.*

1.4.7 Mass-energy equivalence

Regardless of the fame of the relation $E_0 = mc^2$, Aslanides [25] reported that no previous misconceptions were found in literature. Selçuk [22] observed that according to the greatest part of the students, mass is thought as an invariant quantity that, as Dimitriadi and

Halkia [13] said, can be tied with the Hewson's inference on students' mechanistic view of reality. However a part of the learners shows doubts about the concept of "relativistic mass", leading them to think about an increase of the mass in the moving reference frame. As we will discuss in Section 2.2.3 and how it is underlined also by Aslanides [25], the use of a "relativistic mass" is misleading and also incorrect. Selçuk argued that this misuse of γ-factor associated with mass arises from a confusion with the relativistic formula of energy and momentum. Indeed the concept of mass is introduced within the context of energy-momentum and thus students might have associated the γ-factor inside the definition of $P = \gamma m v$ and $E = \gamma m c^2$ with the mass rather then with the definition of the previous quantity. This might have convinced them that mass increases with speed.
Selçuk also noticed that the mass conception has interesting implications on density conception: some students, according to the mechanistic view, stated that as mass does not change, also density remains invariant while other ones specified the volume would have shown a length contraction that was not actually real otherwise the nature of the substance would have changed. Other invariance of density has been achieved with a double γ-factor on both the mass and the length term in the density formula. Finally some students thought of an increase on the density due to the increase of the mass, neglecting the transformation of the volume. Misunderstanding on both mass conception and length one result in a non-correct consideration of the transformation of a body's density.

1.4.8 Conclusion

The result of the analysed researches on the didactic features of Special Relativity in terms of students' misconceptions shows different aspects of how this theory is perceived.
Regardless of the level of instruction (from secondary to university), students seems to have difficulties in understanding and internalising the relevant consequences of Special Relativity also after instruction[5]. If Galileo's theory is not fully understood even if it deals with concrete everyday life situations, even more Einstein's one faces this problem due to its abstract outside daily life contents[6]. Indeed as there is not a chance to experiment relativistic phenomena, students try to appeal to their own common sensory experience in their explanation ([22]). The impossibility of observing these phenomena increases the conflict between their outcomes and students' intuition.
In particular many authors noticed that students tend to adopt the ground reference frame[7] as the preferred one with an absolute sense. Classical Mechanics even after instructions continues to have still a strong influence on the students' reasoning into their explanation

[5] [22, 8, 29, 23, 10, 32, 26]
[6] [22, 23, 10, 24, 32, 13, 26]
[7] [20, 21, 22, 13, 12, 31, 14]

of relativistic phenomena[8], especially in the mechanistic view ([17]) and in the existence of an absolute perception of reality (absolute motion, absolute time, absolute space). Moreover the old misconception already observed by Saltiel and Malgrange [20] between a real and an apparent motion is even more emphasised leading students to consider time dilation and length contraction as an optical illusion [9] as well as asymmetric phenomena ([25]). Therefore some authors[10] strongly emphasised the importance of the appropriate language instructors and teachers should use, paying attention to words as *measurement* and *observation*, trying to avoid words as *to see* or *to appear* in order not to strengthen this incorrect conception ([22]).

Students are not used to consider the two postulates as effective tools for the resolutions of problems and it seems that most of the students' alternative frameworks lays in the not-correct understanding of the concept of reference frame and event[11] which are considered as a prerequisite for Special Relativity. These concepts guide them to a misleading evaluation of relativistic phenomena as for instance simultaneity.

1.5 Teaching Special Relativity

The evidence of persistent and resistant to change ([29]) students' difficulties motivated researchers, instructors and teachers to find different and various strategies for teaching Special Relativity. If the Physics university courses already provide for its instruction also with introductory courses of Modern Physics ([8]), the interest in the secondary education is quite dissimilar. Indeed only in recent years (almost in the last decades) a growing international interest in dealing with Modern Physics has been observed with a particular attention to Special Relativity[12], leading to different didactic proposals. The term *Einsteinian physics* has been coined to gather the most relevant theories of Modern Physics: the General together with the Special Relativity and Quantum Mechanics (Kersting and Blair [42]).

Dimitriadi and Halkia [13] however reported that as well as examples of strategies, there are also some objections to its introduction due to the difficulties it hides. In this context, Alstein et al. [21] noticed that there is not a general agreement on the guidelines to adopt in order to delineate the most suitable approaches for both university and secondary students.

As far as the Italian educational overview is concerned, since 2010 the national guidelines

[8] [23, 9, 22, 17, 8, 13, 26]
[9] [16, 22, 31, 17, 13]
[10] [33, 22, 21]
[11] [17, 8, 18, 9, 13, 34]
[12] [32, 24, 14, 21, 13, 35, 36, 37, 2, 26, 38, 39, 40, 41]

for high-school teaching and learning ([43]) have formalised the require of introducing Special Relativity in the last year of students' formation in Physics:

> *The study of Einstein's Theory of Special Relativity will lead the student to deal with simultaneity of events, time dilation and length contraction. The study of the mass-energy equivalence will allow him to develop an energetic interpretation of nuclear phenomena (radioactivity, fission, fusion).*[13]

The peculiar feature of this theory of being a scientific revolution but at the same time a profound and new reflection on the concept of *space* and *time* places Special Relativity as an important moment of the students' formation. He is invited to reflect and deepen the philosophical reasoning in line with the General Guidelines and Skills for the Physics curriculum of a Scientific High School:

> *By the end of the high school course the student will have learned the fundamental concepts of physics, the laws and theories that make them explicit, acquiring awareness of the cognitive value of the discipline and of the link between the development of physical knowledge and the historical and philosophical context in which it developed.*[14]

In particular a second document, the Normative Table [44], reports a distinction between prerequisites, contents, abilities and competences students have to acquire at the end of their instruction on Special Relativity in order to be prepared for their final exam.
Useful and complete analyses about different teaching strategies have been recently carried out by Prado et al. [7] and Alstein et al. [21]. The former gave a general overview on students' and teachers' reasoning about Galilean and Einstein's Relativity and then illustrated different didactic proposals for teaching Special Relativity. They showed how usual approaches give rise to difficulties in correctly applying mathematical formulas to solve problems. The latter summarised and analysed the amount of educational researches in Special Relativity in the secondary and undergraduate instruction. In particular they studied forty works focusing over the learning difficulties, the different teaching approaches and the research tools. Finally the more recent work of Kersting and Blair [42] reported four inspiring examples of alternative instructional tools for the teaching of Special Relativity: one approach based on relativistic dynamics aims to avoid the issue of changing reference frame. The second one shows the work of Kamphorst et al. [24] while the third one introduces Special Relativity through a different use of the spacetime diagrams. The last one emphasises the employ of Virtual Reality (see further on).

[13] [43] - my translation from Italian.
[14] Ibid. - my translation from Italian.

In order to give an easy classification of the various existing strategies to Special Relativity we can consider the 5 major approaches Besson and Malgieri [19] identified exploring the university and secondary textbooks:

- **kinematic-algebraic approach**: this is the most adopted way of teaching in Italian High Schools ([1]). As pointed out by De Ambrosis and Levrini [2], the textbooks hark back to Resnick's book ([45]) of 1968, tracing Einstein's original paper of 1905 ([46]). At the beginning, the two Einstein's postulates are presented. Then the main aspects of the theory are explained: loss of simultaneity, time dilation, length contraction, the concept of interval, and Lorentz's transformations. This frame is completed by the relativistic formulas of the composition of velocities, momentum, and energy. Often light clocks and figures with drawings representing the succession of events (*diagrams of events*) are used;

- **kinematic-geometric approach**: it is based on the geometric study of 4-dimensional spacetime. It immediately introduces the invariance of the interval ds^2, analogously to the invariance of the distance in the Euclidean metric. The theory of Special Relativity is presented as a particular case of General Relativity, redefining classical momentum and energy through the moment-energy four-vector, constructed with invariant quantities. Levrini [34] strongly emphasised the geometry of spacetime as useful didactic strategy;

- **dynamic-experimental approach**: this methodology first introduces the experimental existence of a limit speed (that one of light). Then it shows how the formula of classical kinetic energy is not correct. This is replaced by the relativistic expression, obtained empirically from the data. A new definition for the momentum is introduced within the dispersion relation ($E^2 = p^2c^2 + m^2c^4$). Its meaning and consequences from a conceptual and physical perspective are finally discussed;

- ***k*-calculus approach**: in this approach, the two postulates of Einstein constitute the starting points. Then it is considered the k value equal to the ratio between the reception time T' from an observer moving at a constant speed v of light signals sent by another observer stationary to time intervals equal to T. It is obtained that $k = \sqrt{(1+\beta)/(1-\beta)}$, providing the formula for the longitudinal Doppler effect. With other manipulations, the expressions of the phenomena such as time dilation, length contraction, the addition of velocities, and Lorentz transformations are obtained. This approach does not allow to explain all the result of the Special Relativity. Thus it may be considered a simple introduction;

- **historical approach**: this way provides a historical reconstruction of the contents of the theory, starting from the analysis of the crisis of Classical Physics in the nineteenth century. In particular, it focuses on the contradictions between Electromagnetism and Mechanics. Often this approach is mixed with some elements of the kinematic-algebraic one. Selçuk [22] in particular does not agree with this ways of teaching while Alstein et al. [21] reported that Levrini's strategy ([34, 37]) involving an historical and philosophical contextualisation is promising, even if it has not been empirically experimented so far. Kamphorst et al. [24] pointed out that in science history and philosophy can be useful clues *to help students bridge the gap between their pre-instructional ideas and physics concepts*. This approach has been tested efficiently in high schools by Arriassecq and Greca [14] who reported result more positive then when adopting traditional teaching.

To these approaches, we add the use of online resources like the *Real Time Relativity* game developed by Savage et al. [47] and the *OpenRelativity* project of Sherin et al. [48], helping students to visualise the effects of Special Relativity through interactive simulations. The former allows students to see what happens to a scenario when high speed are reached with particular emphasis on the optical distortion of objects and on relativistic phenomena. It was used in a first year university course on Special Relativity accompanied to a more classical teaching: the authors suggested that it might be of great importance for students *who prefer the concrete over the abstract*. The latter is a first-person game providing a view of a relativistic world which has an arbitrary value of the speed of light. The authors stated that it can provide an experience of the abstract relativistic concepts, hoping that *intuition about Special Relativity can be built*. As far as the examined literature is concerned, relativistic virtual reality was used by de Hosson et al. [9], suggested by Selçuk [22] and thought to be the next step in the work of Gousopoulos et al. [26]. However Kamphorst et al. [24] showed some perplexities about its use for the analysis of the second postulate, as these simulations hide the constancy of light speed inside the programming, but recommended it for the other aspects of Special Relativity. Maybe a combined didactic actions with other approaches would be suitable.
Villani and Pacca [23] suggested the use of qualitative problems and in particular of *thought experiments* in the form of *paradoxes*, a prerogative of Einstein's reasoning, which Velentzas and Halkia [38] showed to be of great importance for scientific reasoning. This suggestions were implemented by Dimitriadi and Halkia [13] in their experimentation and confirmed by Alstein et al. [21], who expressed a preference with respect to the kinematic-algebraic approach which is excessively focused over the formal mathematical aspects of Special Relativity. Scherr et al. [29], although they used different scenarios involving thought experiments, showed some doubts in their use to generate a cognitive conflict in the students' reasoning. They stated it is useless that the instructor shows the contra-

dictions a thought experiment wants to emphasise: the process of "confront and resolve" should be carried out from students themselves, if a real conceptual change is expected to be achieved.
Velentzas and Halkia [38] used two different thought experiments with high-school students to investigate some aspects of both the Special and the General Relativity. The former contain elements about simultaneity, time dilation and length contraction while the latter was a simple Einstein's elevator. The strong points of this approach are argued to be its narrative form and the minor content of mathematical formalism which motivated students, helping the understanding of abstract concepts.
In the follow-up of their previous work ([32]), Kamphorst et al. [24] used thought experiments in secondary teaching together with *event diagrams* to investigate the understanding of the second postulate, solving the issue highlighted by Scherr et al. [8] about students' difficulty in distinguishing between the occurrence of an event and the reception of a light signal towards an observer coming from it. They noticed that students did not use a preferred reference frame to determine light speed: even if the approach of Kamphorst et al. was successful in their aim, they agreed that it could be difficult to use in classes as it requires teachers to individually guide students in their reasoning. Improvements to make it more suitable for larger groups was stated.
However the multiple approach used by Kamphorst et al. [24] underlines the importance of differentiating the didactic practices which Selçuk [22] stressed to be the way to eliminate or at least to minimise students' misconceptions.

1.5.1 Approaching as Minkowski

The geometrical Minkowski's approach of 1908 ([49]) to Special Relativity based on diagrams that show the existence of an absolute spacetime has already been taken into account in literature. Its importance was firstly recognised some years later by Einstein himself in the preface to his article of 1916 about General Relativity ([50]) as the existence of a spacetime turned to be fundamental for its description. Minkowski's standing about spacetime starts from the matching between reality and a four-dimensional manifold, not a 3+1 one. It has been argued by Levrini [51] that the reality's description as it emerges from Einstein first work of 1905 still has some Newtonian heritages. As quoted in [*ibid.*], citing [52], *space and time travel along correlated paths but no way to unify them is proposed*[15]. This is still a classical approach as Newton considered space and time as separated entities. Minkowski's substantivalist view of spacetime leads to a reality as a four-dimensional manifold that exists as something invariant, observer-independent, thus absolute ([51]).

[15] Original Italian sentence: *spazio e tempo viaggiano sì su percorsi correlati, ma non si propone alcuna forma di riunificazione tra di essi.*

The didactic implications of this assumption are numerous ([37]): it should avoid students to continue sharing the popular and incorrect version of Special Relativity that "everything is relative" ([7, 14]), a feature of "relativity" that Einstein himself did not accept ([13]). Moreover, beyond the epistemological considerations about space and time, Minkowski's geometry paves the way to General Relativity's concepts. Some problems related to the equivalence principle ([2]) or the transition from Special Relativity to the General one ([3]) can be solved thanks to four-dimensional spacetime.

In order to make easier for students the drawing, the understanding and the use of a Minkowski diagram, different attempts were made in the middle of the XIX century. From a didactic point of view, as Benedetto et al. [53] pointed out, to avoid the introduction of a complex notation arising from an imaginary time ict, it is more useful to introduce a real rotation in a spacetime diagram. For the first time in 1948 this approach was applied by Loedel ([54, 55, 56]) and then it was independently found also by Amar ([57, 58]). They all considered two different observers S and S' in relative uniform motion and a coordinate reference system in the form (x, ct') and (x', ct) showing that each one can be obtained from the other one simply with a rotation of a real angle, determined by the relative speed. In the same way relativistic phenomena can be derived.

In 1962, Brehme [59] got an alternative derivation of these diagrams from the invariance of the spacetime intervals with a slightly difference in reporting coordinates of an event on the axes ([60]). Indeed if we consider

$$ds^2 = dx^2 - c^2dt^2 = dx'^2 - c^2dt'^2 = ds'^2, \tag{1.1}$$

one can rearrange as

$$dx^2 + c^2dt'^2 = dx'^2 + c^2dt^2, \tag{1.2}$$

getting an invariance of an Euclidean distance in terms of events (x, ct') and (x', ct).

The main didactic advantage of the picture of Loedel, Amar and Brehme is to have Lorentz transformation $(x, ct') \rightarrow (x', ct)$ as a simple rotation of the coordinate system (x, ct') into (x', ct). As outlined by the previous authors and as we will show in Section 2.3, one can get the same inferences as from a Minkowski diagram with an easier approach to the understanding and resolution of problems asking transformations of events according to different observers. This use of alternative real diagrams has been addressed to be more didactic: Benedetto et al. [53] highlighted that their strengths lay in the symmetric treatment of the reference frames as both of them have orthogonal axes. The direct consequence is that it is not necessary to introduce a scale factor to convert the measures of one observer to those of the other observer as the scale of both the reference frame are equal.

In 1965 Taylor and Wheeler ([61]) following Minkowski's ideas elaborated the first didactic geometrical approach to Special Relativity based on a new geometry, namely the one of spacetime, and on invariant quantities.

Already in 2001, Berenguer and Selles [3] pointed out the necessity of modifying the didactic practice, developing a curriculum for the Bachelor degree in Physics based on the Minkowski diagrams. They aimed to introduce new concepts, as worldline or spacetime event, which are seldom discussed in schools and textbooks, leading to the understanding of all the relativistic phenomena as purely spacetime effects, resulting from a not ordinary Euclidean geometry. From the analysis, they concluded that this alternative didactic approach is possible with positive outcomes related to a greater understanding and ability of the students to visually and graphically represent relativistic effects. But the authors could not deny that students needed to familiarise with the new geometry which may be a drawback. Moreover there is the possibility that students could be lead to an incomplete, thus false, conceptual change if for instance they understood that a Minkowski diagram represents the unfolding structure of reality and not an already given one. Even though the research of Berenguer and Selles [3] was conducted for Bachelor courses, their conclusions showed important considerations that can be applied also to high schools.

However, as Liu and Perera [4] highlighted, despite the age of the Minkowski diagrams and its effectiveness as a quantitative tool in Special Relativity, they are not used in university instruction and *a fortiori* in secondary one to reinforce the algebraic method. They showed that kinematics of Special Relativity can be easily derived from Minkowski diagrams, together with Lorentz transformations and invariant quantities. Moreover they can be used to illustrate efficiently paradoxes. Their research found that spacetime diagrams help students to visualise situations and they facilitates qualitative reasoning and the turn into a quantitative one requires only few steps forward.

Cayul and Arriassecq [5] carried out an experimentation with high-school students investigating the use of Minkowski spacetime diagrams to evaluate the simultaneity of two events among different observers in relative motion. Students understood the meaning of the necessary elements to elaborate a diagram and it is not a problem to connect different concepts of Special Relativity to analyse a diagram. The greatest difficulty remains performing numerical calculations and correctly using algebraic equations to determine the simultaneity of events. In the follow-up ([62]) they added the use of applets in order to overcome some difficulties students encounter with the concept of spacetime. The applets were employed to address the concepts of simultaneity, time dilation and length contraction which revealed to be beneficial for a better understanding.

Prado et al. [63] concluded their inquiry with the formulation of a full program for a teaching proposal for high schools based on Minkowski spacetime diagrams in forms of event diagrams ([64]). Their approach allows to visualise and to explain qualitatively and quantitatively relativistic phenomena as time dilation, length contraction, the existence of a limiting speed and the mass-energy equivalence which are the most relevant topics in secondary education. They tested this strategy with students of a high school in a Spanish town with promising result. They reported also other profitable uses in literature of spacetime diagrams beyond their application for common topics of Special Relativity as length contraction in accelerated reference frames ([65]) or scenario linked with black holes ([66]) or even particle collisions ([67]) which in particular inspires our application in Section 3.3.7.

In his PhD dissertation, Moutet [6] presented an experimentation, involving students of the last year of high school, about Special Relativity through Minkowski spacetime geometry using at first Brehme's diagrams while later on Minkowski's ones and Loedel's ones. In his didactic proposal, Moutet used GeoGebra software to create diagrams and found that, despite some difficulties related to the graphical approach (slope, interpretation of a representation or a represented concept), it is advantageous with promising result showing that students understand and assimilate the concepts of Special Relativity. He also pointed out the need of a full formation for teachers about the use of spacetime diagrams. In this direction goes the work of De Ambrosis and Levrini [2] who conducted a research among teachers of high schools to evaluate the possibility of a didactic project based on Taylor and Wheeler's approach. Despite some initial resistance, teachers *agreed on the reliance and relevance* ([2]) of this new didactic strategy in teaching Special Relativity, thus following Minkowski's original treatise.

1.6 Which approach for Special Relativity?

The previous section has highlighted the most common approaches to teach Special Relativity. They are valid, suggestive, yet providing different *points of view* to start dealing with this theory. But if there are still so many difficulties, it is fair to question about their shortcomings that have not been pointed out. Giving that each student may resonate particularly with one of the suggested approach rather than with another one or a combination of them, there are some critical issue that these ways of teaching implicitly bring. In particular Besson [68] outlined some implications about the first two, being the most common way of teaching.

The kinematic-algebraic approach is the most common among Italian high schools as it is followed by the national textbooks. As we will see in Chapter 4 from the result of our prior analysis, it does not provide a good accommodation of the relativistic theory: it may tend to relay excessively upon some formal aspects, leading students to focus more on the mathematical demonstrations (as for instance using light's clocks) then on the operative

concept of space, time and synchronisation which should be the starting point of this approach. Thus the belief that Special Relativity is extremely mathematical and then only for gifted students ([41]) is enhanced and reinforced.
Conversely the kinematic-geometric approach is well compact but it needs to introduce a very peculiar and abstract geometric structure (the one of the spacetime) without any chance of validating its features *a priori*. Students have to accept as an authoritative imposition a choice which seems to be without any foundations, actually complicating the description of reality but that will be understood only *a posteriori*.

Chapter 2

The Theory of Special Relativity

2.1 Historical-scientific context

By the end of the 1800 all the Physics was thought to have been discovered:

"*There is nothing new to be discovered in physics now.*
All that remains is more and more precise measurement."

said Lord Kelvin in a speech to the British Association for the Advancement of Science in 1900 [69]. He could not know how completely wrong he was...

After Newton saw *further standing on the shoulders of Giants*, Mechanics was a completed theory, describing and predicting the motion of objects over the Earth as well as celestial phenomena, even though some of them were not completely understood like the precession of Mercury's perihelion. Thermodynamics was able to deal with heat's transmission and its successful application to the thermal machines did not question its effectiveness. But the microscopic interpretation was not considered as reliable (molecules were a fancy ideas) and the black body radiation was still an unsolved problem. Finally Maxwell's just-born Theory of Electromagnetism well gathered electric and magnetic phenomena. The interpretation of the optics solved the secular debate over light's nature: the discovery of the electromagnetic waves by Hertz in 1886 together with its theoretical interpretation thanks to Maxwell's theory sanctioned light to be a wave. Nevertheless even this elegant theory had its dark counterpart destined to kick off one of the most important scientific revolution.

In the 1600s Galileo stated that (in a modern way):

Principle of Relativity (of Galileo)**.** *All the laws of Nature must be form invariant under Galilean transformations.*

Galileo empirically observed that all inertial frames were equivalent, thus it was not possible to detect whether an inertial frame was moving with constant speed or not. Since the laws $\mathcal{G}$ of transformation from one inertial frame of reference $\mathcal{K}$ to another one $\mathcal{K}'$ moving with constant speed x with respect to the first one were

Definition 1 (Galilean laws of transformation)**.**

$$\mathcal{G} = \begin{cases} \boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{v}t \\ t' = t \end{cases} , \tag{2.1}$$

Galileo deduced that all the laws of physics must preserved their form under these transformations:

The equations of Mechanics are covariant under Galilean transformation.

When an experiment of Mechanics is performed in the frames $\mathcal{K}$ and $\mathcal{K}'$, having the laws the same mathematical form, the result is the same in both the inertial frames. An experiment can never detect if an inertial frame is moving or not, thus showing their equivalence. Since 1600s Galileo's principle of relativity was always correctly observed up to Maxwell's theory of electromagnetism which turned to be incompatible with Galilean transformations.

The set of equations[1]

$$\begin{cases} \nabla \cdot \mathbf{E} = 4\pi\rho \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \wedge \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \wedge \mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \end{cases} , \tag{2.2}$$

are not invariant under Galilean transformations. Thus performing a simple electromagnetic experiment, one would have been able of detecting an uniform motion with respect to a particular inertial system: that of the *luminiferous ether* at rest.

The motion of light was well explained as long as it was regarded as a wave-field in a completely analogy to the mechanical vibration field in an elastic solid body. But for the latter there was a physical body throughout which vibrations could propagate: what about the former? Thus, it was necessary to introduce a new field existing also if there was no physical matter, in presence of an "empty space": the *ether.*

Electromagnetic fields were regarded as states of ether and light was considered propagating in the same way as elastic vibrations throughout the ether.

[1]Here we use Gauss' CGS system [70].

In 1851 Hippolyte Fizeau used a water-interferometer to study the effect of the movement of a medium over light's speed and to measure the relative speed between light and water. It seemed that light was dragged by the movement of the water: being n the refractive index of the water and v its speed, light's speed w in the water was:

$$w = \frac{c}{n} + v \cdot \left(1 - \frac{1}{n^2}\right). \tag{2.3}$$

But the experiment showed a result significantly lower than the one expected from this formula. Nevertheless it was consider a proof of the truthfulness of Fresnel drag coefficient.

This quantity was introduced in 1818 by Augustin-Jean Fresnel in order to explain Arago's experiment. He concluded stating the existence of a quasi-stationary ether, namely a stationary ether that was dragged by moving objects along with them as they move. If v was the speed of the moving object and n its refractive index, then the ether was dragged with a velocity vf where f is the Fresnel drag coefficient:

$$f = \left(1 - \frac{1}{n^2}\right). \tag{2.4}$$

Thus both the ether and the water brought the light along with them.

Even though Fresnel's theory of the dragging ether well reproduced the experimental result, the famous Michelson-Morley experiment of 1887 questioned the existence of a quasi-stationary ether.

In their experiment, Michelson and Morley used an interferometer to measure a difference in the light's speed, according to the dragging of the ether, after a complete 90° rotation of the instrument. At first one arm was aligned along the direction of Earth's velocity in the ether and then it was rotated of 90°. A shift in the interference pattern of 0.4 fringes was expected to be seen, being the sensitiveness of the instrument of 0.01 fringes. No shift was observed, implying no change in the light's speed and then no dragging from the ether...probably because there was no stationary ether.

In a similar way, the existence of a dragged ether was questioned by the aberration phenomenon, the apparent change of the positions of a star in the sky during the year.

In 1727, James Bradley provided a simple explanation using a finite value for the speed of light and the Earth's motion around the Sun. But if ether was considered, then there would not have been any aberration as the star's light would have been dragged together with the Earth in its motion. The light's rays would have hit the telescope in an orthogonal direction with respect to the Earth's motion.

Thus by the end of 1800s, the ether theory was the only way to understand light's propagation but at the same time there were experimental result undermining this theory.

A first attempt to solve this irreconcilability was unconsciously made by Woldemar Voigt in 1887 [71, 72]: looking for the covariance of light's wave equation between inertial reference frames, Voigt derived a first version of Lorentz transformations.
Voigt made two assumptions:

- the covariance of wave equation implies the form invariance of this equation between inertial reference frames: this is nothing else than Einstein's principle of relativity;
- he asked for the invariance of the speed of light ω (using his notation) in both the at rest and moving reference frame: this will be Einstein's second principle.

These conditions led him to find a set of transformation between the frame of reference at rest (x_1, y_1, z_1) and time coordinate t and the moving frame (ξ_1, η_1, ζ_1) and time coordinate τ:

$$\xi_1 = x_1 - \chi t, \qquad \eta_1 = y_1 q, \qquad \zeta_1 = z_1 q, \qquad \tau = t - \frac{\chi x_1}{\omega^2}, \tag{2.5}$$

where χ is the x-component of the relative speed between the two inertial frames and $q = \sqrt{1 - \chi^2/\omega^2}$ (the inverse of the future Lorentz factor γ).
In a modern notation, these transformations are:

$$x' = x - vt, \qquad y' = \frac{y}{\gamma}, \qquad z' = \frac{z}{\gamma}, \qquad t' = t - \frac{vx}{c^2}, \tag{2.6}$$

very similar to the Lorentz transformations.
However Voigt did not provide any explanation for his assumptions nor theoretical or experimental bases. Actually his aim was to obtain the formula for the Doppler effect, showing that the wave equation was form invariant under transformation (2.5): his attempt was to derive a new set of spacetime coordinate transformation. Moreover it seems that he did not recognise the importance of these equations and in particular of the relation $t' = t - vx/c^2$ introducing for the first time since Newton's age a non-absolute time.
A decisive step forward were the contribution of Hendrik Antoon Lorentz by the end of XIX century[2].
We have to consider 5 main publications:

- *The Electromagnetic Theory of Maxwell and its Application to Moving Bodies* (1892);
- *The Relative Motion of the Earth and the Ether* (1892);
- *Attempt of a Theory of Electric and Optic Phenomena in Moving Bodies* (1895);
- *Simplified Theory of Electrical and Optical Phenomena in Moving Bodies* (1899);

[2]Here we follow Acuña's reconstruction in [73].

- *Electromagnetic Phenomena in Systems Moving with any velocity Less than that of Light* (1904).

With the aim of describing a new transformations of coordinates leaving invariant Maxwell's equations, Lorentz applied a three-step method. He defined:

1. $\mathcal{S}_0$, the frame at rest with respect the ether.
2. $\mathcal{S}$, the Galilean, *real* frame moving at velocity $\boldsymbol{v}$ with respect to $\mathcal{S}_0$: $\mathcal{S}$ and $\mathcal{S}_0$ were connected by Galilean transformations.
3. $\mathcal{S}'$, an *auxiliary* frame which also was moving at velocity $\boldsymbol{v}$ with respect to $\mathcal{S}_0$: $\mathcal{S}'$ and $\mathcal{S}$ were connected by the following transformations[3]:

$$x' = \gamma l x, \qquad y' = ly, \qquad z' = lz, \qquad t' = l\left(t/\gamma - \gamma(v/c^2)x\right), \tag{2.7}$$

where l was a parameter, subsequently set to 1.

Thus, combining these two set of transformations, in 1899 Lorentz wrote the transformation between the frames $\mathcal{S}_0$ and $\mathcal{S}'$:

$$x' = \gamma l(x_0 - vt_0), \qquad y' = ly, \qquad z' = lz, \qquad t' = \gamma l\left(t_0 - (v/c^2)x_0\right). \tag{2.8}$$

They differ from the version of 1892 where:

$$x' = x_0 - vt_0, \qquad y' = y, \qquad z' = z, \qquad t' = t_0 - (v/c^2)x_0. \tag{2.9}$$

The set of transformations (2.8) leaves Maxwell's equations form invariant, even though already the version of (2.9) showed a form invariance up to first order of v/c.
Moreover since 1892, Lorentz introduced the *local time* t' whose physical meaning was provided only in 1904 by Poincaré [74] as the time measured in a frame moving with respect to the ether, with an illustration that resembles Einstein's procedure of synchronisation of clocks. Poincaré also pointed out the impossibility of perceiving the local time as slower due to the slowing of the watch itself; by this way a moving observer was not able to determine if he was in motion or not, as stated by the principle of Relativity.
As said before, the idea of a non-absolute time together with a first version of Lorentz transformations were already introduced by Voigt but Lorentz was unaware of it until 1908 [72]: afterwards he recognised the priority of Voigt.
Lorentz formulated also the *Theorem of Corresponding States* (1892) that more or less is the principle of invariance for electromagnetic fields:

[3]This is the ultimate version of 1899.

> If there is a solution of the source free Maxwell equations[4] in which the real field $\mathbf{E}$ and $\mathbf{B}$ are certain functions of $\boldsymbol{x}_0$ and t_0, the coordinates of $\mathcal{S}_0$ and the real Newtonian time, then if we ignore terms of order v^2/c^2 and smaller, there is another solution of the source free Maxwell equations in which the fictitious field $\mathbf{E}'$ and $\mathbf{B}'$ are those same functions of $\boldsymbol{x}'$ and t', the coordinates of $\mathcal{S}'$ and local time (Janssen [75]).

Briefly, the theorem relates the configuration of an electromagnetic field in two different inertial frames $\mathcal{S}$ and $\mathcal{S}'$, respectively at rest and moving with respect to the ether. If in $\mathcal{S}$ there is a possible configuration for this field, then its form is invariant in $\mathcal{S}'$: it can be written using the same mathematical expression of the field in $\mathcal{S}$ using the variables of $\mathcal{S}'$. Another crucial aspect of Lorentz's work was the attempt to explain the null result of the Michelson-Morley's experiment.
He started ([76]) from the laws of transformation of the electromagnetic forces between inertial frames at rest and moving with respect to the ether. But the forces bringing together the molecules and thus determining the length of a body have an electromagnetic nature. Then Lorentz suggested that the length should transform in the same way, obtaining that objects along the motion get contracted by a factor $\sqrt{1 - v^2/c^2}$.
This expedient, known as *Lorentz-Fitzgerald contraction* because it was proposed independently also by George Francis Fitzgerald in 1889, solved the issue with the Michelson-Morley's experiment. Even though without a plausible explanation, with the hypothesis of the length's contraction Lorentz showed that the difference in the light's travel time across the arms of the interferometer was zero, thus explaining the experimental null result.
In the work of 1899 Lorentz summed up all the previous considerations introducing the final version of the transformations (2.8) together with the theorem of the corresponding states. Moreover he gave a physical interpretation to the γ factor and to the transformations using the so-called *generalised contraction hypothesis* [75]. It states that, changing reference frames, the configuration of the charged particles re-arranges itself so that the electromagnetic field in $\mathcal{S}$ is the corresponding state of the original one in $\mathcal{S}_0$.
The assumptions made in the work of 1899 led Lorentz to new surprising result described in the paper of 1904 in which he developed the model of the electron. He showed that its mass depended upon the velocity, finding an energy-mass relation that, after having beenn modified by Poincaré with an additional term (*Poincaré-pressure*) in 1906, yielded for an electron $m_0 = U_{tot}/c^2$, being U_{tot} the electron's total amount of energy.
Poincaré played a key role in Lorentz's works: he also corrected Lorentz's formula for the velocity transformations obtaining the correct one equal to Einstein's one. The problem

[4]Maxwell equations in vacuum and in the absence of sources.

was rooted in the third-step method used by Lorentz to derive the law of transformations. Poincaré instead used a two-step method, deriving directly the transformations between the frame at rest with respect to the ether $\mathcal{S}_0$ and the moving one $\mathcal{S}'$. These direct transformations were called by Poincaré himself for the first time *Lorentz transformations* and thus the correct form invariance for Maxwell's equations was achieved.
From a purely mathematical point of view, Poincaré showed that Lorentz transformations form a group and that their symmetry has also a physical consequence over the length contraction effect. Finally he found that Lorentz transformations leaves the quantity $x^2 + y^2 + z^2 - c^2t^2$ invariant and that some other physical quantities like the electric charge and the current density can be combined into a four-component object that was Lorentz invariant. In some way Poincaré anticipated Minkowski's geometrical formulation of Special Relativity but in his vision these properties were only mathematical and did not have any physical meaning.
However, even though Lorentz's theory (or better Lorentz and Poincaré's theory) seems to be a pre-theory of the Special Relativity, we need to look at it as it is, namely a ether theory constituting the embodiment of a space absolutely at rest. Lorentz's aim was to find a theory that could explain the aberration of light, the Doppler effect and the Fizeau experiment. He actually succeeded and in addition he introduced new concepts that would have been the protagonists of Einstein's scientific revolution.

2.2 Einstein's Special Relativity

It is not possible to understand the birth of the Special Relativity without taking into account the developments described in the previous section. It would seem only the result of Einstein's brilliant intuition.
Einstein's work arises from a scientific community puzzled over the theoretical hypothesis of the ether. He knew Lorentz's first studies but not the length contraction phenomenon, yet acknowledging him as the first to introduce the hypothesis of the change of electron's shape even if purely by formal points of view. He was aware of the result of Michelson-Morley's experiment ([77, pg. 40]), nevertheless his major inspiration came from Maxwell's work:

> I'm not thinking only of Newton: there would be no modern physics without Maxwell's electromagnetic equations. I owe more to Maxwell than to anyone. [78, pg. 152]

In one of his first publications for general public *Relativity: the Special and General theory* [79] (1920), Einstein wrote that the theory of Special Relativity grew from the electrody-

namics and optics, in particular from the Maxwell-Lorentz theory describing the electromagnetic interaction between classical point charges and their electromagnetic fields. At the light of the new developments in electrodynamics and optics, Classical Mechanics was no more able to provide a foundation to describe all the laws of nature.
He worked without appreciably modifying the prediction of Maxwell-Lorentz theory, moving towards a simplification in the theoretical assumptions, reducing the basis hypotheses and the ad hoc ones. As Einstein quoted [79], the experimental result in favour of Maxwell-Lorentz theory (aberration, Doppler effect, Fizeau's experiment) were the same as the arguments in favour of the Special Relativity but this one did not need artificial constructs like the ether or the Lorentz-Fitzgerald contraction:

> The theory of relativity leads to the same law of motion[5], without requiring any special hypothesis whatsoever as to the structure and the behaviour of the electron [79].

As we will see the contraction of moving bodies follows only by the new principles of the theory without introducing any particular additional hypotheses.
From a epistemological point of view, these considerations (synthesis of old theories and less hypotheses) gave to Einstein's theory an higher degree of *truthlikeness* or verisimilitude [80] with respect to the Maxwell-Lorentz one, thus representing a forward step for science in its development towards the understanding of the laws of nature.
The incompatibility between experiments with the light (Maxwell-Lorentz theory) and the principle of relativity (Newton theory) arose from two hypotheses of Classical Mechanics: the idea that both time-interval and space-interval between two events were independent from the state of motion of the observers. The origin of Special Relativity ([77]) is a complete analysis of the concept of *absolute space* and *absolute time* arising from the attempt of adjusting Maxwell's theory with Newton's one.
The aim of 1905 Einstein's work [46] was to establish a:

> simple and consistent theory of electrodynamics of moving bodies based on Maxwell's theory [81],

where *simple* means without introducing ad hoc hypotheses like ether and *consistent* means without creating conflicts like those arising from the incompatibility with Classical Mechanics and Maxwell's theory.
This work would start a scientific and philosophical revolution, mainly leading to rewrite the concept of space and time on the basis of only two postulates.

[5] Of Maxwell-Lorentz theory.

2.2.1 Only two suns

The two postulates of Special Relativity (1905) can be formulated in different ways. Following Einstein's original work [46], we have:

Postulates of Special Relativity (of Einstein)**.**

1. **Principle of Relativity:** *The laws of nature (mechanics, electrodynamics and optics) are the same in all the inertial frames of reference.*
2. **Principle of light's speed:** *Light propagates in empty space with a finite velocity c independently from the state of motion of the emitting body.*

Just before exploring the meaning as well as the implications of these two postulates, I want to point out that Einstein never used the word *postulate* with a mathematical meaning: the postulates of Special Relativity are not *a priori* but instead have a purely experimental nature. Einstein was constantly open to the possibility of criticising them if they would have proven to be wrong. Thus it is reasonable to ask if it is scientifically correct that these two experimental evidences can be a theory's foundation and therefore if only experimental confirmations can suffice to consolidate its foundation.
In Appendix A.1 Einstein's position with respect to the role of the postulates has been analysed from the epistemological point of view of the Falsifiability.
Now let discuss the two postulates: the original work [46] does not explain in details neither their origins nor their implications in contrast to the following publications [79, 82]. The first principle was already well known throughout the history of physics: if $\mathcal{K}$ is a frame where the law of inertia is hold[6] and $\mathcal{K}'$ is a second frame which is uniformly moving with respect to $\mathcal{K}$, then natural phenomena in $\mathcal{K}'$ occur according to the same general law as in $\mathcal{K}$. Until the XIX century, as all the natural phenomena were described with the help of Classical Mechanics, there were no doubt of it under the assumption of the absoluteness of both space and time. But, as already discussed, this general principle was no longer true as far as electrodynamic phenomena were concerned. Two different inertial frames $\mathcal{K}$ and $\mathcal{K}'$ in relative motion would have been distinguishable thanks to a peculiar physical properties: the light's speed should have been different in the two frames. Thus in the laws of Nature the velocity of the $\mathcal{K}'$ frame should have played a key role: however whenever the Earth was considered as the frame $\mathcal{K}'$ in an electromagnetic or optical phenomenon (as in Michelson-Morley's experiment), no dependence upon Earth's speed was recorded. Even though this is not a demonstration of the principle of relativity, it is *a powerful argument in favour* [79].
This, with the fact that it seems not very probable that *a priori* a principle of such a broad

[6]It is the frame where the laws of Nature are expressed in the easiest form.

generality should be valid only within a certain class of phenomena, led Einstein to extend its range of validity to all the laws of nature, including electrodynamics and optics ones.
As far as the second principle of Special Relativity is concerned, Einstein started stressing [82] the importance of some experimental result: the consequence of the Maxwell equations as well as of Lorentz's electrodynamics was that light's speed c was constant *in vacuo*. This had to be regarded as proved, experimentally confirmed. Indeed in 1913, studying the double stars, the Dutch astronomer De Sitter showed that light speed did not depend on the velocity of the emitting body.
Being this result true for at least one definite inertial system $\mathcal{K}$, according to the principle of relativity, *we must assume the truth of this principle for every other inertial system* [82]. Once again, as previously, Einstein assumed as true this second principle, he did not demonstrate it as it was a necessary assumption.
If the second postulate was false, one could detect uniform motion performing an electromagnetic experiment, thus invalidating also the first postulate, never found to be false. Thus the second postulate is inferred from the first (it is a consequence) as well as the first also from the second, having already accepted the Galilean principle of relativity.

2.2.2 Return of Lorentz

Therefore, Einstein's real problem was to determine the correct equations of transformation from one inertial system $\mathcal{K}$ to another $\mathcal{K}'$, in uniformly relative motion to it. Actually this problem is uniquely settled by means of the two principles of Special Relativity [82].
Consider an inertial system $\mathcal{K}$ and two points P_1 and P_2: if their distance is r, then the propagation of light satisfies the equation:

$$r = c\Delta t, \tag{2.10}$$

where Δt is the time the light needs to propagate from point P_1 to point P_2. Now as $r = \sqrt{\Delta x_1^2 + \Delta x_3^2 + \Delta x_3^2}$, taking the square of Eq. (2.10):

$$\sum_i (\Delta x_i)^2 - c^2 \Delta t^2 = 0. \tag{2.11}$$

For the two principles of Special Relativity, the same equation must be true also in an inertial system $\mathcal{K}'$ moving with respect to $\mathcal{K}$ with a certain constant velocity $\boldsymbol{v}$ relative to it:

$$\sum_i \left(\Delta x_i'\right)^2 - c^2 \Delta t'^2 = 0. \tag{2.12}$$

If we suppose that $\mathcal{K}$ and $\mathcal{K}'$ have the same orientation and that $\mathcal{K}'$ is moving only along the x-direction with speed v, then the transformations between $\mathcal{K}$ and $\mathcal{K}'$ leaving Eq. (2.11) and (2.12) mutually consistent are the *Lorentz transformations*:

Definition 2 (Lorentz laws of transformation)**.**

$$\mathcal{L} = \begin{cases} x' = \gamma\,(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \dfrac{v}{c^2}x\right) \end{cases}, \tag{2.13}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$.
In a more general way, Lorentz transformations can be written as:

$$\mathcal{L} = \begin{cases} \boldsymbol{r}'_{\parallel} = \gamma\left(\boldsymbol{r}_{\parallel} - \boldsymbol{v}t\right) \\ \boldsymbol{r}'_{\perp} = \boldsymbol{r}_{\perp} \\ t' = \gamma\left(t - \dfrac{\boldsymbol{v}}{c^2}\cdot\boldsymbol{r}\right) \end{cases}, \tag{2.14}$$

where $\boldsymbol{r}'_{\parallel}$ and $\boldsymbol{r}_{\perp}$ are the spatial vector respectively parallel and perpendicular to the relative speed $\boldsymbol{v}$ between $\mathcal{K}$ and $\mathcal{K}'$.
If we consider light as not having a finite speed ($c \to +\infty$), this set of equations reduces to Galilean transformations (2.1) (*classic limit).*
Obviously when $\mathcal{K}'$ is moving with respect to $\mathcal{K}$ also along other directions, transformations of coordinates y or z must be taking into account. For a complete description, including also rotation of the frames, we refer to specialise texts as Barone [70].
Thus the two postulates of Special Relativity can be summarised as:

The laws of Nature must be covariant under Lorentz transformations.

Einstein was able to show that both the laws of Mechanics and those of Electromagnetism could be rewritten in a covariant form under Lorentz transformations. The phenomena that gave rise to incongruities like aberration, Doppler effect, Fizeau's experiment and Michelson-Morley's experiment now had a simple explanation without special hypothesis as for instance the presence of an ether.
But new phenomena were about to be brought to light...

2.2.3 Jump at lightspeed

The prominent aspect of Lorentz transformations (2.13) is that of time: Einstein pointed out that time t as measured in an inertial frame $\mathcal{K}$ is not the same compared to the one t'

measured in a moving frame $\mathcal{K}'$.
As previously said, Special Relativity has some of its roots in the deep analysis of the concept of time: far from giving a psychological interpretation of the time, thus subjective, Einstein was interested in an operative definition.

> If, for instance, I say "That train arrives here at 7 o'clock", I mean something like this: "The pointing of the small hand of my watch to 7 and the arrival of the train are simultaneous events." [46]

A judgement in which time plays a role is always a judgement of at least two simultaneous events. In this way, a given observer A with a clock can determine the time values of events in his proximity. If now B is a second observer (in the same frame) with a identical clock, how can they compare their time?
It is thus clear that before the comparison of the two clocks the observers must settle a *procedure of synchronisation* [45]: the two observers A and B have to put their clocks into their position and to use a light signal to synchronise them. For instance, observer A, when its clock reads $t = 0$, sends a light signal to B. This one will receive it after a time $t = L/c$, being L the distance between the two clocks: observer B will set his clock at $t = L/c$ as soon as the light signal arrives.
This procedure accounts for the time of transmission of light whose velocity is finite: it is evident that this problem does not exist within Classical Mechanics. As the light was thought to have an infinite speed, instant transmission was allowed and then a clock was always synchronised with all the other possible clocks, moving or not. This condition is the absolute validity assigned to time by Newton: an observer will always read on his clock the same time of the clock of all the other possible observers.
Now instead there is a procedure to synchronise the clocks in one reference frame that allows to create a temporal order for the events in that frame. The time of an event is that one of a clock whose position coincides with the one of the event. Furthermore events happening in different places with the corresponding clocks reading the same time are *simultaneous*.

Loss of simultaneity

The procedure of synchronisation allows each observer to have a theoretically infinite set of synchronised clocks, being thus able to compare the time of events in each point of the inertial frame he belongs to. But what happens when we consider another inertial observer?
A very classical argument illustrating this case can be found in Resnick's book [45] but here we follow the reasoning of Morin [83]. Let's consider the following setup: $\mathcal{K}$ is an inertial

frame where a light bulb is in the middle between two observers A and B, at distance L from each of them (Figure 2.1). D is a third observer, constituting the $\mathcal{K}'$ inertial frame, moving at speed v to the left with respect to $\mathcal{K}$.

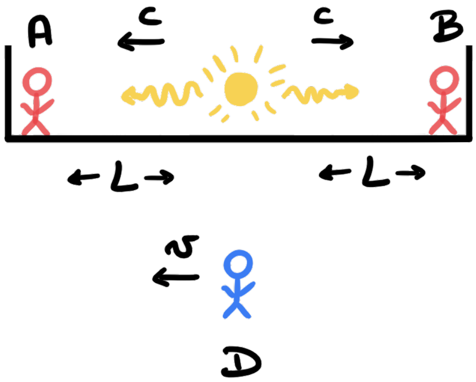


Figure 2.1: Setup.

A and B have synchronised clock: thus when the light bulb emits two different signals arriving to A and B, they would agree that their clock reads the time $t_A = t_B = t = L/c$.
Now let's consider the point of view of observer D who actually sees $\mathcal{K}$ frame moving to the right (Figure 2.2) at speed v. What about the simultaneity of the two events (arriving of the light signal to A and B)?

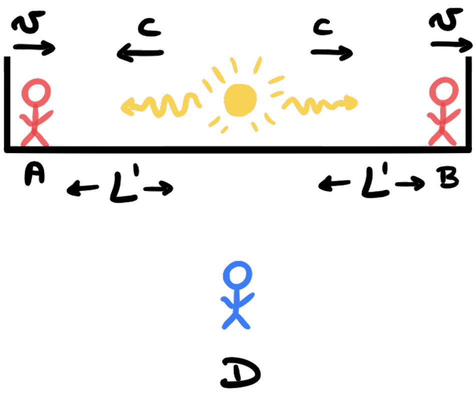


Figure 2.2: Loss of simultaneity: the point of view of observer D.

According to the second principle of relativity, observer D sees light still moving at speed c, while as seen from D the *relative speed* of light reaching A is $v-c$ while that one reaching B is $v+c$.
Now if L' is the distance between the light source and A (or B) in the frame of D[7], then D

[7]A priori we are not sure that the distance is always equal to L: actually it is contracted by γ factor.

says that A receives the light signal *before* B as

$$t'_A = \frac{L'}{v-c} \qquad t'_B = \frac{L'}{v+c} \tag{2.15}$$

and $t'_A < t'_B$.
In other words, if with set a coordinate system x with the same orientation as v with the center O' in the bulb, thus the light's propagation equation is $x' = ct'$ while that one of observer B is $x' = L' + vt'$. Photons from the light bulb reaches B when $ct' = L' + vt'$ that means at time $t'_B = L'/(v+c)$. A similar reasoning for light propagating to the left towards observer A leads to $t'_A = L'/(v-c)$.
Thus if in the inertial frame $\mathcal{K}$ the two events were simultaneous, according to the moving frame $\mathcal{K}'$ there is a time gap of

$$\Delta t' = t'_A - t'_B = 2\gamma^2 \frac{v}{c^2} L', \tag{2.16}$$

that means the two events are no more simultaneous.
More in general, consider a inertial frame $\mathcal{K}$ and there two simultaneous events $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A} = (ct, x_\mathcal{A}, y_\mathcal{A}, z_\mathcal{A}) \qquad \mathcal{B} = (ct, x_\mathcal{B}, y_\mathcal{B}, z_\mathcal{B}), \tag{2.17}$$

where $t = t_\mathcal{A} = t_\mathcal{B}$.
Thus in another inertial frame $\mathcal{K}'$ uniformly moving with respect to $\mathcal{K}$ with speed v along x-axis, according to Lorentz transformations, the new temporal coordinate of the events are

$$t'_\mathcal{A} = \gamma\left(t - \frac{v}{c^2} x_\mathcal{A}\right) \qquad t'_\mathcal{B} = \gamma\left(t - \frac{v}{c^2} x_\mathcal{B}\right). \tag{2.18}$$

Hence the time gap is

$$\Delta t' = t'_\mathcal{B} - t'_\mathcal{A} = \gamma \frac{v}{c^2}\left(x_\mathcal{A} - x_\mathcal{B}\right). \tag{2.19}$$

This result coincides with expression (2.16) as soon as it is set (for the length contraction) $x_\mathcal{A} = \gamma L'$ and $x_\mathcal{B} = -\gamma L'$.
Summarising:

> Simultaneous events with respect to an inertial frame are no more simultaneous in another system in relative motion with respect to the first.

This first result definitely gave up with the absolute nature of the classical concept of simultaneity. Actually it shows that each reference system has its own particular time: if we do not specify the frame which time is referred to, the statement of the time of an event does not have any meaning.

Time dilation

The description of the problem of simultaneity highlights that in some ways the motion can affect the measure of the time.

Consider the following situation: A is an observer on a railway embankment ($\mathcal{K}$ frame) seeing a train ($\mathcal{K}'$ frame) passing nearby him inside which there is a second observer B. Both the observers have a clock and let be v the speed of the train with respect to the embankment (we suppose the two frames moving in parallel each other along x-direction, so we will forget about the other dimensions).

According to A, the tick of one second of his clock is the difference between the position of hand reading 0 and reading 1.

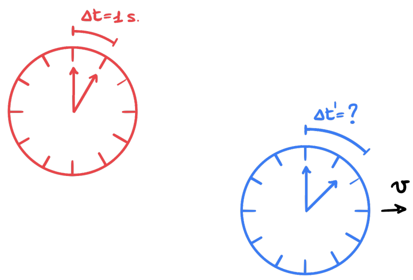


Figure 2.3: How long does one second last as measured from a moving clock (blue one)?

Thus let be $\mathcal{A} = (c \cdot t_{\mathcal{A}}, 0)$ the event "the hand of the clock is pointing to 0" that happens at time $t_{\mathcal{A}} = 0$ in position $x_{\mathcal{A}} = 0$ (we can suppose A to be at the center of the coordinate system). In the same way, $\mathcal{B} = (c \cdot t_{\mathcal{B}}, 0)$ is the event "the hand of the clock is pointing to 1" happening at time $t_{\mathcal{B}} = 1$ in the same position. The time the clock of A needs to tick a second is $\Delta t = t_{\mathcal{B}} - t_{\mathcal{A}}$.

Now let consider the point of view of B: according to him, the two events happen at different time (of his clock); applying a Lorentz transformation and taking into account that $x_{\mathcal{A}} = x_{\mathcal{B}} = 0$:

$$t'_{\mathcal{A}} = \gamma t_{\mathcal{A}} = 0 \qquad\qquad t'_{\mathcal{B}} = \gamma t_{\mathcal{B}}. \tag{2.20}$$

This means that *according to B*, the duration of one tick of A's clock is $\Delta t' = \gamma t_{\mathcal{B}} - \gamma t_{\mathcal{A}} = \gamma (t_{\mathcal{B}} - t_{\mathcal{A}})$.

This phenomenon is called **time dilation** and can be summarised as:

Definition 3 (Time dilation). If in an inertial frame $\mathcal{K}$ an event lasts Δt, then in another inertial frame $\mathcal{K}'$ uniformly moving with respect to $\mathcal{K}$ at speed v along one dimension, the same event lasts $\Delta t'$:

$$\Delta t' = \gamma \Delta t. \tag{2.21}$$

Let now have a deeper look: first, in this phenomenon we have to deal with *two* clocks.

Observer B uses his own clock to measure an event whose duration is Δt in $\mathcal{K}$ frame: if for instance observer B tries to measure the duration of a click of his clock, he will see that this lasts one second. In the same way, when A measures the duration of a click of his clock, he sees that this lasts one second. This is correct according to the principle of relativity: if B was able to see that the duration of a click of his own clock lasts more then 1 second, he would know to be in motion.
On the contrary, observer B uses his clock to measure the duration of a tick of A's clock and he sees that 1 second of A's clock, according to him, lasts more than 1 second of his clock. It is a *relative* measure. In the same way, to prevent the validity of principle of relativity, the same phenomenon occurs in A's frame: if A measures the duration of a tick of B's clock, he will sees that time dilated in the same way B sees.
Another way to express this phenomenon is saying:

> An observer looking to a clock uniformly moving with speed v relative to him measures the clock running slowly by the factor $\gamma = 1/\sqrt{1 - v^2/c^2}$.

The amount of delay ε of the moving clock can be easily quantify as the difference between the duration Δt in $\mathcal{K}$ frame and the duration $\Delta t'$ in $\mathcal{K}'$ frame:

$$\varepsilon = \Delta t' - \Delta t = \Delta t\,(\gamma - 1) = \left(1 - (1 - (v^2/c^2))^{-1/2}\right)\Delta t \simeq \frac{1}{2} \cdot \frac{v^2}{c^2} \cdot \Delta t \tag{2.22}$$

up to the second order in $(v/c)^2$. Being zero the first order, in Classical Mechanics all clocks, moving or not, go at the same rate.
As a final consequence if A and B are two points of a straight line and a clock in A starts moving towards B with speed v relative to another clock in A, if they have been synchronised, at the end of the journey lasted t, they are no longer synchronised by $1/2\,t\,(v/c)^2$.

Length contraction

Strictly connected with the loss of simultaneity there the is length contraction.
Consider two observers A and B in the same condition depicted in the time dilation phenomenon. They do not have a clock but a rigid rubber. Together with observer A consider also a rigid body, a cat for instance.

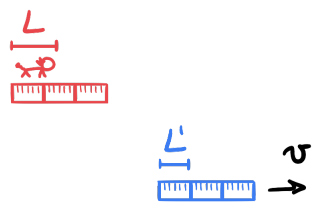


Figure 2.4: How is it long an object as measured with a moving rubber (blue one)?

Observer A measures the length of the cat using his rubber: let focus the attention over the measurement process. The comparison between the object and the instrument is carried out at one instant of time: that is, the object must stay at rest during the measurement process. Observer A needs to align the cat's tail with one notch of the ruler and its head with another, to read the value of each notch and to calculate their difference. This is the whole measurement process: it is clear that if, after having taken the first reading, while moving to align the head to a notch, the cat moves, the measure will not give the true length of the animal because the body was moving during the measurement process. Once again the key word is simultaneity: the reading of the head and tail's position with respect to the rubber has to be carried out at the same time.
According to observer A at time $t = t_{\mathcal{A}}$ the tail is aligned with the zero notch of the rubber: we can assign the 0 coordinate to the tail $x_{\mathcal{A}} = 0$. The event $\mathcal{A} = (c \cdot t_{\mathcal{A}}, x_{\mathcal{A}})$ is thus "the observer A reads the tail's position". At the same time, $t_{\mathcal{B}} = t_{\mathcal{A}}$, observer A reads the position of the cat's head, $x = x_{\mathcal{B}}$: the event "the observer A reads the head's position" is $\mathcal{B} = (c \cdot t_{\mathcal{A}}, x_{\mathcal{B}})$. Observer A determines the cat's length L as the difference between the head and the tail's position $L = x_{\mathcal{B}} - x_{\mathcal{A}}$.
Let now investigate the point of view of observer B. According to the operational definition of the length measure, the position of the extremities of the object has to be measured at the same instant of time. If we simply transform the event $\mathcal{A}$ and $\mathcal{B}$, the two x-coordinates would depend respectively on $t'_{\mathcal{A}}$ and $t'_{\mathcal{B}}$. As $t'_{\mathcal{A}} \neq t'_{\mathcal{B}}$, the coordinate of $x'_{\mathcal{A}}$ and $x'_{\mathcal{B}}$ would be determined at two different instants of time and thus the difference $x'_{\mathcal{B}} - x'_{\mathcal{A}}$ would not correspond to the true length of the cat as measured by the observer B.
For this reason, it is easier to consider $\mathcal{K}'$ reference frame: according to observer B, at certain time $t'_{\mathcal{A}}$ the cat's tail is in a certain position $x'_{\mathcal{A}}$ while, at the same instant of time, the cat's head is in a certain position $x'_{\mathcal{B}}$. By definition, the quantity $L' = x'_{\mathcal{B}} - x'_{\mathcal{A}}$ is the length of the cat as measured by observer B.
Now let link L' to L; using the inverse Lorentz transformations:

$$x_{\mathcal{A}} = \gamma\left(x'_{\mathcal{A}} + vt'_{\mathcal{A}}\right) \qquad x_{\mathcal{B}} = \gamma\left(x'_{\mathcal{B}} + vt'_{\mathcal{A}}\right). \tag{2.23}$$

Thus $L = x_{\mathcal{B}} - x_{\mathcal{A}} = \gamma\left(x'_{\mathcal{B}} - x'_{\mathcal{A}}\right) = \gamma L'$.
This phenomenon is called **length contraction** and can be summarised as:

Definition 4 (Length contraction). If in an inertial frame $\mathcal{K}$ an object is long L, then in another inertial frame $\mathcal{K}'$ uniformly moving with respect to $\mathcal{K}$ at speed v along one dimension, the same object is long L':

$$L' = \frac{L}{\gamma}. \tag{2.24}$$

In particular the dimension that is contracted is the one along the motion while the other ones remain the same as Lorentz transformations leave unchanged the coordinates that are not in relative motion.
Another time, we need to give a deeper look to this phenomenon: we have to deal with *two* rubbers. For instance, let consider observer A: he has a rubber $l_m = 1$ m long, with notches 1 mm distant each others. If he has an object $L = 2.50$ m long, it means that the rubber is contained 2 times and half into the length of the object (L/l_m).
If we consider a second observer B travelling with respect to A with speed v and having the same rubber of A, he will see observer A together with the object moving at speed $-v$. If B using his own rubber measures the moving object, he will get a contracted length $L' = L/\gamma$. In the same way, to prevent the validity of principle of relativity, when A uses his ruler to measure an object which is stationary with respect to observer B, he will get a contracted length. It is a *relative* measure.
However if B tries to use the rubber of the observer A, he will see no contraction: from the point of view of B, both the ruler and the object are moving, thus both are contracted. In particular, observer B will measure the ruler of A as long $l'_m = l_m/\gamma$ and the notches will no longer be at distance 1 mm each others but $1/\gamma$ mm. Thus if observer B uses observer A's ruler, comparing the length of the object with that one of the ruler, he will get:

$$L'/l'_m = (L/\gamma)/(l_m/\gamma) = L/l_m, \tag{2.25}$$

i.e. always the same length.
The process of measure involves the object in the moving reference system as measured with a rubber at rest with respect to it. Moreover as we can interpret this phenomenon as arising from a contraction of the distance between the notches of the ruler, another way to express the length contraction is saying:

> An observer looking to a ruler uniformly moving with speed v relative to him measures the ruler shorter of the factor $\gamma = 1/\sqrt{1 - v^2/c^2}$.

From a historical point of view, Einstein was able to give the correct interpretation of the Lorentz-Fitzgerald contraction phenomenon. Indeed Lorentz firstly found that the shape of the electron faces a contraction in the direction of the motion but he was guide by a purely formal and mathematical spirit, nor he had a theoretical support for his argument. Einstein showed that his theory lead to the same law of motion but without introducing any ad hoc hypothesis about the structure of the electron [79]: the contraction arises from the two principles of his theory.

Addition of velocity

With the introduction of the Lorentz transformations, velocity also has to transform in a different way from how Galileo discovered.

Consider the two inertial observers A and B in uniform relative motion with speed u along the x coordinate. The laws of transformation from A (at rest frame $\mathcal{K}$) to B (moving frame $\mathcal{K}'$) are:

$$\mathcal{L} = \begin{cases} x' = \gamma\,(x - ut) \\ t' = \gamma\left(t - \dfrac{u}{c^2}x\right) \end{cases} . \tag{2.26}$$

Now we consider (x, t) as the coordinate of an object which is moving in the reference frame $\mathcal{K}$ with speed $v = dx/dt$ along the direction of the relative motion of $\mathcal{K}$ and $\mathcal{K}'$. In order to derive how B measures the speed of the object in his reference system, we differentiate the Eq. (2.26):

$$\begin{cases} dx' = \gamma\,(dx - u\,dt) \\ dt' = \gamma\left(dt - \dfrac{u}{c^2}\,dx\right) \end{cases} . \tag{2.27}$$

Taking the ratio between the first and the second equation, we get

$$\frac{dx'}{dt'} = v' = \frac{dx - u\,dt}{dt - \dfrac{u}{c^2}\,dx} = \frac{dt\,(dx/dt - u)}{dt\left(1 - \dfrac{u}{c^2}\,dx/dt\right)} = \frac{v - u}{1 - \dfrac{u}{c^2}\,v}, \tag{2.28}$$

being dx'/dt' the speed v' of the object as measured in $\mathcal{K}'$.

This is the law of addition of velocity:

Definition 5 (Addition of velocity). If in an inertial frame $\mathcal{K}$ an object is uniformly moving with speed $\boldsymbol{v}$, then in another inertial frame $\mathcal{K}'$ moving with respect to $\mathcal{K}$ with speed $\boldsymbol{u}$ along the direction of $\boldsymbol{v}$, the measured speed $\boldsymbol{v}'$ of the object is:

$$\boldsymbol{v}' = \frac{\boldsymbol{v} - \boldsymbol{u}}{1 - \dfrac{\boldsymbol{v}\cdot\boldsymbol{u}}{c^2}}. \tag{2.29}$$

This expressions reduces to the Galileo's Classical one in the non-relativistic limit as if $v \ll c$ then $1 - \boldsymbol{v}\cdot\boldsymbol{u}/c^2 \to 1$ and $\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{u}$.

Einstein's formula contains the solution to the apparent irreconcilability of the principle of relativity with the one on light's speed constancy. Indeed if we consider a light beam as seen from a moving inertial frame $\mathcal{K}'$, moving in the same direction of the beam at speed

u, an observer will measure a speed c':

$$c' = \frac{c-u}{1-\frac{c\cdot u}{c^2}} = c\cdot\frac{c-u}{c-u} = c. \tag{2.30}$$

No object can move faster then light.

Using Eq. (2.29), Einstein gave an explanation to the result of Fizeau's experiment (Eq. (2.3)): if $u = c/n$ is the speed of light in a liquid of refractive index n, when the water is moving with speed v, then light's speed with respect to an external inertial observer is:

$$w = \frac{u+v}{1+\frac{u\cdot v}{c^2}} \simeq (u+v)\cdot\left(1-\frac{uv}{c^2}\right) = \left(\frac{c}{n}+v\right)\cdot\left(1-\frac{v}{cn}\right) = \tag{2.31}$$

$$= \frac{c}{n} - \frac{v}{n^2} + v - \frac{1}{n}\frac{v^2}{c} \simeq \frac{c}{n} + v\left(1-\frac{1}{n^2}\right). \tag{2.32}$$

Thus starting from the relativity addition of velocity, with simple approximations to the first order in v (the water's speed is very small with respect to that of light), one gets the relation obtained by Fizeau.

Finally, contrary to the Galilean addition formula, Einstein formula predicted a new phenomenon about the composition of velocity. In particular if we consider the two inertial frames $\mathcal{K}$ and $\mathcal{K}'$ in relative motion with speed u and an object moving in $\mathcal{K}$ with speed $v_\perp$ in orthogonal direction with respect to $\boldsymbol{u}$, the Galilean transformation will give that an observer in $\mathcal{K}'$ will measure always $v'_\perp = v_\perp$. Consider in $\mathcal{K}$ an object moving with respect to $\boldsymbol{u}$ with speed $\boldsymbol{v}$, then remembering Eq. (2.14), the perpendicular components $\boldsymbol{v}_\perp$ of $\boldsymbol{v}$ transform as:

$$\frac{\boldsymbol{dr'}_\perp}{dt'} = \boldsymbol{v}'_\perp = \frac{\boldsymbol{dr}_\perp}{\gamma\left(dt - \frac{\boldsymbol{u}}{c^2}\cdot\boldsymbol{dr}\right)} = \frac{\boldsymbol{dr}_\perp}{\gamma dt\left(1-\frac{\boldsymbol{u}}{c^2}\cdot\boldsymbol{dr}/dt\right)} = \frac{\boldsymbol{v}_\perp}{\gamma\left(1-\frac{\boldsymbol{u}}{c^2}\cdot\boldsymbol{v}\right)} \tag{2.33}$$

This term, absent in the Galilean relativity, arises clearly from the transformation of the time. In the classical limit it is easy to show that $\boldsymbol{v}'_\perp \to \boldsymbol{v}_\perp$.

Summarising, the two components of a velocity parallel and perpendicular to the relative motion between $\mathcal{K}$ an $\mathcal{K}'$ transform as:

$$\boldsymbol{v}'_\parallel = \frac{\boldsymbol{v}_\parallel - \boldsymbol{u}}{1-\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{c^2}} \qquad \boldsymbol{v}'_\perp = \frac{\boldsymbol{v}_\perp}{\gamma\left(1-\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{c^2}\right)} \tag{2.34}$$

Mass-energy equivalence

Under Galilean transformations the total momentum $\boldsymbol{P}$ of a system assumes the same value when inertial frames are changed. But as far as Lorentz transformations are taken into account, the relation $\boldsymbol{dP} = 0$ is no longer true when we go from a $\mathcal{K}$ inertial system to another one $\mathcal{K}'$ in relative motion. A new definition for momentum has to be introduced in order to have the conservation of momentum invariant under Lorentz transformation. Following [70], we can hypothesise a momentum in the form of $\boldsymbol{p} = mf(v^2/c^2)\boldsymbol{v}$ where the function f depends on $\boldsymbol{v}$ via v^2 assuming isotropy of space. The factor $1/c^2$ makes the entry of the function dimensionless.
Then, analysing an elastic collision with two bodies with the same mass in different inertial frame (center of mass, body 1 and body 2), one gets that $f(v) = 1/\sqrt{1 - v^2/c^2} = \gamma$ obtaining:

Definition 6 (Relativistic momentum)**.**

$$\boldsymbol{p} = m\gamma\boldsymbol{v}. \tag{2.35}$$

For this reason we can think of rewrite Newton's second law of motion as:

Definition 7 (Minkowski's law)**.**

$$\frac{d}{dt}(m\gamma\boldsymbol{v}) = \boldsymbol{F}. \tag{2.36}$$

Einstein then made a strong assumption, namely the validity of the theorem of kinetic energy also in Special Relativity, obtaining a new definition of the kinetic energy. Indeed if we consider a particle subjected to a force $\boldsymbol{F}$ along the direction of motion, the work done by the force in a displacement $\boldsymbol{ds}$ is $dW = \boldsymbol{F} \cdot \boldsymbol{ds}$. If all this work increases the kinetic energy dT of the particle, using the Eq. (2.36), we get:

$$dT = \frac{d}{dt}(m\gamma\boldsymbol{v}) \cdot \boldsymbol{v}dt \longrightarrow \frac{dT}{dt} = \frac{d}{dt}(m\gamma\boldsymbol{v}) \cdot \boldsymbol{v} = m\gamma\frac{\boldsymbol{dv}}{dt} \cdot \boldsymbol{v} + m\frac{d\gamma}{dt}v^2. \tag{2.37}$$

With some algebra:

$$dT = \frac{1}{2}m\gamma^3 d(v^2). \tag{2.38}$$

Integrating the above expression, the final form for the relativistic kinetic energy is:

Definition 8 (Relativistic kinetic energy)**.**

$$T = mc^2(\gamma - 1). \tag{2.39}$$

Before going into the analysis of the formula, we can see that if the speed of a free particle is lower then that of the light, we can expand the parenthesis obtaining:

$$T = mc^2(\gamma - 1) = mc^2\left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right) \simeq mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right) = \frac{1}{2}mv^2. \tag{2.40}$$

Thus if the speed is small with respect to light's one, the relativistic kinetic energy yields the Classical kinetic energy.
Now, giving a deeper look to the expression (2.39), we find that the relativistic kinetic energy is made up of two contributions:

- The total energy E which is the total energy of the particle:

  Definition 9 (Total energy).

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{2.41}$$

- The rest energy E_0 which is the energy of the particle when its speed is $v = 0$:

  Definition 10 (Rest energy).

$$E_0 = mc^2 \tag{2.42}$$

This last equation is one of the most known consequences of Special Relativity and it shows the equivalence between the mass of a body and its energy at rest:

> *Mass and energy are therefore essentially alike; they are only different expressions for the same thing* ([82]).

Thus the relativistic kinetic energy is made up of the difference between these two quantities:

$$T = \underbrace{\frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}}_{\text{total energy } E} - \underbrace{mc^2}_{\text{rest energy } E_0} \tag{2.43}$$

In the same way we can write the total energy of a particle:

$$E = T + E_0 = T + mc^2 \tag{2.44}$$

Einstein's new definition of the energy yielded into some consequences. From Eq. (2.35) to preserve a Newtonian aspect for the momentum, it has been usual to introduce a relativistic mass in the form of:

$$m_r = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.45}$$

where m_0 is the *rest mass*, i.d. the *inertial mass* as measured in an inertial frame at rest with respect to the body. In this way, the momentum can be write in the form of $\boldsymbol{p} = m_r\boldsymbol{v}$, very similar to the Classical momentum $\boldsymbol{p} = m\boldsymbol{v}$.

However, from this definition it arises a mass dependence upon the velocity: the inertial mass should vary according to a change in the energy of the body. Thus it is necessary to define a rest mass m_0 for a body at rest to be distinguished from the "mass" γm_0 for the moving body: the greater the speed, greater the mass of the object.

This is a leading astray definition which is no more adopted and it is only a historical heritage. Okun [84] pointed out an analysis of the propagation of concept of the relativistic mass: in 1899 Lorentz introduced the notion of dependence of mass on velocity and in particular in his work of 1904 ([85]), he defined a *longitudinal mass* m_l and a *transverse mass* m_t for an electron in motion. Later on, in 1921 Wolfgang Pauli published the book "The Theory of Relativity" which would have been an introduction in Special Relativity in the following years for many generation of physicists. He went beyond the concept of transverse and longitudinal, defining conversely a rest mass and a relativistic mass, a distinction employed also in other important works like Robert Resnick's book [45] or Richard Feynman's lectures [86]. This distinction today is no more used and it is considered wrong.

What about Einstein's opinion? He himself in the work of 1905, in order to maintain the form of the equation "mass × acceleration = force", obtained that $m_l = m\gamma^3$ and $m_t = m\gamma$, thus actually assuming a mass dependence upon velocity. It seems that he did not understand the implication of the equations he derived as he did not report any comment to these relations. But in the same year, Einstein wrote another article "Does the inertia of a body depend upon its energy-content?" ([87]) where he stated that:

> *If a body gives off the energy L in the form of radiation, its mass diminishes by L/c^2* [...] The mass of a body is a measure of its energy-content.

The equation, rewritten with the current notation as $\Delta E_0 = \Delta mc^2$, means that if a body varies its mass, then also the amount of the (at rest) energy varies and vice versa. Indeed Einstein never introduced a relativistic mass. In the following years (1906-1914), Einstein wrote many articles ([88, 89, 90, 91]) clarifying the meaning of the previous sentence where he begun to talk about inertia of a body or inertial mass: "the inertial mass and the energy of a physical system appear in it as things of he same kind" leading "to consider any inertial mass as a reserve of energy" ([90]). Thus the term mc^2 means that a body with inertial mass m is an energy store of magnitude mc^2: it is the measure of its energy.

But the rest-energy of a body can be modified for instance by cooling it: if a body at rest emits a radiation of energy ε, then the amount of energy at rest decreases by $\Delta E_0 = E_0 - \varepsilon$

and also the inertial mass m must change during this process of $\Delta E_0/c^2$. Conversely let us consider in $\mathcal{K}$ a body at rest with mass m and an observer moving in $\mathcal{K}'$ with uniform relative speed v along the x-direction. In $\mathcal{K}'$ the object is no longer at rest as it will be seen as moving with speed $-v$. If in $\mathcal{K}$ the total energy of the body is $E = E_0 = mc^2$ then in $\mathcal{K}'$, as it acquires a relative speed thus a kinetic energy, the total energy is $E' = T + E_0 = mc^2(\gamma - 1) + mc^2 = \gamma mc^2$. In this process the intrinsic body's amount of energy does not change, thus its inertial mass is not changing.
Thus Einstein started talking about the *inertia of the energy* to indicate that the process involves directly the energy of a body and thus its inertial mass, which properly quantifies the resistance of the body to variation in the motion. This inertia depends upon the amount of at-rest energy and thus can increase or decrease according to the variation of at-rest energy. Then if gravitational mass is equal or proportional to the inertial mass, we are also able to measure with a balance this variation (as for example in the radioactivity process or in the chemical reactions).
In the book "The meaning of relativity" ([82]), Einstein wrote that:

> We see that the energy, E_0 of a body at rest is equal to its mass.[...]
>
> $$E_0 = mc^2 \tag{2.46}$$
>
> Mass and energy are therefore essentially alike; they are only different expressions for the same thing. The mass of a body is not a constant; it varies with changes in its energy.

It is true that the (inertial) mass of a body can change but this is not to ascribe to its velocity, rather to the variation in the energy amount. Thus it is not correct to write the previous equation using E instead of E_0 because we are referring to the inertia of the body which is quantified by the at rest-energy. Nor it is correct to use m_0 instead of m: this difference may arise considering inertial and gravitational mass. Let consider an inertial frame $\mathcal{K}$ with observer A and a frame $\mathcal{K}'$ with a ball moving along x-direction with speed v. Because of the law of transformation of the force, an observer A will measure that the force along y-direction increases by a factor γ. Thus the weight P of the object is measured to be equal to $P' = \gamma P = \gamma mg$ and then the mass m can be thought to have increased as $m' = \gamma m$, together with the inertia of the body. But as we are measuring a weight, we are dealing with the gravitational mass while Einstein referred to the variation of the inertial one. If a relativistic mass $m_r = \gamma m$ is introduced, then the variation of mass with the velocity will be ascribed to the gravitational mass as the body will be measured heavier than if at rest. The dependence upon the energy is a feature of the inertial mass, not of the gravitational one: in this context the fact that the gravitational mass coincides with

inertial one is an "accident" that however can be used to show this equivalence. Indeed it is true that if a chemical reaction is considered, one can weight the reagents and the products to compute the released energy E_r as $E_r = (m_r - m_p)c^2$.
Another consequence of using m_0 instead of m regards the non-relativistic limit. If we write the kinetic energy T as $T = m_0 c^2(\gamma - 1)$, when $v \ll c$ then $T \to (1/2)m_0 v^2$. Thus we will have defined a moving classical body in terms of a mass at rest m_0 and we will have to precise that this mass tends to the mass m we use in Classical Mechanics. But actually this is the same as the one defined in the Theory of Special Relativity. It is misleading to use m_0 as it would introduce another unnecessary clarification.
Einstein will write in 1948 in a letter to Lincoln Barnett ([84]):

> It is not good to introduce the concept of the *mass* $M = m/(1 - v^2/c^2)^{1/2}$ of a moving body for which no clear definition can be given. It is better to introduce no other mass concept than the "rest mass" *m*. Instead of introducing M it is better to mention the expression for the momentum and energy of a body in motion.

It is necessary to define once a rest mass or simply the mass m of a body as the measure of its inertia and then to treat about the amount of momentum and energy of a moving body. The body's inertia depends on its at-rest energy content.
We point out that Einstein always seems to refer only to a body at rest in order to evaluate the inertia of the energy, namely considering always the reference system at rest with respect to the object. Actually this is not a problem for uniform motion, that is, we can always find an inertial frame where a moving body is at rest. Einstein equivalence gives a measure of the energy stored into a body, regardless of its state of motion. As we show in Appendix A.2, the at-rest energy and thus the mass is invariant for a change of reference frames. Then the variation in the mass and thus the variation in the inertia of the energy is always the same.
Maybe rewriting Eq. (2.41) as $E = \gamma E_0$ it would be clearer that, since $\gamma > 1$, the *total* amount of a body's energy increases while moving due to the contribution of kinetic energy, without requesting any hypothesis about the mass and its dependence upon velocity. In this way the at-rest energy would not show any variation being E_0 always equal to mc^2: it is important to underline that from a moving object we will not get more energy with respect if it is at rest. This is also true according to the relativity principle: being uniform motion only depending on the choice of the reference frame, an observer that sees the object moving or at rest can not get more energy from the same object otherwise he would detect uniform motion. If then in $\mathcal{K}'$ the object is seen as at rest, the observer will get an amount of energy equal to mc^2. In the same way, an observer in $\mathcal{K}'$, who sees the object

as moving, will always get the same amount of energy mc^2 from the mass of that body. Introducing a relativistic mass where it seems that the mass of a body undergoes to a modification due to velocity will lead to believe in the possibility of getting more energy from it. The only way to have a real modification of the body's mass is with a process involving exchange of energy (like emission or absorption).

Another very important consequence of this equivalence is that the theory of relativity gathers the principle of the conservation of mass with the principle of the conservation of energy. This leads to a more compact theory that uses less propositions then the classical one and then it has an higher degree of verisimilitude ([80]).

Before closing this paragraph, we report a short investigation about the treat of the energy in the most used Italian scholastic textbooks. We found that "Amaldi Blu" ([92]) writes $E = mc^2$ whence it derives that for at-rest particle $E_0 = m_0c^2$. Thus it introduces the relativistic mass $m = \gamma m_0$ where m_0 is the mass at-rest, adopting this notation throughout the text. However another edition of the Amaldi book "Il nuovo Amaldi" ([93]) shows the equation $E_0 = m_0c^2$, pointing out that m_0 is an old heritage slowly falling into disuse. It does not use the relativistic mass but that m_0, the one measured in a reference frame at-rest with respect to the object. As we have argued, also this approach is misleading. Finally both the textbooks "La fisica di Cutnell e Johnson" ([94]) and "Fondamenti di Fisica" ([95]) correctly write at-rest energy as $E_0 = mc^2$ and m as the mass for the definition of total and kinetic energy.

2.3 The rise of Minkowski

Einstein's work of 1905 solved the issues about Classical Electrodynamics bringing to light new ideas about space and time, challenging the Newtonian view of reality with the existence of an absolute space and an absolute time.
However, as argued by Levrini [51], the reality's description emerging from Einstein first work still has some Newtonian heritages. It is not relativistic at all as it will be in the following years. As quoted in [*ibid.*], citing [52], "space and time travel along correlated paths but no way to unify them is proposed[8]". This is still a classical approach that, as Newton did, considers space and time as separated entities. Einstein's approach was to use a three-dimensional language to describe something which is more complex and actually four-dimensional as it will be clearer with General Relativity.
The greatest contribution to Special Relativity came from Einstein's mathematics professor, Hermann Minkowski, who was the first to announce a new four-dimensional vision of the world with three articles ([49]):

- *The Relativity Principle* (1907)
- *The Fundamental Equations for Electromagnetic Processes in Moving Bodies* (1908)
- *Space and Time* (1909)

Minkowski's geometrical approach to Special Relativity through the introduction of the four-entity *spacetime* was[9] not a great success among the physicist. Einstein himself at first considered Minkowski's ideas only a "superfluous learnedness" [49, pg. 2], even though he had to quickly change his mind writing his theory of gravitation that would have been impossible to be formulated without the great discovery of Minkowski:

> The generalisation of the theory of relativity has been facilitated considerably by Minkowski, a mathematician who was the first one to recognise the formal equivalence of space coordinates and the time coordinate, and utilised this in the construction of the theory. [96]

Einstein's work of 1905 entered into the flow of attempts to define an experimentally confirmed theory of Electrodynamics: he was more interested in understanding how to *measure* space and time, a little bit overlooking the real implications, even of the principles themselves. For instance he needed to postulate the relativity principle but he did not

[8] My translation from original Italian sentence: "*spazio e tempo viaggiano sì su percorsi correlati, ma non si propone alcuna forma di riunificazione tra di essi*".

[9] Maybe still today [49, pg. 3].

explain its physical meaning as well as he did not explain why it was superfluous the existence of the luminiferous ether.
It is important to highlight that already in 1906 Henri Poincaré introduced Minkowski's ideas in one of his work ([97]), which Minkowski was certainly aware of [49, pg. 19]. Poincaré noticed that Lorentz transformations left the quantity $x^2 + y^2 + z^2 - t^2$ unchanged, similarly to the transformations from a three-dimensional coordinate system to another one with the same origin ($x^2 + y^2 + z^2$ is invariant). Then one could have defined a four-dimensional coordinate system x, y, z, t and vectors with four components, using Lorentz transformations to change reference frames. However this is just an hypothesis for Poincaré, an abstract four-dimensional space without a real importance on the Physics. Instead Minkowski's works are firstly deep considerations about the physical consequences of Einstein's relativity leading to clearer highlight its inferences over our world and to define a new geometry, that one of spacetime, worldlines and quadrivectors.

2.3.1 A new spacetime

The concept of spacetime is a direct consequence of Einstein's relativity principle (but actually also of Galileo's one) that means the impossibility of detecting absolute uniform motion. From this statement one can derive two important consequences.
First, with a simple experiment with light signals it can be shown ([98, p. 40-45]) that the speed of light is constant and simultaneity is not absolute. This is a first sign of a break with Einstein since he needed to *postulate* the constancy of light's speed whereas it is inside the principle of relativity itself.
The second logical implication is that *there is no absolute space.* Here space means the common space in which the Earth moves. Hence the space is *absolute* as it is *one* entity, only one, common for everyone. An absolute uniform motion is therefore an uniform motion with respect to a single space but, as it does not exist, there exist more spaces. Thus an object can be at rest with respect to one space but still moving with constant velocity in other spaces.
How to interpret this idea of different spaces?
A physical three-dimensional space is defined by all the space points that exist simultaneously at a given moment of time. Now, since light has a finite speed[10], it turns out that the objects we see each moments are all *past images* of that objects since light needs time to travel from them to our eyes. Thus at each instant of time, the objects do not exist simultaneously with us but have existed in different three-dimensional spaces belonging to different instants of time. This is to say that space-time in Classical Mechanics is foliated by hyper-surfaces that are planes: reality exists over different three-dimensional spaces,

[10] As moreover Römer experimentally showed in 1676.

each one parallel to the others.

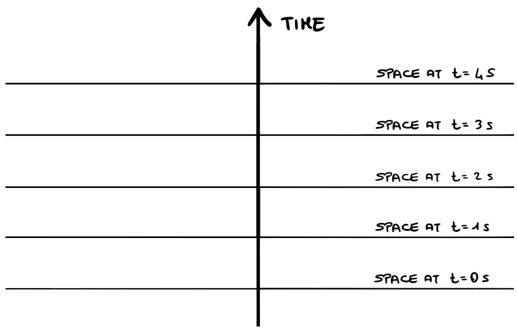


Figure 2.5: The space-time foliation in Classical Mechanics: three-dimensional spaces correspond to different instants of time. The spaces are shown with only one dimension in this image.

We do not perceive a *space* as "now" corresponds to all points over all the three-dimensional spaces belonging to the path of a light ray toward us.
Nevertheless it is necessary that at *each instant of time* there must exist different three-dimensional spaces that there are different three-dimensional spaces belonging to different instants of time (Figure 2.5).
Petkov [98] showed that the only possible way is to assume that these three-dimensional spaces are cross-sections of an at least four dimensional space. Therefore as the world, the reality itself is everything that coexists simultaneously at the present moment, it becomes possible to have different observers with different "present" (simultaneity is no longer absolute), each of them having different three-dimensional worlds, only if the world is four-dimensional, having time as the fourth dimension. Otherwise in a three-dimensional world different observers would have a common three-dimensional space, with a common set of simultaneous events and therefore simultaneity would be absolute, which means the absolute uniform motion does exist. We do not perceive the four-dimensional reality but only the three-dimensional cross-sections that help us to reconstruct the four-dimensional space in the same way as the two-dimensional images we perceived, which our brain processes in order to give us the perceiving of the three-dimensionality of the sensitive reality.
Thanks to Minkowski, Einstein's relativity principle acquires a physical meaning in its more genuine sense (i.e. referring to nature): the world is four-dimensional. It is nor then a consequence of this. Also the covariance principle of the physical laws finds in the four-dimensional world an explanation: each inertial observer describes the phenomenon exactly in the same way being he at rest in his own reference frame, to be considered made up of its own space and its own time. In the same way, the principle of constancy of light's speed is no longer a principle but a necessary consequence since each observer measures the speed of light in his own space (which is at rest) with his own time.

We need to introduce a four-dimensional space called *spacetime* or *Minkowski spacetime*, which Minkowski referred to as simply the *world.* Each point of the spacetime is a *world point* or *event* which is defined by four numbers: three of them define the position of an event in the three-dimensional space whilst the fourth the moment at which it happens. The concept of event is a building block of the spacetime but has a slight different meaning: in the spacetime, the fourth dimension, time, is already all given, it is entirely given at once as the three spatial dimensions otherwise we would not have a four-dimensional world. As a consequence, being a four-dimensional entity, the whole history of everybody in spacetime is already entirely given: history is not unfolding in spacetime.

The four-dimensional object corresponding to a physical point-like object (like a particle) is called *worldline* whilst if the object has an extended shape, it is a *worldtube.* As the worldline[11] is the collection of the particle's events, an event *is* that particle at a certain given instant of its history in time: a worldline contains all the moments of a particle. As a consequence, *in spacetime there is no motion*: for time is already given, a worldline is not a trajectory in spacetime. A position that changes with time is a three-dimensional world where time is a parameter that flows and allows to see the changes. The worldline of a particle is a built four-dimensional entity and spacetime is something like a frozen world.

To relate these considerations to our sensitive experience, we can start by introducing a reference frame: in particular to create an inertial frame we can choose the worldline of a uniformly moving particle to be the time axis whilst the three-dimensional space is orthogonal to the object's worldline. This particle is of curse at rest in its own three-dimensional space but actually it can be a moving particle with respect to other particles.

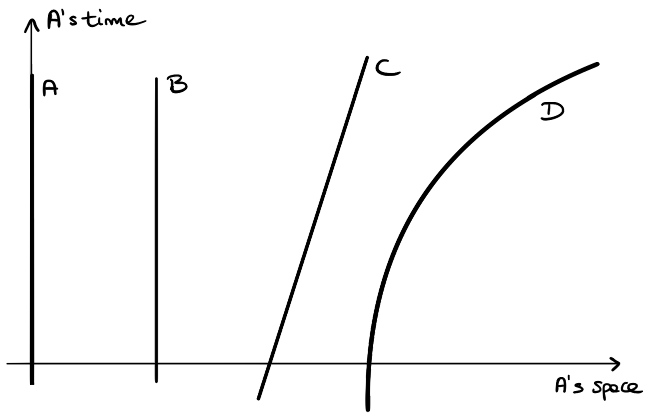


Figure 2.6: Examples of different worldlines in spacetime: the particle A determines the inertial frame in whose space it is at rest. B also is at rest with respect to A's space while C is uniformly moving in A's space. Particle D has an accelerated motion. The figure shows only one spatial dimension of A's space.

[11]The same consideration is true also for a worldtube: if not specified, we will assume the validity of a result also for a worldtube, thus omitting to write it.

Indeed we can have different relations between worldlines as shown in Figure 2.6. If A is a uniformly moving particle and we choose its worldline to be the time direction, then:

- if B has a worldline parallel to A's one, then in A's space the distance between the two particles does not change. They are at rest with respect to each other. Then B's worldline defines the same time direction of A's one and it makes no difference to take one worldline or the other one to define the inertial frame: they share the same time and three-dimensional space;
- particle C has an inclined worldline with respect the one of A (or B): the distance from particle A increases with time which reveals C to be an uniformly moving particle in A's space but not in its own space. Therefore as vertical worldline means particle at rest, there must be an angle between two worldlines to talk about motion;
- particle D has a curved worldline which means that the distance between particles A and D increases with time but in such a way different from C: it is an accelerated motion.

This distinction has an important consequence as the accelerated motion turns to be *absolute*, where absolute means detectable by everyone and not a motion with respect to an absolute space. If the shape of a worldline is curved, then this particle is accelerated in all the inertial frames, namely with respect to all the three–dimensional spaces. An uniformly moving particle is not moving in all the inertial frames as it is at rest in its own space. This is not true for an accelerated particle because it does not have its own space.

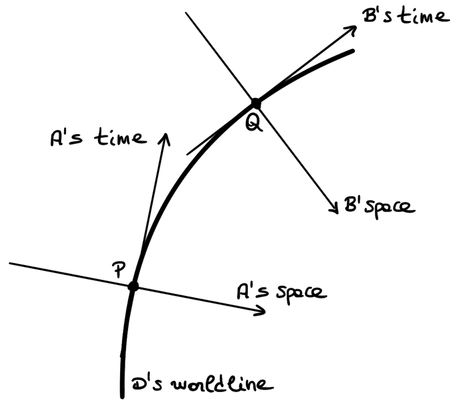


Figure 2.7: An accelerated particle D has a curved worldline: at each moment of its story it is at rest with respect to an inertial frame whose time axis is defined by the tangent direction to D's worldline. In P and Q there are two different inertial frames determined by the uniformly moving particles A and B.

When we say that a uniformly moving particle has its own space we are stating that the spaces at different instant of time are all orthogonal to the same time direction defined by the particle's worldline, as the particle defined *only one* time direction (in a similar way to Figure 2.5).

Indeed if we consider a worldline of an accelerated particle D, at each instant of time of its history the time direction is different, being defined by the tangent lines to the worldline (Figure 2.7). This means that at each time of the particle's history we can define a different space, orthogonal to the time direction: thus at each time a *different* inertial frame is associated with this particle, whose time axis is along the time direction in that point of spacetime. Physically this means that the accelerated particle is instantaneously at rest with different inertial frame corresponding to different uniformly moving particles which in that instant of time share the same space and the same time with particle D. Thus at events P and Q, we can find two particles A and B whose time axis coincides with the tangent direction to D's worldline: the inertial frame associated with particle A and B are called *comoving* inertial frame.

But spacetime is four-dimensional: the distinction between three-dimensional space and time is not real and it is only a way we use to describe the invisible spacetime, a *shadows* as Minkowski said ([99]). Similarly motion in the three-dimensional space is a description of the way we perceive the particles' worldlines ([98]).

The conclusion that spacetime is four-dimensional entity could lead to interpret time as equivalent to three-dimensional space: coordinates x, y and z are of the same nature of t or ct (in order to have all of them homogeneous). Time coordinate actually is not equivalent to spatial ones: Einstein explicitly wrote that Minkowski recognised only the *formal equivalence of space coordinates and the time coordinate* ([96]). They are not of the same nature, there is always a distinction among them.

Indeed the spatial coordinate are interchangeable, namely there is an arbitrary in choosing which direction is x, which y and which z. Moreover we are free to create non-orthogonal spatial planes between each others and also a three-dimensional non-orthogonal spatial space with respect to the time axis. This is not true for time direction: there are some constraints. We have seen already that we can not choose the worldline of an accelerated particle to define the time direction as it changes in each event (Figure 2.7).

Other differences arise when we examine the behaviour of light in spacetime: light's equation of motion for a ray propagating along x-axis is $x = ct$ which is a line forming an angle of 45° with the time axis. If now we consider an isotropic light source in $x, y, z = 0$ which emits at $t = 0$, we will observe in the three-dimensional space a light-sphere expanding. However in spacetime there is no light-sphere expanding as the entire history of the light signal is already all given as a crystallised entity in the four-dimensional world. Neglecting the z component of the three-dimensional space, in spacetime we have the structure shown in Figure 2.8.

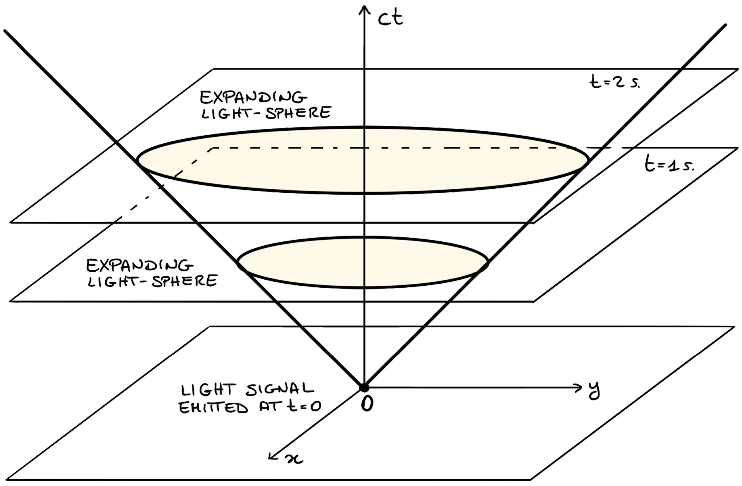


Figure 2.8: A source in O emitting a light signal generates a cone in the four-dimensional spacetime whose cross-sections with our three-dimensional space are the sphere we perceive as the expanding light sphere.

Light propagates from the point O in all directions generating a hyper-surface that is a *cone* in the four-dimensional spacetime: at each instant of time our three-dimensional space cuts the four-dimensional space creating a three-dimensional cross-section that is a sphere, the light-sphere we interpret as expanding from O at time $t = 0$ in following moments. In Figure 2.8 as we suppressed one spatial dimension, the cross sections are the highlighted circumferences. This light cone is the future history of the light signal emitted in the point O.
In the same way it can be defined another sheet of this cone that refers to the past of point O: at certain time, what we see is defined by all the light rays that have reached our eyes.

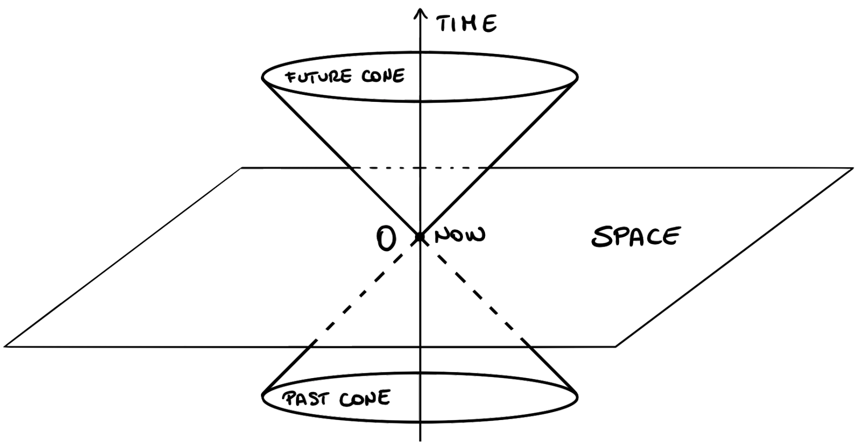


Figure 2.9: An event defines two region of spacetime delimited by two four-dimensional cones: one contains all the future history of event in O whilst the other contains all its past history.

Thus as shown in Figure 2.9, if O represents the present moment, the lower sheet of the cone is the *past* history of all the light rays reaching O at $t = 0$ whilst the upper sheet of the cone contains the *future* history of all the light rays emitted from O at $t = 0$. The light cone however does not define a space in the sense of a three-dimensional region simultaneously existing at a given instant of time. Whilst the past sheet of the light cone contains events corresponding at different instants of time.
An observer in O thinks that he sees a three-dimensional space but this is defined at one instant of time as a cross-section orthogonal to time direction: as shown in Figure 2.9 the three-dimensional space is not intercepted by any light rays, thus the observer does not see anything. It is the past region of the light cone that actually the observer perceives.
The light cone is a powerful instrument that allows to link the four-dimensional spacetime with our apparent three-dimensional world and it is independent from the introduction of a reference frames: it is a property of each event of spacetime.

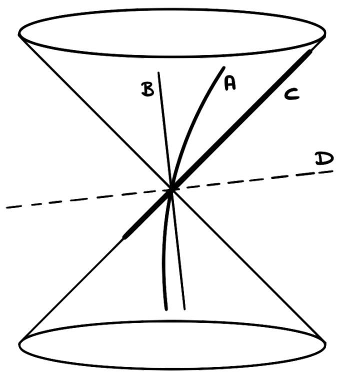


Figure 2.10: Different kinds of worldlines depending on their position with respect to a light cone.

With respect to a light cone, we can distinguish three kind of worldlines (Figure 2.10):

- worldlines inside the cone like A and B, defining respectively a particle at rest and moving with speed less then that one of light;
- a worldline that lying on the surface of the cone like C that represents a light ray or a particle moving at speed of light;
- a worldline outside the cone like D that corresponds to a particle moving at speed grater that one of light.

This distinction tells us more about the non-freedom in choosing the time direction. For instance, we can think of setting an inertial frame whose time axis is along D's worldline (Figure 2.11).

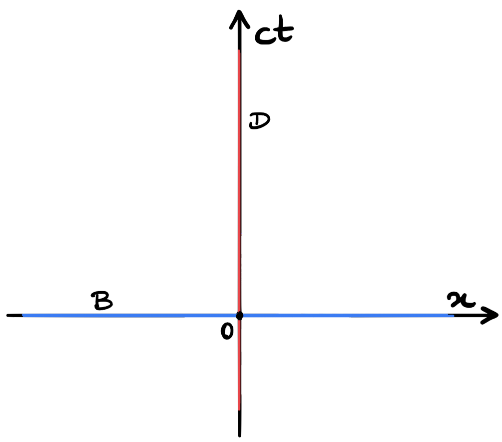


Figure 2.11: Setting the inertial frame on the worldline of particle D (red line) of Figure 2.10: particle B (blue line) will appear and disappear momentarily in all the space of particle D.

Then, B's worldline will be contained inside D's space but, as a worldline contains all the history of a particle, this means that particle B simultaneously at one instant of time would appear and disappear from nowhere. Indeed we can directly exclude the existence of a worldline outside the cone (like D's one): if we consider the inertial frame whose time direction is orthogonal to D's worldline like B's one, then the superluminal particle's worldline would lie entirely on the three-dimensional space, simultaneously existing at all instants of its history.
Thus we can not choose the time direction lying on a worldline outside the light cone nor on its surface. Being the speed of light constant in all the reference frames, then also the inertial frame associated with worldline C (Figure 2.10) would itself move at speed c and in this way it would be a privileged reference frame. Moreover it would lead to a contradiction as, since a particle is at rest in the inertial frame associated with its own worldline, light will itself be at rest while we know that light has always the same speed.
The conclusion is that although we are free to decide the direction of spatial coordinates, time direction is constrained to lie inside the light cone: therefore the worldlines inside the light cone are called *time-like*, the ones lying on the surface of a light cone *light-like* whilst the ones outside the light cone *space-like*.

2.3.2 The geometry of spacetime

The best way to study the geometry of the spacetime is starting from what we already know about geometry of the Euclidean three-dimensional space. In particular we can take advantage from the transformation between two coordinate systems which are rotated with respect to each other to find the expression of the transformations between two inertial frame, since the worldlines corresponding to these observers are rotated with respect

to each other in spacetime.
If two coordinate systems $\mathcal{K}$ and $\mathcal{K}'$ are rotated by an angle α (taking with respect to y axis), then the law of transformation $\mathcal{R}$ from $\mathcal{K} \to \mathcal{K}'$ is:

$$\mathcal{R} = \begin{cases} x' = x\cos\alpha - y\sin\alpha = \dfrac{x - y\tan\alpha}{(1+\tan^2\alpha)^{1/2}} \\ y' = x\sin\alpha + y\cos\alpha = \dfrac{y + x\tan\alpha}{(1+\tan^2\alpha)^{1/2}} \end{cases} . \tag{2.47}$$

Now we introduced the *spacetime* $\mathcal{M}$ (or $\mathcal{M}_4$) as a four-dimensional manifold: in addition to the three-dimensional spatial coordinates x, y and z we use a fourth coordinate ct where the coordinate t has been multiplied by c in order to have all the four coordinates homogeneous. But we must keep in mind that time coordinate ct is not equivalent to the spatial ones and we must keep this constrain. A solution that shows their different nature, it is to introduce an imaginary time ict (following Minkowski's approach in his papers [49]) and real spatial coordinates (or a real time and imaginary spatial coordinates) in a similar way as the imaginary numbers are distinct from the real one in a complex plane and can not be interchanged. The formalism of a complex coordinate is an helpful way to *describe* mathematically the spacetime and that allows to derive its properties. However this does not have to lead to misunderstandings about the *reality* of spacetime, being real (both in mathematical and ontological sense) the quantities we measure in spacetime on its three-dimensional cross-sections (distances, time intervals, velocities...): the Physics is held only from real quantity. We can formally look at Minkowski's world as a four-dimensional Euclidean space with an imaginary time: this will help us to derive the main features of spacetime only considering its three-dimensional Euclidean counterpart and extending the results we will get to the four-dimensional space.
Thus we are going to describe the rotation in spacetime looking to the three-dimensional ones: we can think of neglecting two spatial coordinates (y and z) and to use Eq. (2.47) where y will be replaced by ct as time is orthogonal to x, in the way y is to x. The complex substitution[12] is $x \to ix$. If then we consider two inertial frames $\mathcal{S}$ and $\mathcal{S}'$ in uniformly relative motion with speed v, since $\tan\alpha = \Delta x/\Delta y = i\Delta x/(c\Delta t) = i\,v/c = i\beta$, making the substitutions in Eq. (2.47), we get that:

$$\begin{cases} x' = \dfrac{x - \beta ct}{\sqrt{1-\beta^2}} \\ ct' = \dfrac{ct - \beta x}{\sqrt{1-\beta^2}} \end{cases} . \tag{2.48}$$

[12]Minkowski adopted the substitution $ct \to ict$ but our choice is more suitable in order to get the interval defined as $c^2t^2 - x^2$.

We recognise in Eq. (2.48) the law of Lorentz transformations (Eq. (2.13)):

Lorentz transformations are rotations in Minkowski spacetime.

Similar to these considerations, we know that in Euclidean space the length between two points $l^2 = x^2 + y^2$ is a conserved quantity, independent of the reference frame we use to measure it. Then making our substitutions, we get that the conserved quantity is $s^2 = c^2t^2 - x^2$. Then in the complete four-dimensional spacetime we have:

$$s^2 = c^2t^2 - x^2 - y^2 - z^2. \tag{2.49}$$

This quantity, called *interval*, is the *distance in spacetime* between two events and it is an invariant under spacetime rotations, namely under change of inertial frames.
It is evident the difference from the Euclidean length $l^2 = x^2 + y^2 + z^2$, arising from the minus sign on the fourth coordinate. This feature is called *signature* of a space: the signature of Euclidean space is $(+, +, +)$ whilst the signature of spacetime is $(-, +, +, +)$ or either $(+, -, -, -)$: the difference is only in the choice of which coordinates are imaginary, the temporal one or the spatial ones. We will use the signature $(+, -, -, -)$ that corresponds to the set of substitutions $y \to ct$ and $x \to ix$.
Due to the similarity with Euclidean space, the spacetime is a *pseudo-Euclidean* space. In the pseudo-Euclidean space $\mathcal{M}$ an event, as a given point in spacetime, is defined by a four dimensional vector $x^\mu = (x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}) = (ct, x, y, z) = (ct, \boldsymbol{x})$. Among the four dimensional vector that we can build in spacetime, importance is given to vectors that changing reference frames transform according to Lorentz transformations: the *four-vectors* or *quadrivectors*.
If the two reference frames are in uniformly motion with respect to each other at speed v along one direction (x), then this transformation is described by the matrix Λ of element $\Lambda^\mu{}_\nu$:

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.50}$$

being $\beta = v/c$.
Formally speaking, Minkowski spacetime is a manifold with a norm and then with the concept of length (see Appendix A.3): one can use it to measure the distance between two events in the spacetime and in the same way to define the length of a quadrivector (its norm) that, for what we have just said, it is a conserved quantity. However if two events are connected by a displacement four-vector $\Delta x^\mu = (c\Delta t, \Delta x, \Delta y, \Delta z)$, then the interval is $\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$. It is clear that this scalar product is not

positive definite as it can be negative and also zero even when the two points in spacetime are not coincident. We can use this metric to evaluate the distance between two points on each kinds of worldlines, thus deriving some important consequences. Consider as the first event the point in spacetime (we simplify taking into account only one spatial dimension) $O = (0, 0)$ while as second point $A = (ct, x)$, belonging to a certain worldline. Moreover, since the length is invariant under change of inertial frame, we will choose a suitable inertial frame to determine the properties of the worldlines:

- **distance over a time-like worldline**: since it represents a real particle, we can choose an inertial frame in which event A lays over the time-direction. Then $x_A = 0$, $\Delta x = 0$ (the two events happen in the same spatial position) and since $c^2\Delta t^2 > 0$, we have that $\Delta s^2 = c^2\Delta t^2 > 0$. In general, for a time-like worldline in the four-dimensional spacetime:

$$\Delta s^2 > 0. \tag{2.51}$$

  This means that since the particle moves at speed $v < c$, then the time Δt it needs to travel the distance Δx is smaller then the distance $c\Delta t$ light can travels in the same interval of time.

- **distance over a light-like worldline**: since it represents a particle moving at speed of light, then event A lays on the light cone. Thus the distance $\Delta x = x_A - x_O$ is travelled at speed of light in an interval of time Δt so that $c\Delta t = \Delta x$ and then taking the square $\Delta s^2 = 0$. In general, for a light-like worldline in the four-dimensional spacetime:

$$\Delta s^2 = 0. \tag{2.52}$$

- **distance over a space-like worldline**: it represents an hypothetical particle moving at speed greater then that of light and this means that it does exist an inertial frame whose three-dimensional space simultaneously contains the entire worldline. The event A is simultaneous with respect to event O, thus $\Delta t = 0$ and $\Delta s^2 = -\Delta x^2 < 0$. In general, for a space-like worldline in the four-dimensional spacetime:

$$\Delta s^2 < 0. \tag{2.53}$$

  This means that since the particle moves at speed $v > c$, then the time Δt it needs to travel the distance Δx is greater then the distance $c\Delta t$ light can travels in the same interval of time.

The important consequence of this classification is that it exists an objective difference between worldlines and this difference is absolute in the way that it is the same for every

inertial observer.
Moreover we have seen that on a light-like worldline, the distance between two points O and A in the spacetime is always zero even if the two events do not coincided. This is another reason why an inertial frame whose time axis lays along a light-like worldline can not be chosen: in this case, the two events O and A would lay over the time axis and, since Δx would be 0, then $\Delta s^2 = c^2 \Delta t^2$. This would lead to a contradiction since for a light-like particle $\Delta s^2 = 0$ and it should be 0 also in this particular reference frame. Moreover since $\Delta s^2 = 0$ and since in this frame $\Delta s^2 = c^2 \Delta t^2$, we will have that $\Delta t = 0$ giving a non-clear definition of time interval.

2.3.3 Minkowski strikes back

Minkowski's geometrical approach to Special Relativity reproduces the same result of Einstein's theory, thanks to the *spacetime diagrams*. In particular, he used always bi-dimensional diagrams representing a bi-dimensional spacetime with a time axis t and one spatial coordinate (the *space*) x, orthogonal to the first.
Textbooks (for instance [70]) explain how to draw a Minkowski spacetime diagram but we are going to adopt a more didactic strategy, following the approach of Loedel and Amar (see Section 1.5.1). Indeed to draw the diagram it can be useful to think in terms of the invariance of the interval: given an event $A = (ct, x)$ in spacetime according to reference frame $\mathcal{S}$, the distance from the center O is $s^2 = c^2 t^2 - x^2$. Then, if we consider an uniformly moving observer $\mathcal{S}'$, the event A is mapped in $A' = (ct', x')$ whose distance from O[13] is $s'^2 = c^2 t'^2 - x'^2$. Because of the invariance of the interval

$$c^2 t^2 - x^2 = c^2 t'^2 - x'^2, \tag{2.54}$$

we get ([98]):

$$c^2 t^2 + x'^2 = c^2 t'^2 + x^2. \tag{2.55}$$

This expression looks like the invariance of Euclidean distance $x^2 + y^2$, telling us that, as x and y are orthogonal in the Euclidean space, then we can draw t axis orthogonal to x' axis and t' orthogonal to x. The ct' axis is rotated with respect to ct axis by an angle α, being $\tan \alpha = \beta = v/c$. The rotation of the two reference frames indicates that they represent inertial observers in uniformly relative motion.

[13] O' coincides with O, having supposed that the two inertial frames coincide at $t = t' = 0$.

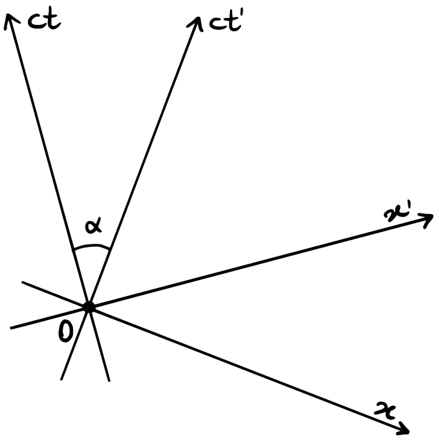


Figure 2.12: Spacetime diagrams of two observers in uniformly relative motion: the time axes are chosen along the observers' worldlines.

We just have to remember that all the considerations we deduce from a spacetime diagram can be obtained also by making use of the Lorentz transformations.

Loss of simultaneity

Consider an inertial frame $\mathcal{S}$ and a second one $\mathcal{S}'$ uniformly moving with respect to the former.

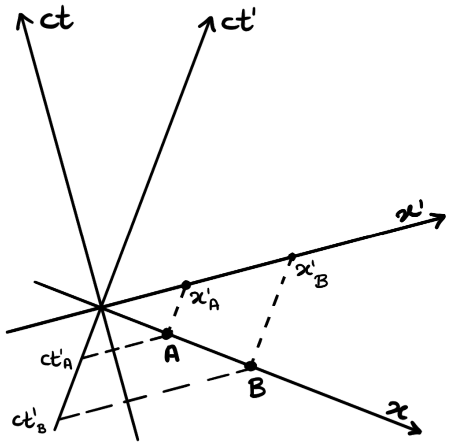


Figure 2.13: Minkowski diagram concerning loss of simultaneity: events A and B, simultaneous in $\mathcal{S}$, are no longer simultaneous in $\mathcal{S}'$.

Let A and B be two events in spacetime: according to observer $\mathcal{S}$, events A and B are simultaneous as they occur at the same time $t = 0$ in two different positions in space x_A and x_B (Figure 2.13). Conversely, according to observer $\mathcal{S}'$ the two event are no longer simultaneous: event B happens at time t'_B before event A, happening at t'_A.

Relativity of simultaneity allows to reflect about the four-dimensionality of spacetime: consider two observers A and B in uniformly relative motion whose time axis lays on their worldlines. We suppose that there are also two clocks C_1 and C_2 at rest with respect to A and thus moving with respect to B. The two observers meet at event M and try to determine the instant of time of their meeting with respect to the two clocks.

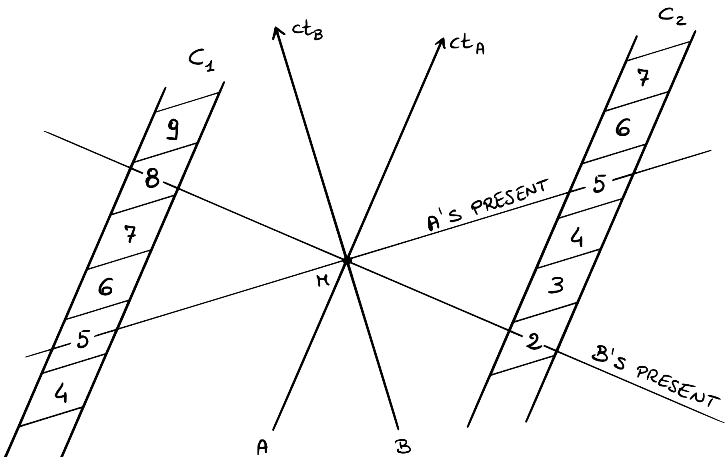


Figure 2.14: Relativity of simultaneity implies that observers A and B in uniformly relative motion have two different pairs of three-dimensional clocks C_1 and C_2, each one reading different time for each observer.

According to observer A, drawing his three-dimensional space at event M, they meet when both the clocks are reading the 5th second: A-observer's set of simultaneous events is made up of the clock C_1 reading $t_{C_1} = 5$ s and clock C_2 reading $t_{C_2} = 5$ s. However observer B sees that clock C_1 is reading $t_{C_1} = 8$ s while clock C_2 is reading $t_{C_2} = 2$ s.
The common view of the world based on the absolutely of simultaneity (presentism), defines the present as all that exists simultaneously at the moment "now". But Figure 2.14 shows that there are two different sets of simultaneous events and thus two different presents: observers A and B, meeting at event M, disagree on what is "present". A pair of clocks exists simultaneously with M according to A, whereas B affirms that another pair of clocks exists simultaneously with M. Relativity of simultaneity represents thus a problem for our current presentism's view: the only solution to keep the presentist idea of the world is the *ontological relativisation of existence* ([98, pg. 131]). This does not mean that observers in uniformly relative motion describe the *same* three-dimensional object in a different way according to their reference frame. Instead it means that different observers have *different* three-dimensional objects: reality becomes frame- or observer- depending. Thus at event M there exist both one pair of clocks simultaneously with M according to A and another pair of clocks simultaneously with M according to B. Clock C_1 exists as two different three-dimensional objects, being one according to A the clock at its own time

$t_1(A) = 5$ s and the second according to B the clock at its own time $t_1(B) = 8$ s. In the same way, clock C_2 simultaneously to clock C_1 exists according to A at its proper time $t_2(A) = 5$ s and according to B at its proper time $t_2(B) = 2$ s.

This implies the even if observer A has to deny the existence of another present[14] and then the existence of other two clocks C_1 and C_2 reading different times, being the relativity of simultaneity a real phenomenon, observer A knows that clock C_1 is reading the time 8 s according to observer B as a different three-dimensional object. The same considerations hold also for observer B and for the second clock. This results in a contradiction in terms as each observer knows that his own present is the only one but each one knows also (for relativity of simultaneity) that the clock exists as a different three-dimensional object for the other observer at the event of meeting. If the clock C_1 does not exist as *two* different three-dimensional object, no relativity of simultaneity would be possible.

However if we assume that spacetime exists and then the two clocks exist as a four-dimensional objects, namely a worldtubes within the spacetime, reality can still be absolute, no longer observer-dependent, obtaining non-contradictory conclusions.

Being clocks' worldtubes real four-dimensional entities, all the moments of their existence are already entirely given in spacetime: as a consequence, observers A and B, having their own plane of simultaneity at event M (their own present), cut off different three-dimensional "slice" from the worldtubes of the two clocks. As all the worldtubes already exist, the three-dimensional space of observer B crosses the clocks' worldtube in two events "clock C_1 is reading 8 s" and "clock C_2 is reading 2 s", while observer A's present in other two events "clock C_1 is reading 5 s" and "clock C_2 is reading 5 s" because all these events already exist in spacetime, being part of the worldtubes. There is no need of introducing different copies of the three-dimensional clocks. For both the observers clocks C_1 and C_2 are always the same four-dimensional objects but not the same three-dimensional objects.

Time dilation

Consider the classical Euclidean bi-dimensional space and two reference frame $\mathcal{K}$ and $\mathcal{K}'$ rotated with respect to each other by an angle α. An observer in $\mathcal{K}$ measures the length of a road OA laying entirely along y axis and he finds that it is, say, y long.

[14] B's one because of the presentist view (only his own present exists).

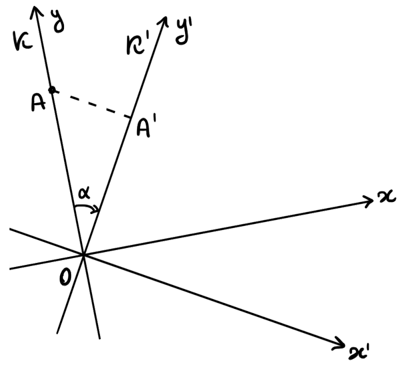


Figure 2.15: In Euclidean space the length OA of a road all along the y-axis is projected towards a shorter road OA' in the $\mathcal{K}'$ reference frame.

Now observer in $\mathcal{K}'$ wants to perform the same measure but it discovers that the end's part of the road A has both x- and y- components in $\mathcal{K}'$. Since the Euclidean length is an invariant, the components x' and y' of the end's part of the road are such that ${x'}^2 + {y'}^2 = y^2$.

However the height of the road, namely the y' component, appears shorter than in $\mathcal{K}$ frame. Observer in $\mathcal{K}'$ has only to project the point A over y'-axis, obtaining (using Eq. (2.47)):

$$y' = \frac{y}{(1 + \tan^2 \alpha)^{1/2}} \tag{2.56}$$

It is evident that $y' < y$ as the observer in $\mathcal{K}$ is measuring the real height of the road (that coincides with its length), that we called *proper*, whilst the observer in $\mathcal{K}'$ measures its apparent height, since the road in $\mathcal{K}'$ is inclined and thus it has x- and y- components. What the observer in $\mathcal{K}'$ is measuring is the *projection* of the road only along the y'-axis without taking into account the x-component.

Now consider the spacetime and two inertial observers $\mathcal{S}$ and $\mathcal{S}'$ in uniformly relative motion; making the substitution $y \to ct$ e $\tan \alpha = i\beta$ we have that

$$ct' = \frac{ct}{(1 - \beta^2)^{1/2}}, \tag{2.57}$$

which means

$$t' = \gamma t = \frac{t}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}. \tag{2.58}$$

We have obtained the formula of the time dilation phenomenon.

Consider two inertial frames $\mathcal{S}$ and $\mathcal{S}'$ having in their common origin two identical clocks; their time axis is chosen to lay along the worldlines of the two clocks like in Figure 2.16.

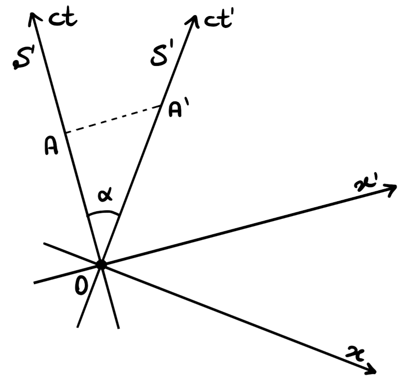


Figure 2.16: The spacetime diagram of the time dilation phenomenon: the duration an event of proper time τ in $\mathcal{S}$ reference frame is greater in $\mathcal{S}'$ reference frame ($\gamma\tau$).

The observer in $\mathcal{S}$ measures on the four-dimensional worldline a process that starts in O and finishes in A: according to him it has only a time component that we will call *proper time*:

Definition 11 (Proper time)**.** *Proper time* τ is the time of an event as measured in an inertial frame at rest with respect to it.

Observer in $\mathcal{S}$ measures the proper time of the process OA that is, say, $\tau = 5$ s. Then when observer in $\mathcal{S}'$ tries to perform the same measurement, he will find that event A has not only a time component but also a spatial one. If he determines the time coordinate of the event A in his reference frame, he will measure a greater time, say $t' = 6$ s as he is not measuring the whole four-dimensional distance from event O. He is measuring only the time component of the four-dimensional process OA in order to compare his time component with the $\mathcal{S}$-observer one: as observer in $\mathcal{K}'$ did in the Euclidean example, observer in $\mathcal{S}'$ is projecting event A on the event A' along the worldline of his clock, thus determining a different time coordinate of event A. Observer in $\mathcal{S}'$ will find that event A' is simultaneous with event A, as this one lays on the instantaneous three-dimensional space[15] corresponding to event A'. He will find that when clock in $\mathcal{S}$ reads $t = 5$ s, simultaneously his clock reads $t' = 6$ s, concluding that the process OA lasts 6 s in $\mathcal{S}'$ frame. Event A' has an apparent or dilate time as actually in $\mathcal{S}'$ the worldline of the clock, that was at rest in $\mathcal{S}$, is *inclined.* As a consequence, A has both a spatial and temporal coordinate: observer in $\mathcal{S}'$ is not measuring proper time as his worldline is not laying along the one of the process OA (the clock of the observer in $\mathcal{S}$).
Another important feature of this phenomenon is that it is reciprocal according to the relativity principle: observer in $\mathcal{S}$ can determine the duration of a process OB', being the proper time $\tau = t' = 5$ s, that is the duration of OB' is 5 s as measured with $\mathcal{S}'$-observer's

[15]It is the three-dimensional plane over which lays the segment AA' in Figure 2.16.

clock, at rest with respect to the process OB'. Then observer $\mathcal{S}$ will find that the simultaneous projection of event B' on event B lies on the three-dimensional plane at time $t = 6$ s. This means that the proper time of the two observers flows in the same way, being then no difference in the distance between the marks of two following seconds. This also means that

proper time τ is invariant

in the change of inertial frame, being proportional to the length of a time-like worldline $\tau = ct$.
However in a different way from the Euclidean world, the projection of the event A does not lead to a shorter road ($y' < y$) but a greater component of the time coordinate ($t' > t$), which is a consequence of the pseudo-Euclidean nature of the spacetime. This difference is also evident in the way a spacetime diagram and an Euclidean diagram are drawn: comparing Figures 2.15-2.16, it is evident that for the former event A is projected onto A' towards smaller value of the y-coordinate while for the latter event A is projected onto higher value of the ict-coordinate. This is due to how the two coordinated axes are drawn on the Euclidean surface of a paper sheet: as shown in the beginning of Section 2.3.3, for the Euclidean space the two axes are orthogonal while for the pseudo-Euclidean one the x-axis and the ct-axis are not orthogonal. Due to the invariance of s, the ct-axis is orthogonal to x' one.
Actually time dilation phenomenon is an experimental evidence of the four-dimensional nature of the spacetime. Consider Figure 2.17, depicting the worldline of two observers S_1 and S_2, avoiding the notation with $'$ in order to be as general as possible.

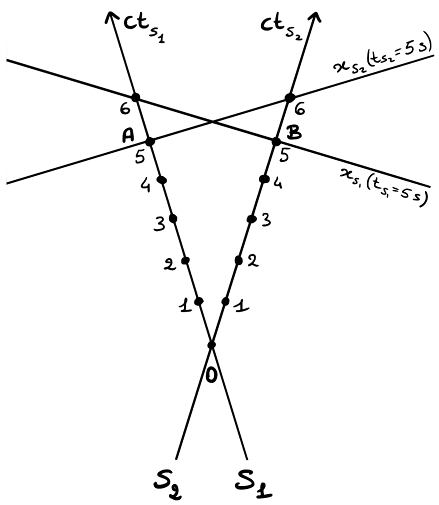


Figure 2.17: Reciprocity of time dilation: this is guaranteed only admitting either two different pairs of three-dimensional clocks (presentism) or two clocks as four-dimensional entities.

For the observer S_1 a process OA lasts 5 s that means when his clock reads $t_{S_1} = 5$ s the process finishes. Observer S_2 however will say that the event OA lasts $t_{S_2} = 6$ s due to the time dilation phenomenon: this means that when the S_2-clock reads $t_{S_2} = 6$ s, simultaneously observer S_2 knows that S_1-clock reads $t_{S_1} = 5$ s.
In the same way the duration of the event OB is $t_{S_2} = 5$ s according to S_2-clock but observer in S_1 will determine the duration of process OB to be $t_{S_1} = 6$ s.
Then S_1-clock reading $t_{S_1} = 6$ is simultaneous with the S_2-clock reading $t_{S_2} = 5$ s belonging to the same three-dimensional plane $x_{S_2}(t_{S_2} = 5)$; in the same way, the S_2-clock reading $t_{S_2} = 6$ s is simultaneous with the S_1-clock reading $t_{S_1} = 5$ s belonging to the same three-dimensional plane $x_{S_1}(t_{S_1} = 5)$. This means that:

- according to observer S_1 all that exists for him at his proper time $t_{S_1} = 6$ lays on the instantaneous three-dimensional plane at $t_{S_1} = 6$: S_1-clock reading 6 s and S_2-clock reading 5 s (the end of OB process);

- according to observer S_2 all that exists for him at his proper time $t_{S_2} = 6$ lays on the instantaneous three-dimensional plane at $t_{S_2} = 6$: S_2-clock reading 6 s and S_1-clock reading 5 s (the end of OA process).

Then in the presentist view if clocks are three-dimensional objects, both S_1 and S_2 should agree with the vision of only one of the two observers, say S_1, as there is only one present: the clock of S_1 existed reading the time 5 s while the clock of S_2 existed reading time 6 s. But in this way we will not have a reciprocal phenomenon and hence we would be able to detect uniform motion.
Once again one arrives to the conclusion that in order to prevent this possibility there should exist two different pair of three-dimensional clock (as in the case of relativity of simultaneity), one for each observer. Observer S_1 will say that it is real for him that S_1-clock exists at time $t_{S_1} = 6$ s while S_2-clock exists at time $t_{S_2} = 5$ s. Observer S_2 will say that it is real for him that S_2-clock exists at time $t_{S_2} = 6$ s while S_1-clock exists at time $t_{S_1} = 5$ s. But both of them will say that only his own pair of clock exists, thus denying the existence of the other observer's pair. One more time, we would lead to an ontological relativised existence in which reality is observer-dependent.

Length contraction

Length contraction phenomenon is a little more tricky to treat and the following analysis will clarify the adopted strategy in the Section *Length contraction*[16].
As for the proper time, we can define a *proper length*:

[16]Section 2.2.3 pg. 48.

Definition 12 (Proper length)**.** *Proper length* is the length of an event as measured in an inertial frame at rest with respect to it.

We can now suppose to have an inertial frame $\mathcal{S}$ in which a road has proper length L: the road lays entirely along the x-axis, thus having only a spatial component like in Figure 2.18 (where its complete worldtube is also shown).

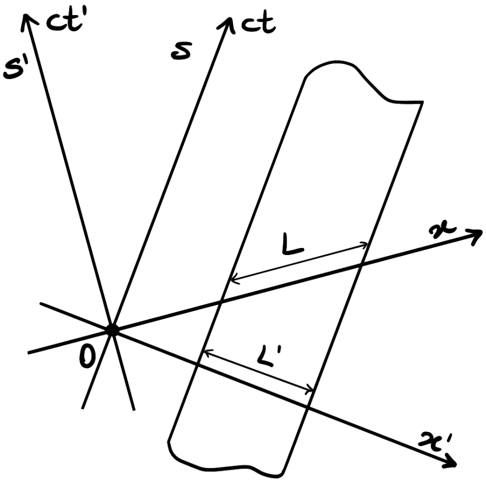


Figure 2.18: The spacetime diagram of the length contraction phenomenon: in $\mathcal{S}$ reference frame the length of the road is L while in $\mathcal{S}'$ reference frame the road is shorter (L').

This means that, if we consider the road as a three-dimensional object, all its parts are given *simultaneously* as they lay on the same temporal component ($ct = 0$): the road is all given at once, at only one moment of its history, namely at the present time of the observer in $\mathcal{S}$.

In the same way, when we consider an inertial reference frame $\mathcal{S}'$ moving with respect to $\mathcal{S}$, he also must determine the length of the road with respect to his own reference frame *at one moment* in $\mathcal{S}'$. All the parts of the road also in $\mathcal{S}'$ must exist *simultaneously*. As we see from Figure 2.18 the length L' of the road in $\mathcal{S}'$ is shorter than in $\mathcal{S}$: it is contracted.

Here it comes the trouble: we know that simultaneity is no longer preserved in Special Relativity, hence the observer in $\mathcal{S}'$ can not measure the length of the *same* road of observer $\mathcal{S}$. Being in uniformly relative motion, observers $\mathcal{S}$ and $\mathcal{S}'$ have *different* set of simultaneous events. As the road is an extended three-dimensional object, it is part of the three-dimensional plane of present but the plane of present for observer in $\mathcal{S}'$ is not the same three-dimensional plane of present for the observer in $\mathcal{S}$. This means that observer in $\mathcal{S}'$ is measuring a different road from the one of observer in $\mathcal{S}$.

This can be nonsense since the road is just one, if we consider it as a three-dimensional object. However in the view of a four-dimensional spacetime there is no contradiction: the road is a four-dimensional worldtube that entirely exists in spacetime. Its cross-sections with the three-dimensional plane of both $\mathcal{S}$- and $\mathcal{S}'$- observers' present are interpreted as the *two different* roads each observer measures in his own reference frame. The road is yet *only one* object but it is a four-dimensional entity, not a three-dimensional one.

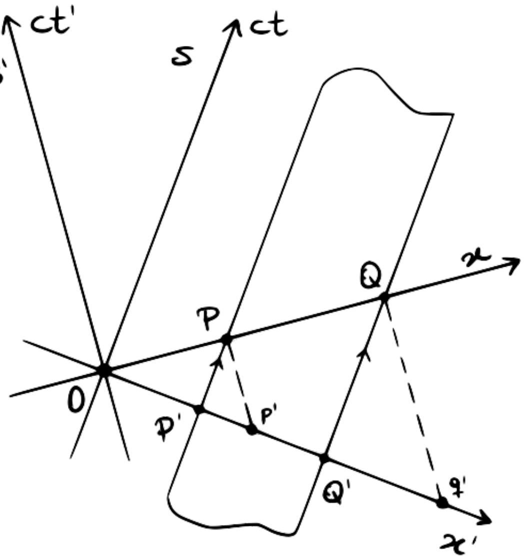


Figure 2.19: Lorentz transformation $\mathcal{S} \to \mathcal{S}'$ maps events P and Q of the road in the events p' and q' that are not the starting (P') and end (Q') point of the road with respect to $\mathcal{S}'$-observer. His three-dimensional cross-section of the road's worldtube is $P'Q'$.

Now we need to get the formal relation between the length of the road in $\mathcal{S}'$ and in $\mathcal{S}$. In this reference frame we can say that the road lays between event P and Q, having coordinate x_P and x_Q: the proper length is $L = x_Q - x_P$. Then if we use the Lorentz transformation $\mathcal{S} \to \mathcal{S}'$ to obtain the length of the road in $\mathcal{S}'$, we are going to determine the unknown coordinates x'_p and x'_q from the known coordinates x_P and x_Q. This will give

$$x'_q - x'_p = \frac{x_Q - x_P}{(1-\beta^2)^{1/2}}, \tag{2.59}$$

which is a dilated length, not a contracted one. Looking to Figure 2.19, we see that actually the cross-section between the road's worldtube and the plane of present of observer $\mathcal{S}'$ is in the point P' and Q', not in p' and q'. Lorentz transformation does not project points P and Q' onto points P' and Q' but onto p' and q'. Here the problem is what we are measuring: the $\mathcal{S}$-observer is looking to the three-dimensional cross-section PQ as his own road while the true three-dimensional cross-section of $\mathcal{S}'$-observer is $P'Q'$ not $p'q'$. We actually have to use the inverse Lorentz transformation $\mathcal{S}' \to \mathcal{S}$ since this one correctly maps $P' \to P$ and $Q' \to Q$ as we can see from Figure 2.19. It is an unusual approach to transform unknown coordinates x'_P and x'_Q into the known ones x_P and x_Q to obtain the

former: however it demonstrates that this transformation is linking two *different* three-dimensional roads, which is possible only in a four-dimensional world.
Then we obtain that

$$x_Q - x_P = \frac{x'_Q - x'_P}{(1-\beta^2)^{1/2}}, \tag{2.60}$$

hence, being $x_Q - x_P = L$ and $x'_Q - x'_P = L'$:

$$L' = L(1-\beta^2)^{1/2} = \sqrt{1 - \frac{v^2}{c^2}} L. \tag{2.61}$$

It can be shown that the same considerations held for the Euclidean space: if the segment OP represents in $\mathcal{K}$ the length L of a road, then the rotated one in the $\mathcal{K}'$ reference frame is long:

$$L' = \frac{L}{\cos\alpha} = L(1 + \tan^2\alpha)^{1/2}. \tag{2.62}$$

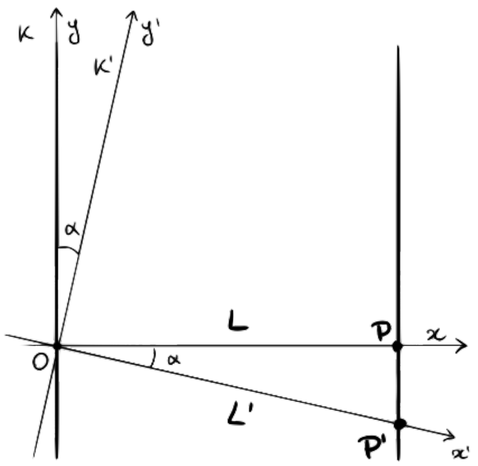


Figure 2.20: In Euclidean space the length "contraction" is a length dilation.

This relation shows that in the Euclidean space there is a length dilation, not a contraction being $L' > L$. Form this relation one can obtain the formula of the length contraction by making the substitution $\tan\alpha \to i\beta$. But more important, as for the case of the spacetime, the only correct transformation that yields to the expression (2.62) is the rotation $\mathcal{K}' \to \mathcal{K}$, not the rotation $\mathcal{K} \to \mathcal{K}'$.
In Appendix A.4 we reported the implication the phenomenon has over the three-dimensional objects we commonly deal with.

Kinematic in spacetime

When we consider the motion of particles (or extended three-dimensional objects), as they move with a speed smaller than light's one, they follow a time-like worldline. Their position on the worldline is given by the position four-vector $x^\mu = (ct, x, y, z)$. Then if ds is a small displacement along the time-like worldline, we know that in the inertial frame whose time axis lays along the worldline's direction[17] the interval is $ds^2 = c^2 dt^2$. As the time t is the time as measured along the worldline, it is the proper time of the particle. Being the interval an invariant quantity, we can say that for a real particle:

$$ds^2 = c^2 d\tau^2. \tag{2.63}$$

Then we can use proper time to parametrize a time-like worldline:

$$x^\mu = x^\mu(\tau) = (ct(\tau), x(\tau), y(\tau), z(\tau)). \tag{2.64}$$

Here the coordinate are considered with respect to a general inertial frame $\mathcal{S}'$ uniformly moving with a speed v relative to frame $\mathcal{S}$, whose time axis lay on the worldline of the particle. The general coordinate time t is linked to the proper time by $t = \gamma\tau$ (time dilation effect) so that if $\mathcal{S}'$ is at rest with respect to $\mathcal{S}$, the time coordinate is measuring the proper time (being $v = 0$, $\gamma = 0$ and $t = \tau$).
If now we have the position x^μ of a particle following a time-like worldline as a function of the proper time τ, we can introduce the four-velocity u^μ as the derivative of the four-dimensional position with respect to its the proper time:

Definition 13 (Four-velocity).

$$u^\mu = \frac{dx^\mu}{d\tau}. \tag{2.65}$$

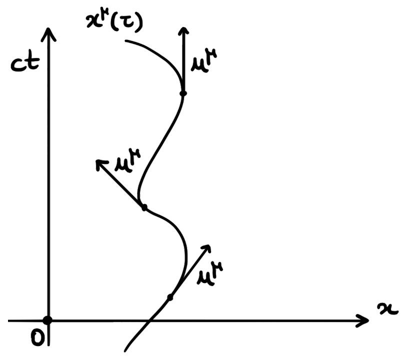


Figure 2.21: The four-velocity u^μ of a particle is the tangent vector to its worldline: the four-velocity's module is constant.

[17] Section 2.3.2 pg. 70.

Due to its definition as derivative of the four-position, the four-velocity is tangent to the worldline (Figure 2.21). Its components are:

- temporal component:

$$u^0 = \frac{dx^0}{d\tau} = \frac{cdt}{d\tau} = \frac{cdt}{dt}\frac{dt}{d\tau}. \tag{2.66}$$

  The relation between the coordinate t and the proper time is $t = \gamma\tau$ and $dt = \gamma d\tau$ (γ is not a changing quantity). Then:

$$u^0 = c\frac{\gamma d\tau}{d\tau} = \gamma c. \tag{2.67}$$

- spatial components:

$$u^i = \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} = v^i\gamma. \tag{2.68}$$

  where $v^i = dx^i/dt$ are the components of the Classical three-dimensional velocity $\boldsymbol{v}$ as measured with respect to the inertial reference frame in which it is defined.

Then the four-velocity is:

$$u^\mu = (\gamma c, \gamma\boldsymbol{v}) \tag{2.69}$$

where we use $\boldsymbol{v}$ meaning the three components $\boldsymbol{v} = (v^1, v^2, v^3) = (v^{(x)}, v^{(y)}, v^{(z)})$.
It is evident that the four-velocity can not be a zero vector: the spatial part of u^μ can be null if the ordinary velocity $\boldsymbol{v}$ is zero but the temporal component $u^0 = \gamma c$ is always different from zero.
Consider the spacetime diagram in Figure 2.22: a particle is at rest with respect to his own reference frame.

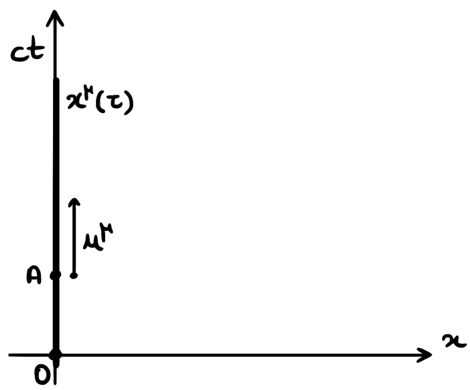


Figure 2.22: The worldline of a particle at rest is a vertical line, having only a temporal component. The four-velocity in event A has the direction shown by the arrow u^μ: it is not null and has only a temporal component $u^\mu = (c, 0)$.

Differently from the Classical Mechanics where a particle at rest has null trajectory, in

spacetime a particle at rest has still a worldline. It has only a temporal component and it is orthogonal to the three-dimensional space. In the same way, being u^μ its tangent vector, it can not be a null vector: in the reference frame at rest with respect to the particle, the four-velocity must have only temporal component in order to be tangent to a vertical line in the spacetime.
The module of the four-velocity is

$$u_\mu u^\mu = \eta_{\mu\nu} u^\mu u^\nu = \gamma^2(c^2 - \boldsymbol{v} \cdot \boldsymbol{v}) = \gamma^2(c^2 - v^2) = c^2, \tag{2.70}$$

that in the natural units $c = 1$ yields to $u_\mu u^\mu = 1$. The module of the four-velocity is constant and it does not depends upon the three-velocity as in Classical Mechanics. This four-dimensional velocity tells us that particles travel through spacetime always at the same rate. Moreover, being the $u_\mu u^\mu > 0$, the four-velocity is a time-like vector. Finally, being the four-velocity a ratio between the infinitesimal four-vector position dx^μ, which is a quadrivector, and the infinitesimal proper time $d\tau$, which is invariant, the four-velocity is a quadrivector.
From the four-velocity as in Classical Mechanics it can be defined the four-momentum p^μ of a particle whose mass, as measured with respect to its rest frame, is m (*rest mass*):

Definition 14 (Four-momentum)**.**

$$p^\mu = m u^\mu. \tag{2.71}$$

It easy to derive the component of the four-momentum, just multiplying the four-velocity by m:

$$p^\mu = (m\gamma c, m\gamma \boldsymbol{v}). \tag{2.72}$$

The module of the four-momentum is:

$$p_\mu p^\mu = m^2 u_\mu u^\mu = m^2 c^2 \tag{2.73}$$

Since $p_\mu p^\mu$ is invariant then the mass at rest m is itself invariant too. Then when a particle is moving according to a reference frame its mass does not change as this is an invariant physical quantity. Moreover the dependence of the four-momentum upon γ, which leads to the idea of a relativistic mass γm, arises from the relativistic three-velocity $\gamma \boldsymbol{v}$. Then the spatial component of the four-momentum is $m \cdot \gamma \boldsymbol{v}$ not $m\gamma \cdot \boldsymbol{v}$. If we admit the hypothesis of a relativistic mass, the spatial component of the four-velocity would be $u^i = v^i$, which is not consistent with Eq. (2.68) as it does not consider the derivation with respect to proper time and actually shows a pre-relativistic aspect (absolute time). From this consideration, being m an invariant quantity and u^μ a quadrivector, also p^μ is a quadrivector.
The definition of p^μ shows that the spatial component is nor then a relativistic definition

of classical momentum $\boldsymbol{p}_c = m\boldsymbol{v}$: $p^i = \gamma p_c^i = \gamma m v^i$.
For temporal component of p^μ, we can expand its expression for small velocities ($v \ll c$), obtaining that

$$p^0 = \frac{mc}{(1-\beta^2)^{1/2}} = mc\left(1 + \frac{1}{2}\frac{v^2}{c^2} + o(v^4/c^4)\right) \simeq$$
$$\simeq mc + \frac{1}{2}m\frac{v^2}{c} + \ldots = \frac{1}{c}\left(mc^2 + \frac{1}{2}mv^2 + \ldots\right). \tag{2.74}$$

We recognise the classical kinetic energy $(1/2)mv^2$ plus a new relativistic quantity $E_0 = mc^2$, the energy at rest as it is the only contribution we have when the particle is at rest (the energy in the particle's reference frame).
Then we can consider for small velocity the total amount of energy E of a body as:

$$E = mc^2 + \frac{1}{2}mv^2 + o(\beta^4). \tag{2.75}$$

From Eq. (2.75) we notice that the expansion of the energy formula at low velocity is not the classical energy $(1/2)mv^2$ because of the energy at rest. Indeed, regardless of the presence of a potential energy, Classical Mechanics does not calculate the *total* amount of energy but only the kinetic energy ([61]).
Comparing Eq. (2.75) with expression (2.74), we have that the temporal component of four-momentum is:

$$p^0 = \frac{E}{c}. \tag{2.76}$$

Since also $p^0 = \gamma mc$, we have that the expression for the total energy of a particle is:

$$E = \frac{mc^2}{(1-\beta^2)^{1/2}} \tag{2.77}$$

which in the particle's rest frame reduces to:

$$E\big|_{\beta=0} = E_0 = mc^2. \tag{2.78}$$

In terms of the relativistic energy we can write the expression of the four-momentum as:

$$p^\mu = (E/c, m\gamma\boldsymbol{v}). \tag{2.79}$$

Because of this definition, the four-momentum is also called *energy-momentum* vector.
The energy-momentum quadrivector's components show also the great difference between energy and mass: energy is only the temporal part of the four-vector while mass is its four-dimensional length. The energy of the particle is mc^2 only when it is observed in

its at rest frame ([61]): the relation between the energy and the mass is in general more complex and it depends on the speed of the particle that increases its kinetic energy according to Eq. (2.77).
Finally we can calculate explicitly the value of the module of the four-momentum:

$$p_\mu p^\mu = \eta_{\mu\nu} p^\mu p^\nu = (E^2/c^2 - \boldsymbol{p} \cdot \boldsymbol{p}) = \frac{1}{c^2}(E^2 - p^2 c^2). \tag{2.80}$$

But according to Eq. (2.73) the module of p^μ is $m^2 c^2$: then comparing this expression with the last equation we get the *relativistic dispersion relation*:

$$E^2 = m^2 c^4 + p^2 c^2, \tag{2.81}$$

where E is the particle's total energy (rest energy + kinetic energy), m is the particle's rest mass and p is the particle's relativistic momentum $m\gamma v$.

Chapter 3

The Special Relativity's project

3.1 On spacetime diagrams

In Section 2.3.3 we have seen that the spacetime diagrams of two bodies uniformly moving with respect to each other can be represented as shown in Figure 3.1:

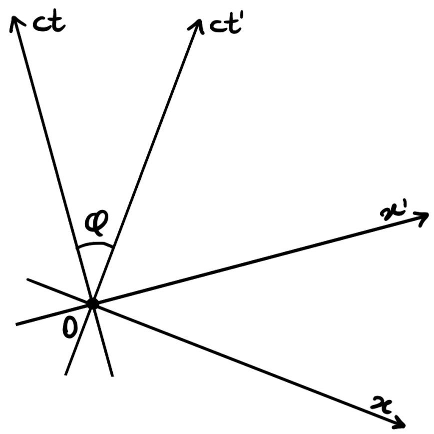


Figure 3.1: Spacetime diagrams of two observers in uniformly relative motion: the time axes are chosen along the observers' worldlines. The angle φ is so that the relative speed is $\beta = \tan\varphi$.

But how can we correctly scale the x'- and ct'-axis so that we can directly read from the new axes the coordinates of an event in another reference frame?
Let take a step backwards. Consider our two observers $\mathcal{S}$ and $\mathcal{S}'$ in uniformly relative motion along one direction, say x. Lorentz transformations link the spacetime coordinates between the two reference frames:

$$\mathcal{L} = \begin{cases} x' = \gamma\,(x - \beta ct) \\ ct' = \gamma\,(ct - \beta x) \end{cases} . \tag{3.1}$$

If we want to describe the x'- and ct'-axis in $\mathcal{S}$ observer's reference frame, then:

- The new temporal axis ct' is the locus of events having $x' = 0$; then from Eq. (3.1) we get that:
$$ct' \text{ axis} \Longleftrightarrow ct = \frac{1}{\beta}x \tag{3.2}$$
- The new spatial axis x' is the locus of events having $ct' = 0$; then from Eq. (3.1) we get that:
$$x' \text{ axis} \Longleftrightarrow ct = \beta x \tag{3.3}$$

Relations (3.2) and (3.3) describe the x'- and ct'- axis in the reference frame (x, ct). The two axes can be drawn in the same diagram (Figure 3.2): the angle φ between ct'-axis and ct-axis is the same as the one between x'-axis and x-axis and it is such that $\beta = \tan\varphi$. The angle δ between ct'-axis and x-axis is obviously

$$\delta = \frac{\pi}{2} - \varphi. \tag{3.4}$$

Thus we can depict both $\mathcal{S}$ and $\mathcal{S}'$ reference frame in one picture as shown in Figure 3.1.

Now we return to our problem of scaling x'- and ct'-axis. In his original paper of 1909 *Space and Time* ([99]), Minkowski looked to following equation:

$$c^2t^2 - x^2 - y^2 - z^2 = 1. \tag{3.5}$$

The left-hand side of Eq. (3.5) corresponds to the interval s^2, the distance of an event $P = (x, ct)$ from the origin O of the reference frame. Being the interval an invariant quantity under Lorentz transformations, Eq. (3.5) means that in all the reference frames the distance of event P from the origin is just 1.
Now in order to simplify the reasoning, we consider a bi-dimensional spacetime where the only spatial coordinate is x; then we have that:

$$c^2t^2 - x^2 = 1. \tag{3.6}$$

For instance we can consider the most suitable reference frame $\mathcal{S}$, the one in which event P has only the temporal coordinate $A = (0, 1)$: it can be regarded as a clock in $x = 0$ beating time with a tick of 1 second. The vector OA represents the unit along the ct-axis: the temporal unit.
The relation 3.6 can be represented in a spacetime diagram and it is nor than the hyperbola $c^2t^2 - x^2 = 1$ in the (x, ct) plane as shown in Figure 3.2:

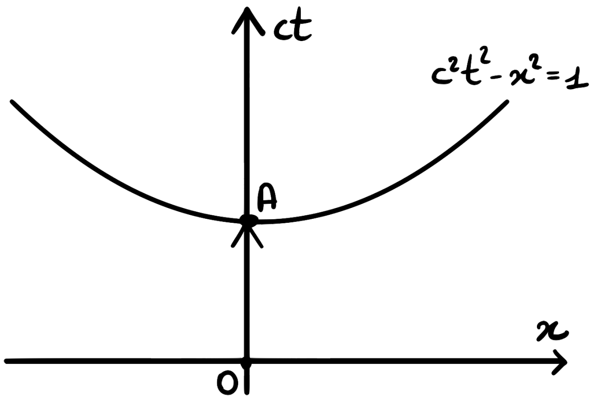


Figure 3.2: The relation $c^2t^2 - x^2 = 1$ is a hyperbola in the spacetime diagram. The event $A = (0, 1)$ and the vector OA is the temporal unit in $\mathcal{S}$ reference frame.

If we now consider an uniformly moving observer $\mathcal{S}'$ with respect to $\mathcal{S}$, we can determine A', the intersection between the hyperbola and the ct'-axis of Eq. (3.2). From the previous considerations and from Figure 3.3, its coordinate $A' = (ct_{A'}, x_{A'})$ are so that $x_{A'}/ct_{A'} = \tan\varphi = \beta$. Moreover it belongs to hyperbola $ct^2 - x^2 = 1$, thus we have to solve

$$\begin{cases} \beta = x_{A'}/ct_{A'} \\ c^2t_{A'}^2 - x_{A'}^2 = 1 \end{cases} . \tag{3.7}$$

It comes that $A' = (\gamma\beta, \gamma)$.

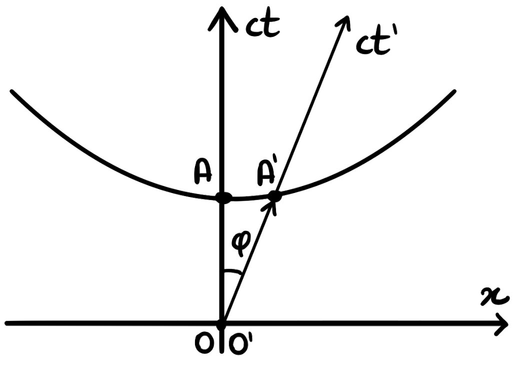


Figure 3.3: The calibration hyperbola $c^2t^2 - x^2 = 1$ allows to find the scale over ct'-axis. The event $A' = (\gamma\beta, \gamma)$ and the vector $O'A'$ is the temporal unit in $\mathcal{S}'$ reference frame.

Let us underline the physical meaning of these coordinates: in $\mathcal{S}'$ reference frame, event A' lays entirely along ct' axis. Indeed if we substitute the coordinates of event A' into Lorentz transformations (Eq. (3.1)), we get that $x'_{A'} = 0$ and $ct'_{A'} = 1$.

Its spacetime distance from the common origin $O' = O$ is 1 by construction:

$$s' = \sqrt{c^2 t_{A'}^2 - x_{A'}^2} = \sqrt{c^2 t'^{\,2}_{A'} - x'^{\,2}_{A'}} = 1. \tag{3.8}$$

Moreover, being in $\mathcal{S}'$ reference frame $x'_{A'} = 0$ and $ct'_{A'} = 1$, event A' can be regarded as the tick of a clock at rest with respect to $\mathcal{S}'$ since its time duration is 1 s: it is the new temporal scale unit in $\mathcal{S}'$ reference frame. Thus hyperbola 3.6 is a calibration guide that allows to correctly find the time scaling on ct'-axis.

This re-scaling has a natural interpretation: according to $\mathcal{S}$-observer, because of time dilation, the tick of at rest clock in $\mathcal{S}'$ lasts more than 1 s: it lasts $\gamma \cdot 1$ that is actually the time coordinate of event A'.

We can proceed in the same way to find the new length unit, looking to another hyperbola $c^2t^2 - x^2 = -1$ which represents an event P having four-dimensional distance $s^2 = -1$ from the origin. We choose as reference frame $\mathcal{S}$ the one in which P has the easiest expression: $B = (1, 0)$. The vector OB represents in $\mathcal{S}$-reference frame the length unit as shown in Figure 3.4.

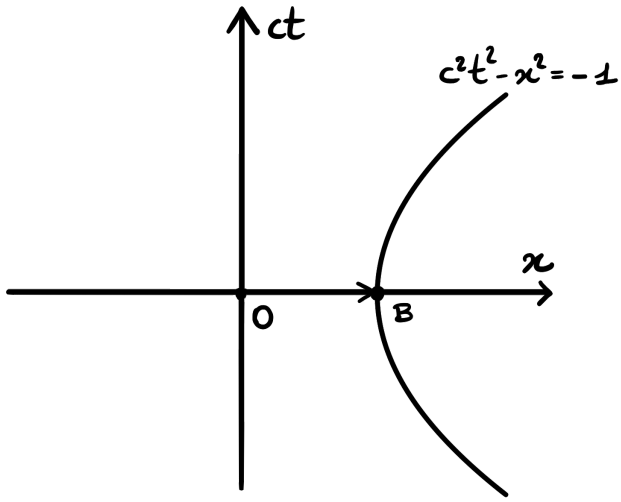


Figure 3.4: The event $B = (1, 0)$ and the vector OB is the length unit in $\mathcal{S}$ reference frame. The hyperbola has equation $c^2t^2 - x^2 = -1$.

Now once again we consider an uniformly moving reference frame $\mathcal{S}'$ and we draw the x'-axis: the intersection between the hyperbola and the x'-axis is the event $B' = (\gamma, \gamma\beta)$ (that still has $s^2 = -1$) and, substituting into Lorentz transformations in Eq. (3.1), we get that in $\mathcal{S}'$ reference frame the event B has coordinate $B' = (1, 0)$ (Figure 3.5).

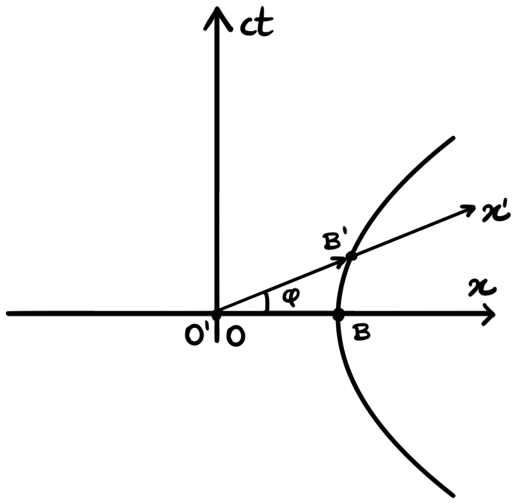


Figure 3.5: The calibration hyperbola $c^2t^2 - x^2 = -1$ allows to find the scale over x'-axis. The event $B' = (\gamma, \gamma\beta)$ and the vector $O'B'$ is the length unit in $\mathcal{S}'$ reference frame.

Thus the vector $O'B'$ is the length unit in $\mathcal{S}'$ reference frame, being an unitary vector: here the link with length contraction is not the same as previously between the temporal unit and the time dilation. The latter is directly a consequence of the rotation in spacetime that is a movement along this hyperbolas. The former, as we already explained in Section 2.3.3, can not be derived with a direct rotation in spacetime: this is the reason why according to $\mathcal{S}'$-observer $x_{B'}$ is not $1/\gamma$, as one could expect from length contraction, but instead $1 \cdot \gamma$.

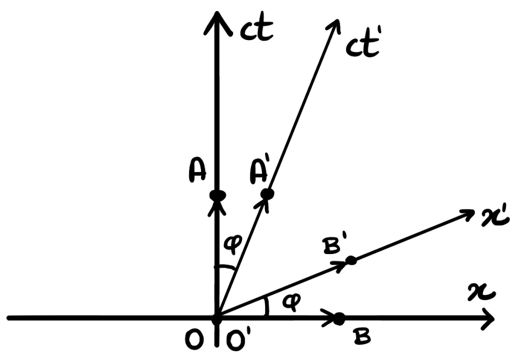


Figure 3.6: The vectors OA and OB are the units in $\mathcal{S}$-reference frame whilst $O'A'$ and $O'B'$ the units in $\mathcal{S}'$-reference frame.

Calibration hyperbolas $c^2t^2 - x^2 = \pm 1$ are important in spacetime as they represent a graphical and geometrical way to find the scale of the new axes and then to map events according to both the observers $\mathcal{S}$ and $\mathcal{S}'$. Whatever is the relative speed between $\mathcal{S}$ and $\mathcal{S}'$, the upper branch of hyperbola $c^2t^2 - x^2 = 1$ will give us the temporal unit, while the right branch of hyperbola $c^2t^2 - x^2 = -1$ the length unit.

3.1.1 Properties of hyperbolas

We know that the coordinates of an event in two different reference frames $\mathcal{S}$ and $\mathcal{S}'$ are linked by:

$$c^2t'^2 - x'^2 = c^2t^2 - x^2. \tag{3.9}$$

This relation shows the invariance of the interval under Lorentz transformations; it points out that the geometrical *locus* of a Lorentz transformation is a hyperbola of equation $c^2t^2 - x^2 = s^2$, with s a real number.

Proposition 1. *If $P = (x_P, ct_P)$ and $P' = (x'_P, ct'_P)$ are two events in spacetime linked by a Lorentz transformation, describing the same event according to two different observers $\mathcal{S}$ and $\mathcal{S}'$ in uniformly relative motion with speed $v = \beta c$, then P and P' lay over the hyperbola:*

$$c^2t^2 - x^2 = s^2, \tag{3.10}$$

where $s^2 = c^2t_P^2 - x_P^2 = c^2{t'_P}^2 - {x'_P}^2$.

Proof. Consider the spacetime diagram of an observer $\mathcal{S}$ and an event $P = (x_P, ct_P)$. From the equation of a rectangular hyperbola $c^2t^2 - x^2 = a^2$, we add the condition that the curve crosses P:

$$c^2t_P^2 - x_P^2 = a^2. \tag{3.11}$$

Then the equation of the hyperbola is:

$$c^2t^2 - x^2 = c^2t_P^2 - x_P^2. \tag{3.12}$$

Consider now an observer $\mathcal{S}'$ in uniformly relative motion with respect to $\mathcal{S}$ with speed $v = \beta c$. Then event P is mapped into event $P' = (x'_P, ct'_P)$ from a Lorentz transformation (3.1). We can now consider the rectangular hyperbola crossing P':

$$c^2t^2 - x^2 = c^2{t'_P}^2 - {x'_P}^2. \tag{3.13}$$

Now looking to the right-hand side of Eq. (3.12) and Eq. (3.13), we know that

$$c^2t_P^2 - x_P^2 = c^2{t'_P}^2 - {x'_P}^2 \tag{3.14}$$

as the quantity $c^2t_P^2 - x_P^2$ represents the four-dimensional distance of event P from O which is equal to the four-dimensional distance $c^2{t'_P}^2 - {x'_P}^2$ of event P' from O' (coinciding with O).

Thus hyperbolas (3.12) and (3.13) are the same hyperbola and then P and P' lay on the same curve of equation

$$c^2t^2 - x^2 = s^2, \tag{3.15}$$

where $s^2 = c^2t_P^2 - x_P^2 = c^2{t'_P}^2 - {x'_P}^2$. □

If now we consider two different values of β, β_1 and β_2 and the two different images of P, $P'_1 = (x'_1, ct'_1)$ and $P'_2 = (x'_2, ct'_2)$, we can apply Preposition 1 to each pair of events P, P'_1 and P, P'_2. We get that as they are the Lorentz-transformed of the *same* event P, both P'_1 and P'_2 lay on the same hyperbola, namely the one crossing P.

Corollary 1. *Given an event $P = (x_P, ct_P)$ in the spacetime diagram of an observer $\mathcal{S}$, all the events on the hyperbola of equation*

$$c^2t^2 - x^2 = s^2 \tag{3.16}$$

where $s^2 = c^2t_P^2 - x_P^2$ are the Lorentz-transformed P' of the event P according to a particular value of β.

We can thus image that in $\mathcal{S}$ reference frame all the events P of the spacetime are crossed by a rectangular hyperbola defining the locus of all the possible events P' that are the Lorentz-transformed of event P. Any event P' we consider that is the Lorentz-transformed of events P for some value of the parameter β lays along the hyperbola crossing P.
By the end of this Section we want to point out a physical condition over the Lorentz transformation we are dealing with. Two generic events laying on the same hyperbolas are always connected by a Lorentz transformation. But it is not true in general that a transformation leaving two events on the same hyperbola has the form of Eq. (3.1).
Indeed ([100]) the most complete set of a Lorentz transformation is defined by:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu} \tag{3.17}$$

which is called *inhomogeneous Lorentz group* or *Poincaré group*. The subset with $a^{\mu} = 0$ is the *homogeneous Lorentz group*: $x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}$.
A whatever transformation of Poincaré group leaves the interval square ds^2 invariant since, differentiating Eq. (3.17), being both $\Lambda^{\mu}{}_{\nu}$ and a^{μ} constant, we have that $dx'^{\mu} = \Lambda^{\mu}{}_{\nu}dx^{\nu}$. Then the infinitesimal displacement ds^2 between two quadrivectors and between the transformed quadrivectors under (3.17) have the same expression as if they are linked by a transformation of the homogeneous Lorentz group. As a consequence the surface obtained as the geometrical locus of transformation (3.17) is still a hyperbola and then the transformation connects two events laying over the same hyperbola. But however it is no true that this transformation is described by Eq. (3.1).

So we are looking for the only transformations that have the physical meaning of a change between reference frames. These transformations are called *boost* and the condition under that a general Lorentz transformation corresponds to a boost is that the transformation's matrix Λ belong to the *proper orthochronous* Lorentz's group, i.e. it respects the conditions $\det \Lambda = 1$ and $\Lambda^0{}_0 > 0$.
We will always refer to this kind of transformation even though not explicitly expressed.

3.1.2 Rotating worldlines in spacetime diagrams

Consider now an observer $\mathcal{S}$ at rest[1]: according to him, he will see himself as stationary. Observer $\mathcal{S}$ can build his own reference frame: the most suitable is the one in which he is at center, in $x = 0$. As a consequence, his worldline is a vertical straight line with equation $x = 0$ (as shown in Figure 3.7).

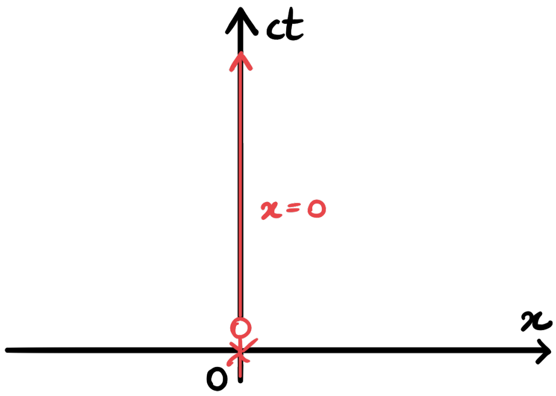


Figure 3.7: The worldline of an observer at rest in his own reference frame is a vertical straight line of equation $x = 0$.

Now consider a second observer $\mathcal{S}'$ (the blue cat in Figure 3.8) in uniformly relative motion with respect to observer $\mathcal{S}$ with speed $v = \beta c$. In $\mathcal{S}$-observer's reference frame the worldline of the moving observer $\mathcal{S}'$ has equation

$$ct = \frac{x}{\beta} \tag{3.18}$$

and it is represented by an inclined line with slope $1/\beta$.

[1]Let's say with respect to the Earth.

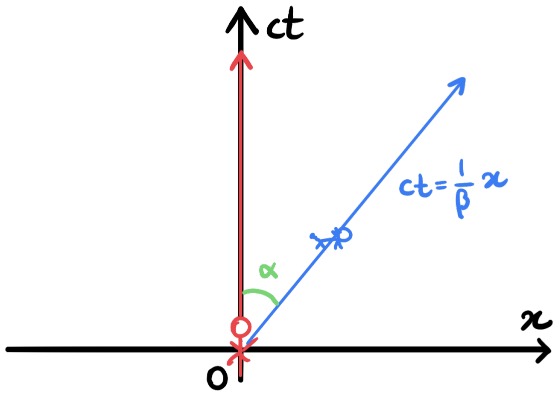


Figure 3.8: The worldline of an uniformly moving observer with respect observer $\mathcal{S}$ with speed $v = \beta c$ is an inclined straight line (blue line) of equation $ct = x/\beta$ in $\mathcal{S}$ reference frame.

Then we are able to describe both the observers $\mathcal{S}$ and $\mathcal{S}'$ in $\mathcal{S}$ reference frame through their worldlines: here we use the "common" meaning of *reference frame* as *point of view*. To be in $\mathcal{S}$ reference frame means to adopt $\mathcal{S}$-observer's point of view: indeed from the perspective of observer $\mathcal{S}$, he is at rest and he sees the cat going forward away from him at speed v. Thus Figure 3.8 is depicting spacetime from the point of view of observer $\mathcal{S}$.
We can however ask which the point of view of observer $\mathcal{S}'$ is. At this purpose, we can build a diagram very similar to the one of Figure 3.8. According to the principle of relativity, from the point of view of observer $\mathcal{S}'$ we know that he sees himself as stationary while he sees observer $\mathcal{S}$ going away from him with the same speed but in the opposite direction ($\boldsymbol{v} = -v\hat{v}$), namely backwards. As a consequence, in $\mathcal{S}'$-observer's reference frame, his worldline has equation $x' = 0$ while $\mathcal{S}$-observer's one $ct' = -x'/\beta$.
Then if we represent $\mathcal{S}'$-observer's point of view, we would have as in Figure 3.9:

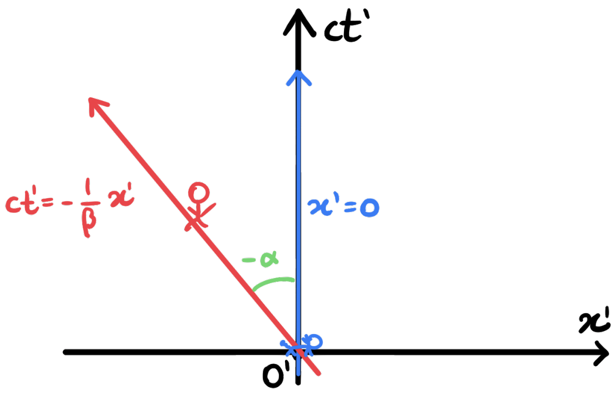
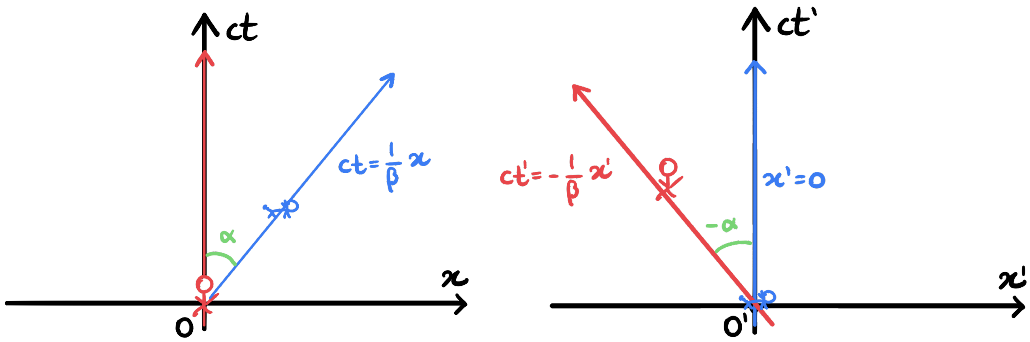


Figure 3.9: $\mathcal{S}'$-observer's point of view: it is at rest (blue worldline with equation $x' = 0$) while the observer $\mathcal{S}$ is moving backwards with respect to it (red worldline with equation) $ct' = -x/\beta$.

Figure 3.9 well represents $\mathcal{S}'$-observer's perspective which means $\mathcal{S}'$-observer's reference

frame in which it (the blue cat) is at rest while $\mathcal{S}$-observer (red man) is moving backwards as his position x' increases in module but in the negative semi-axis.
Then if we gather together Figures 3.8 and 3.9 as in Figure 3.10 we can understand this shift from $\mathcal{S}$-observer's perspective (Figure 3.10a) to $\mathcal{S}'$-observer's perspective (Figure 3.10b):



(a) $\mathcal{S}$-observer's point of view (b) $\mathcal{S}$-observer's point of view

Figure 3.10: Transition from $\mathcal{S}$-observer's perspective (3.10a) to $\mathcal{S}'$-observer's perspective (3.10b).

Comparing the two Figures, we see that the change in the point of view of the two observers is obtained just "rotating" the worldlines of the two observers. But we have to explain how this "rotation" is performed.
First we have to require that the angle between the two worldlines keeps constant during the rotation. Indeed the angle α is such that $\tan\alpha = \beta$: this requirement means that the relative speed between the two observers does not change while we are changing the reference frame. In this way, once the rotation is completed, $\mathcal{S}'$-observer will see $\mathcal{S}$-observer still moving with the same speed $\mathcal{S}$-observer sees $\mathcal{S}'$-observer moving, according to the principle of relativity. We can say that the requirement of the invariance of relative angle between the two observers is a consequence of the relativity principle.
Now we remind that a rotation in spacetime is actually a Lorentz transformation (Section 2.3.2): this is a right conclusion as physically our rotation of worldlines is describing a change between reference frames that is given by a Lorentz transformation. This helps us to understand the movements of the events in the spacetime diagrams when this rotation is performed.
Consider Minkowski spacetime $\mathcal{M}$ with the complex formalism where an event P has coordinate $P = (ix_P, ct_P)$ according to an observer $\mathcal{S}$ at rest with respect it. The angle in the spacetime between the segment OP and ct-axis is δ such that $\tan\delta = ix_P/ct_P$ as shown in Figure 3.11.

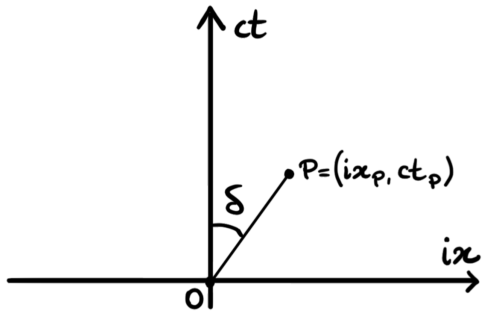


Figure 3.11: The angle δ is the one formed between the segment joining together the origin O with an event $P = (ix_P, ct_P)$ and the ct-axis.

Now consider another observer $\mathcal{S}'$ in uniformly motion with respect to observer $\mathcal{S}$ with relative speed $\beta = vc$. Then applying a Lorentz transformation, in $\mathcal{S}'$-observer's reference frame the coordinates of event P are $P' = (ix'_P, ct'_P) = (i\gamma(x_P - \beta ct_P), \gamma(ct_P - \beta x_P))$ and the new angle δ' between segment $O'P'$ and ct'-axis is such that:

$$\tan\delta' = \frac{i\gamma(x_P - \beta ct_P)}{\gamma(ct_P - \beta x_P)} = i\frac{1 - \beta ct_P/x_P}{ct_P/x_P - \beta} = i\frac{1 - i\beta/\tan\delta}{i/\tan\delta - \beta} \tag{3.19}$$

Remembering that the angle α between $\mathcal{S}'$-observer's worldline and ct'-axis is such that $\tan\alpha = i\beta$, then:

$$\begin{aligned}\tan\delta' &= i\frac{1 - i\beta/\tan\delta}{i/\tan\delta - \beta} = i\frac{1 - \tan\alpha/\tan\delta}{i/\tan\delta - \tan\alpha/i} = \\ &= i\frac{\tan\delta - \tan\alpha}{\tan\delta} \cdot \frac{i\tan\delta}{i^2 - \tan\alpha\tan\delta} = \\ &= -\frac{\tan\delta - \tan\alpha}{-1 - \tan\alpha\tan\delta} = \frac{\tan\delta - \tan\alpha}{1 + \tan\alpha\tan\delta} = \tan(\delta - \alpha)\end{aligned} \tag{3.20}$$

Neglecting the periodicity, we have:

$$\delta' = \delta - \alpha \tag{3.21}$$

This relation tells us how to perform the rotation: starting from Figure 3.11, if we want to draw the reference frame of the moving observer, events in spacetime rotate until the new angle δ' between the segment $O'P'$ and the ct'-axis is equal to $\delta - \alpha$, that is the difference between the angle δ the segment OP forms with ct-axis in $\mathcal{S}$-observer's reference frame and the angle α, defined by the relative speed between the observers $\mathcal{S}$ and $\mathcal{S}'$.
As a consequence, consider the two situations in Figure 3.10:

- according to its physical meaning, changing the reference frames from $\mathcal{S}$-observer's one to $\mathcal{S}'$-observer's one, the inclination of $\mathcal{S}$-observer's worldline goes from $\delta = 0°$

(in Figure 3.10a the worldline is vertical) to $\delta' = -\alpha$ ($\mathcal{S}$-observer is moving in Figure 3.10b). This agrees with Eq. (3.21) as from $\delta = 0°$ we get that $\delta' = \delta - \alpha = 0 - \alpha = -\alpha$;

- in the same way, the inclination of $\mathcal{S}'$-observer's worldline goes from $\delta = \alpha$ (Figure 3.10a) to $\delta' = 0$ ($\mathcal{S}'$-observer's worldline is vertical in Figure 3.10b). This agrees with Eq. (3.21) as from $\delta = \alpha$ we get that $\delta' = \delta - \alpha = \alpha - \alpha = 0$.

We need finally to point out another important feature of this rotation, namely the curves events have to follow. In Euclidean space, when a 2D-rotation is performed, geometrical entities rotate around a central point by a certain angle. Each point track an arc of a circumference whose length depends on the amount of the angle of rotation. This is not true for Minkowski spacetime which is pseudo-Euclidean. We have seen that two or more events in spacetime that are linked by a Lorentz transformation lay on the same hyperbola. Thus P', the Lorentz-transformed of event P with a relative speed $\bar{v} = \bar{\beta}c$, lays on the hyperbola $c^2t^2 - x^2 = ct_P^2 - x_P^2$, where also P lays on.
Consider now a very large succession of n events P_i that are the Lorentz-transformed of event P with a relative speed β_i such that $0 < \beta_i < \bar{\beta}$: as a boundary conditions we have that for $\beta = 0$, $P_0 = P$ (there is no transformation at all) while for $\beta = \bar{\beta}$, $P_n = P'$, the Lorentz-transformed of P which we are interested in. For Corollary 1 we know that all these events lay on the same rectangular hyperbola that crosses P.

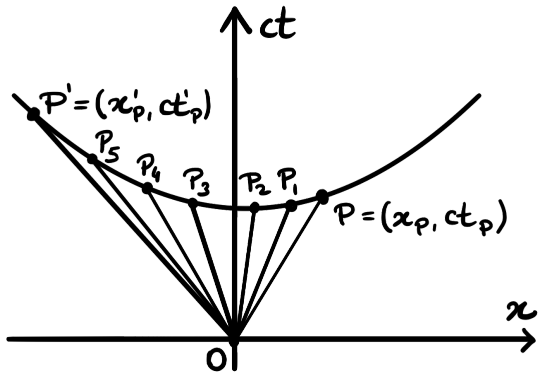


Figure 3.12: Between an event P and its Lorentz-transformed P' according to a relative speed $\bar{\beta}$, there are infinite other events P_i that are Lorentz-transformed of event P with a parameter $\beta < \bar{\beta}$, all laying on the same hyperbola.

Then between P and P' there are a number n of events as large as desired all laying on the same hyperbola linking P and P' as shown in Figure 3.12. This means that we can think of the Lorentz transformation $P \to P'$ as constituting of $n \to \infty$ steps of Lorentz transformations $P \to P_1 \to P_2 \to \ldots \to P'$. In the first step the segment OP is inclined with respect to ct-axis by an angle δ, in the second step the inclination of OP_1 is δ_1' determined by the value β_1, in the third is δ_2' and so on until in the last step the inclination of OP'

reaches the desired value of $\delta' = \delta - \alpha$. As a result we have that event P is transformed into event P', the segment OP is rotated from angle δ to angle $\delta' = \delta - \alpha$ and during this rotation event P follows the hyperbolic path defined by the hyperbola it belongs to.

Proposition 2. *If $P = (x_P, ct_P)$ and $P' = (x'_P, ct'_P)$ are two events in spacetime linked by a Lorentz transformation, describing the same event according to two different observers $\mathcal{S}$ and $\mathcal{S}'$ in uniformly relative motion with speed $v = \beta c$, then P' is obtained moving P along the hyperbola of equation $c^2t^2 - x^2 = ct_P^2 - x_P^2$ until the angle δ' between the segment OP' and the vertical axis is*

$$\delta' = \delta - \alpha, \tag{3.22}$$

begin δ the angle between the segment OP and the vertical axis and $\tan\alpha = \beta$.

In this way we have a geometrical way to draw a spacetime diagram like that ones in Figure 3.10, following two rules.

Rules of a spacetime diagram:

1. when the frame of reference of an observer $\mathcal{S}$ is adopted, his worldline is $x = 0$, a straight line in the origin along the ct-axis;
2. when the previous frame of reference is changed into the one of an observer $\mathcal{S}'$ uniformly moving with speed $v = \beta c$, all the events[2] are rotated by a total angle of $\delta' + \delta = 2\delta - \alpha$ along the rectangular hyperbola crossing them.

3.2 The spacetime globe

Minkowski diagrams are very powerful but not immediate to draw. Even the different technique shown in Section 3.1.2 has many difficulties. Thus we looked for an easier way to employ them within a didactic aim.
On the YouTube channel *MinutePhysics* ([101]) we found an interesting instrument called *spacetime globe* by the guy who built it. An image of the device taken from one of his videos is shown in Figure 3.13.

[2] Worldlines too that are a set of events.

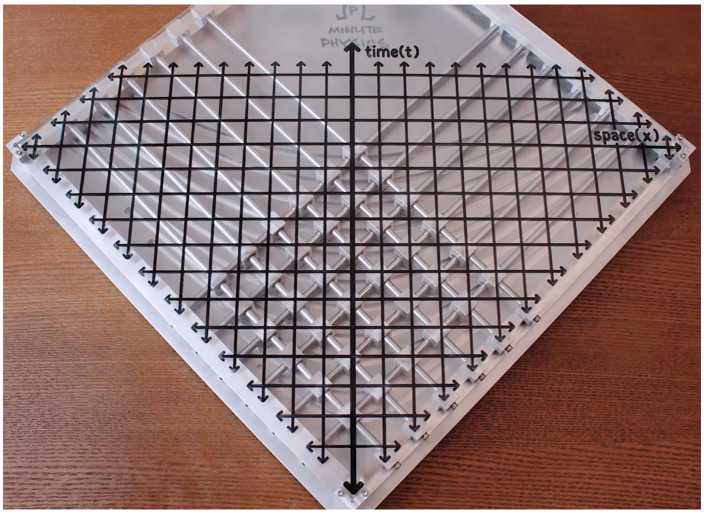


Figure 3.13: A snapshot from the video *Relativity of Simultaneity | Special Relativity Ch. 4* showing the original spacetime globe of MinutePhysics's channel.

Since there were not available any instructions allowing us to reproduce it, we begun drawing an AutoCAD project and after some months it was ready to be built. Thanks to the work of the mechanical laboratory of the Department of Mathematics and Physics of Roma Tre University, we managed to reproduce it and our version of the spacetime globe is shown in Figure 3.14.

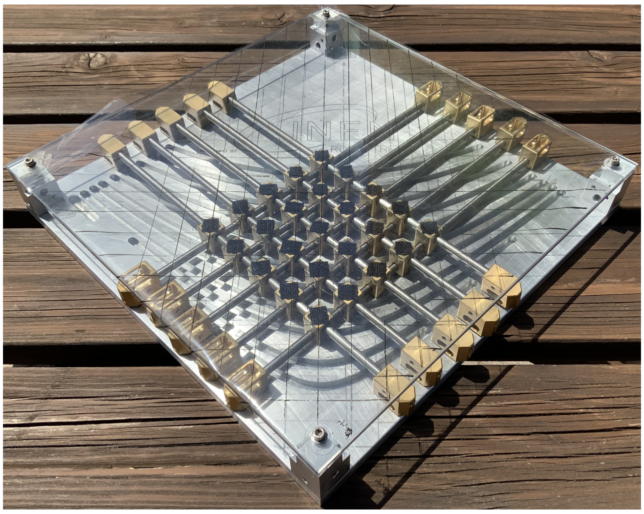
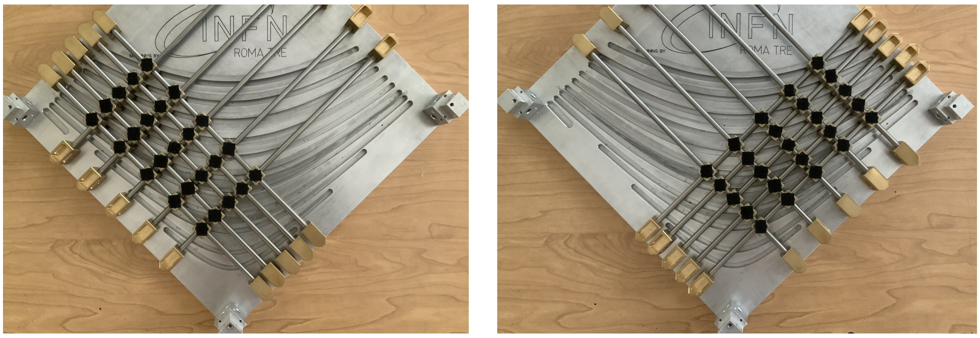


Figure 3.14: The spacetime globe as reproduced by the mechanical laboratory of the Department of Mathematics and Physics of Roma Tre University.

The instrument is made up of a large aluminium squared base of side 35 cm and height 1 cm with four small blocks of height 5 cm sustaining a Plexiglas sheet with a grid printed. Over the base 11 hyperbolas-shaped tracks are engraved, allowing 15 dice to slide inside them while the other 10 dice are forced to follow the global movement thanks to ten circular bars, five perpendiculars to the other five. The dice can be moved acting manually on the handles at the far end of the bars: the possible movements are included between the outer "squeezing" in the left (Figure 3.15a) and in the right (Figure 3.15b) side of the base.



(a) Pushing the dice along the guides until it is flush in the left side.

(b) Pushing the dice along the guides until it is flush in the right side.

Figure 3.15: The possible movements of the dice: from the left side of the base to the right one.

The spacetime globe is a large Minkowski spacetime diagram that allows to represent a bi-dimensional spacetime and in particular to study the change in reference frames and thus Lorentz transformations. Each die represents an event that took place at time t in the x position: the coordinates can be easily identified from the upper grid printed over the Plexiglas square.

To identify both the temporal and spatial coordinate and then to perform physical measure, we built a paper ruler with such a division that the vertical distance between one die and the upper one in Figure 3.14 is 1. Then the ruler has been divided into four part and each one of them in half. The result is a ruler with a instrumental sensitivity of ± 0.0625. As this device was used for both temporal and spatial measure, its unit measure depends on the quantity we are measuring. We choose to assign to the direct measures taken with this ruler an uncertainty of ± 0.02, namely the instrumental sensitivity divided by the square root of 12.

Now we are going to explain how the spacetime globe can be used to describe Special Relativity.

3.2.1 Changing reference frames

We start from the draw in Figure 3.8 of Section 3.1.2: an observer $\mathcal{S}$ (red man) is at rest with respect his own reference frame, seeing an other observer $\mathcal{S}'$ (blue cat) moving with respect to him with speed $v = c/3$. We add also a box 0.5 meters distant from observer $\mathcal{S}$

that after 4.75 seconds catches fire. Using natural units, we can represent the box's event in Figure 3.16 as the event in spacetime $A = (x_A, t_A) = (0.5, 4.75)$:

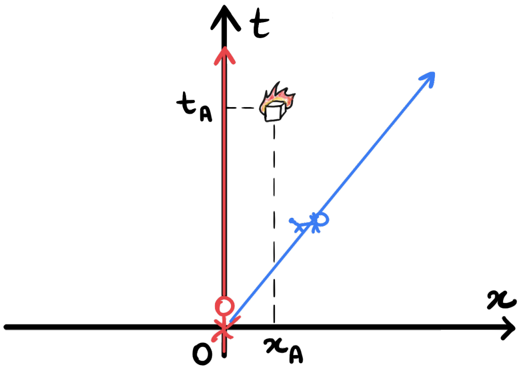


Figure 3.16: Spacetime diagram of an observer $\mathcal{S}$ (red man) in his reference frame with an observer $\mathcal{S}'$ (blue cat) moving with respect to him and a box $x_A = 0.5$ meters distant from him catching fire after $t_A = 4.75$ s.

The same situation represented with our spacetime globe[3] is shown in Figure 3.17.

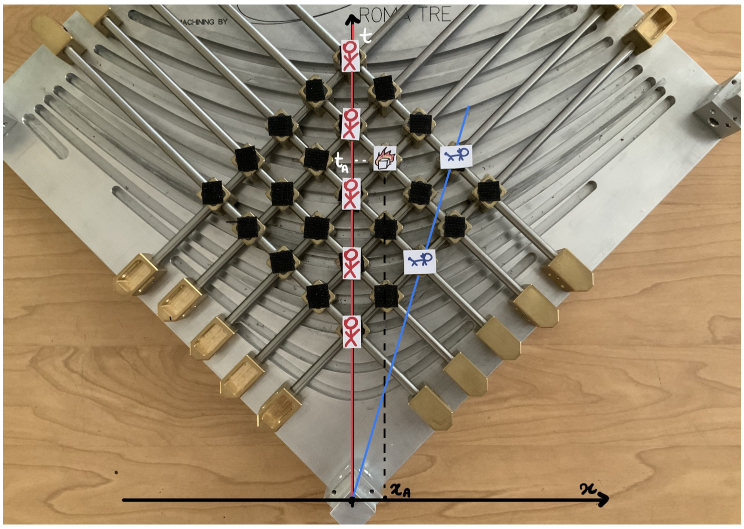


Figure 3.17: Representation of the situation of Figure 3.16 with the spacetime globe. The red man is standing for the observer $\mathcal{S}$ at rest while the blue cat for the moving observer $\mathcal{S}'$. At time $t_A = 4.75$ s the box in position $x_A = 0.5$ m catches fire.

As it is not possible to draw a continuous line on the spacetime globe, each worldline undergoes to a discretization. In Figure 3.17 the set R_i of figures of a red man represents the worldline of the observer at rest $\mathcal{S}$. In the same way, the set B_i of figures of a blue cat represents the worldline of the moving observer $\mathcal{S}'$. In Figure 3.17 we also added

[3]Hence on, to have a better photos of the instrument, we preferred to remove the Plexiglas sheet. As a consequence a Cartesian frame was added graphically reproducing the position of the grid.

graphically a red and a blue line respectively connecting the set of figures R_i and B_i in order to help the visualisation of worldlines: we should image to have an infinite figures of a red man and of a blue cat all over their worldline. Finally a figure of a box catching fire is stuck over a die: its coordinates are $A = (0.5, 4.75)$ measured[4] from the origin O.
Figure 3.17 is showing the red man's perspective: his worldline is a vertical line in $x = 0$, thus laying over the time axis. Then according to him at time $t = 4.75$ s. a burning box appears instantaneously and then disappears. Finally the red man is seeing the cat moving with speed $v = c/3$, thus its worldline is $t = (3/c)\, x$.
Now we want to change the reference frame, adopting $\mathcal{S}'$-observer's one: formally we can write down Lorentz transformations of the two worldlines and of event A obtaining that:

- for observer $\mathcal{S}$ the worldline from $x = 0$ becomes $t' = -(3/c)\, x'$;
- for observer $\mathcal{S}'$ the worldline from $t = (3/c)\, x$ becomes $x' = 0$;
- event A goes from $A = (0.5, 4.75)$ to $A' \simeq (-1.15, 4.86)$.

Then we can draw a diagram like the one in Figure 3.9.
Instead, we can use the mechanical properties of the spacetime globe to visualise the change of reference frame. Indeed we can act over the bars of the instrument, moving the dice until we reach the configuration shown in Figure 3.18.

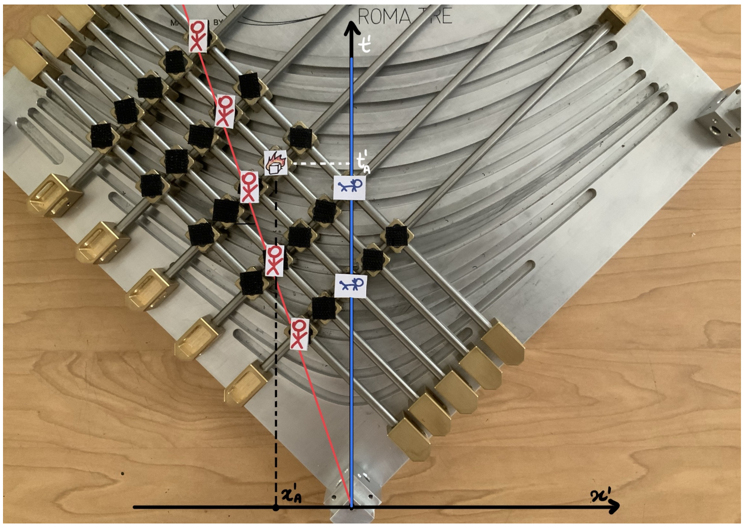


Figure 3.18: Lorentz transformation of the situation in Figure 3.17. From the blue cat's point of view, the red man is moving away from it while event A happens at time $t'_A = 4.875$ s in position $x'_A = -1.125$ m.

In Figure 3.18 we see the result of the change of reference frame. We are now into $\mathcal{S}'$-observer's reference frame: the blue cat is stationary (vertical worldline) in $x' = 0$. Its

[4]We have implicitly assumed a scale for both the axes according to which event A has these coordinates.

worldline is $x' = 0$.
Then according to the principle of relativity, the cat sees $\mathcal{S}$-observer (the red man) moving away from it with the same speed[5] but in the opposite direction. His worldline is $t' = -(3/c)\,x'$: we can measure the angle between the red line and the blue one, checking that it has the same width of the angle between the two worldlines in Figure 3.17.
In the same way we can see that event A is transformed and measuring its coordinates on the spacetime globe with the grid and our ruler we find that $A' = (-1.13, 4.88)$ with a given uncertainty of $\pm\,0.02$ (for both the measure of time and distance). The correct values obtained from Lorentz transformations are $A' \simeq (-1.15 \pm 0.02, 4.86 \pm 0.02)$.
Thus with an easy action on the bars of the instruments we can change the reference frame from one observer to another, getting the same qualitative and quantitative results obtained with the application of Lorentz transformations.
In Appendix B.1 we applied the theorems found in Section 3.1.2 directly to the spacetime globe showing that the movement of the dice corresponds to a real Lorentz transformation.

3.3 Special Relativity with a spacetime globe

In the previous section, we have shown that spacetime globe can be used to investigate the change of reference frame. From a phenomenological point of view many consequences arise when the perspective is shifted from one observer to another one (loss of simultaneity, time dilation etc).
We can use spacetime globe to describe both in a qualitative and quantitative way the wide phenomenology of Special Relativity. In particular we are going to show how we can be describe with spacetime globe the following phenomena: the loss of simultaneity, the time dilation, the length contraction, the addition of velocities, the light speed invariance, the Doppler effect and the mass invariance.
For each situation, we give the physical description together with the representation with the spacetime globe.
We point out an important clarification about the unit of measure: along horizontal axis the position in space is represented through the x-coordinate measured in metres. Along the vertical axis we are going to represent time t of an event which is expressed in seconds/c. However we are going to suppress the c-factor and assign to the temporal coordinate the second unit of measure. On the contrary speed are expressed explicitly in terms of c.

[5]The speed it has with respect to the red man.

3.3.1 Loss of simultaneity

Consider an observer $\mathcal{S}$ who is at rest with respect to the Earth. Near him, at his right and at his left, x_B metres distant, there are also two boxes that at a certain time t_f catch fire and then at time t_a that become ash. There is also another observer $\mathcal{S}'$ who is moving away from observer $\mathcal{S}$ with constant speed $v = c/3$ with respect to him.
In spacetime, observer $\mathcal{S}$ is represented by a straight worldline of equation $x = 0$ and in the same way the box at $\mathcal{S}$-observer's right and left has equation $x = x_B$ and $x = -x_B$, where x_B is the spatial coordinate identifying the position of the two boxes with respect to observer $\mathcal{S}$. Observer $\mathcal{S}'$ indeed has as worldline an inclined line of equation $t = (3/c)x$. In Figure 3.19 we used the spacetime globe to depict this situation.

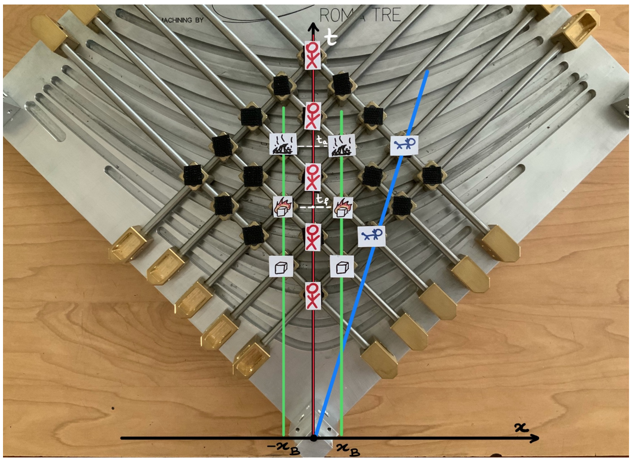


Figure 3.19: According to the red man the two boxes at rest with respect to him catch fire simultaneously.

The two green straight lines starting from x coordinate x_B and $-x_B$ with the figures of the boxes represent the worldline of the boxes, which are at rest with respect to the red man. In particular it is shown that at t_f the boxes catch fire and, after having burned, at subsequent time t_a they become ash. These phenomena happen simultaneously according to the red observer. For instance the two events R "the box on the man's right catches fire" and L "the box on the man's left catches fire" happen at *same time* t_f with respect to the red man, sharing the two images of the boxes the same temporal coordinate[6].
Now we want to understand what happens to these two simultaneous events when we change the reference frame, going into the moving observer's one. We just need to move the dice until cat's worldline becomes a vertical straight line. The result is shown in Fig. 3.20.

[6]We take as landmark the corner of the die.

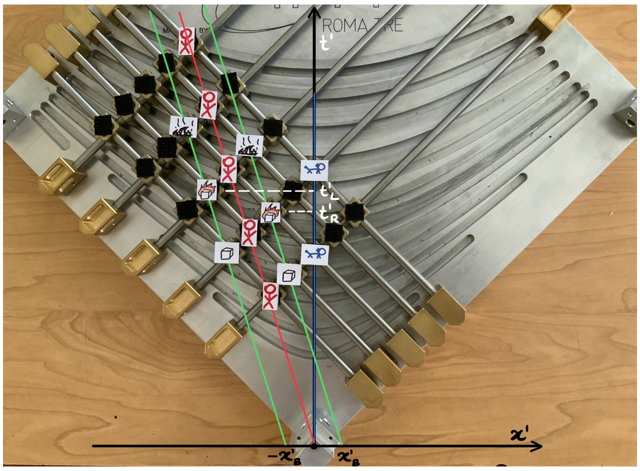


Figure 3.20: Having changed the reference frame, according to the moving blue cat the two boxes do not catch fire at the same time.

According to the principle of relativity we see that the red observer is moving with respect to the blue cat at same speed v but in the opposite direction. Indeed $\mathcal{S}$-observer's worldline in Figure 3.20 has the same slope of $\mathcal{S}'$-observer's one in Figure 3.19. Moreover the cat sees also the two boxes moving with respect to itself at speed v, together with the red man.
If we now look to the transformed of L and R events we see that according to the blue cat (the moving observer) they are no longer simultaneous. Indeed from Figure 3.20 we see that the box at the red man's right catches fire at time $t' = t'_R$, before the one at his left that burns at $t' = t'_L$.
In this simple way, we can show that events simultaneous according to an observer are no more simultaneous in another moving reference frame.

3.3.2 Time dilation

Consider an observer $\mathcal{S}$ who is at rest with respect to the Earth: the red man. He has a clock with him (hence on the red clock) beating the time in $\mathcal{S}$-observer's reference frame. There is also another observer $\mathcal{S}'$, the blue man, who is moving away from observer $\mathcal{S}$ with constant speed $v = c/3$ with respect to him aboard a spaceship. He has a clock beating $\mathcal{S}'$-observer's time: the blue clock. We can say that both the red and the blue clock are beating the proper time of each observer.
In Figure 3.21 we used the spacetime globe to depict this situation.

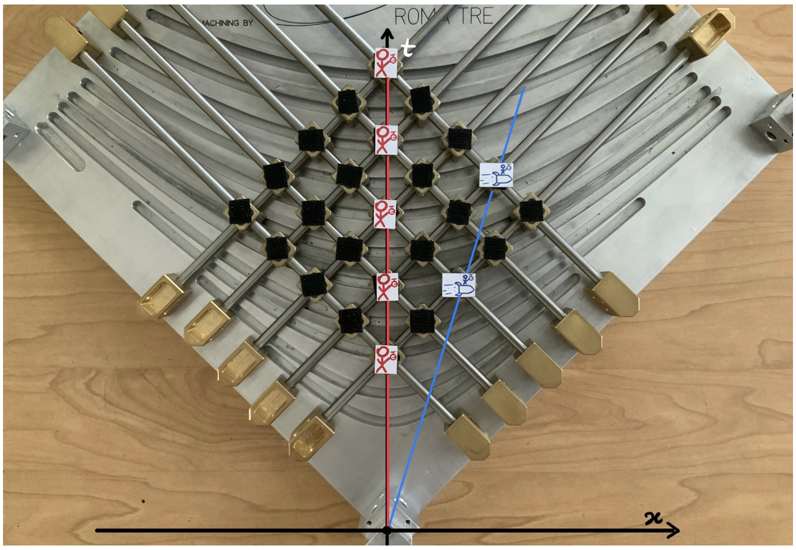


Figure 3.21: A blue observer is moving with respect to a red man. Each observer has his own clock beating the proper time.

Now we can assign to each figure of the red man the meaning of an event: looking to Figure 3.22, we see that the time interval between each image of the red men is 1 second. We take as landmark to measure time the corner of the die over which lays a figure of the red man.

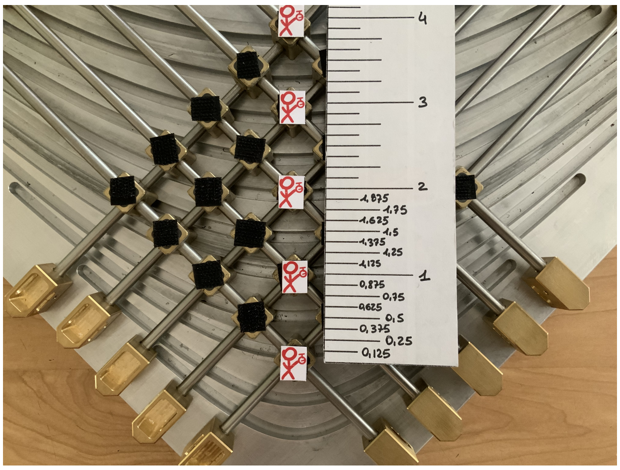


Figure 3.22: Measuring one click of the red clock in the reference frame of the red man. One red click last $\Delta\tau = 1$ s.

We can thus say that each figure of the red man is the event "the red clock ticks one second" happening at $x = 0$. The time interval between two of these events is the temporal duration of a tick of the red clock and, as the clock is at rest with respect the red man, it is actually beating the red man's proper time τ.

Then using a clock at rest with respect to the red man (the red clock), one tick of the red clock lasts $\Delta\tau$:

$$\Delta\tau = (1.00 \pm 0.02)\ \text{s}. \tag{3.23}$$

Now we want to know how the blue man measures the same time interval $\Delta\tau$ using its own clock (blue clock).

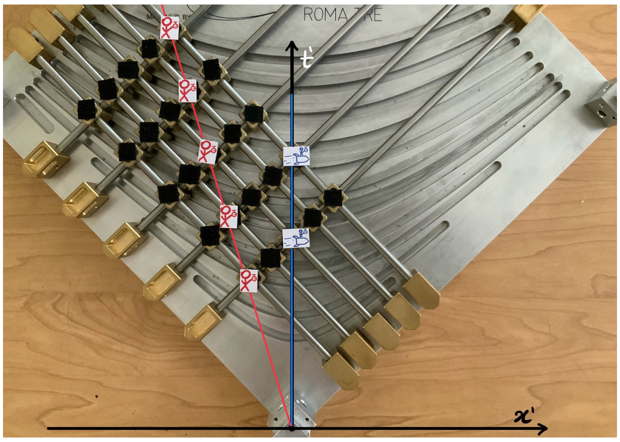


Figure 3.23: Changing reference frame. The blue man's point of view is adopted: the red man is now moving with respect to the other observer.

In Figure 3.23 we have applied a Lorentz transformation to boost into the moving $\mathcal{S}$-observer's perspective (blue man).

Now we want to measure (Figure 3.24) the time interval $\Delta\tau = 1$ s according to the blue man (Figure 3.24).

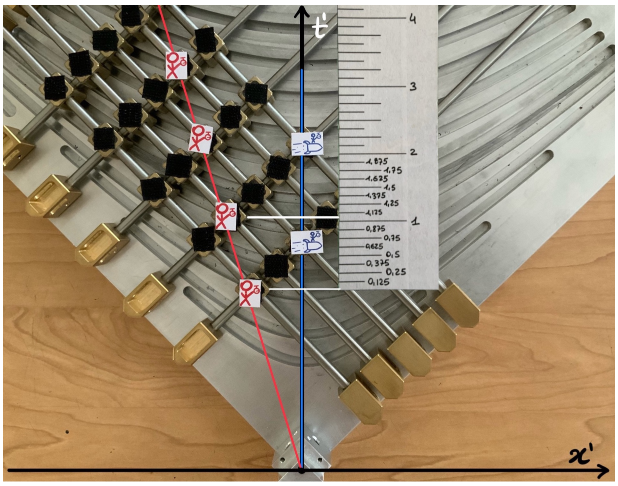


Figure 3.24: Measuring one click of the red clock in the reference frame of the blue man. One red click lasts more than 1 second: $\Delta t' = 1.06$ s.

In $\mathcal{S}'$-observer's reference frame, he is at rest (his worldline is a vertical straight line) while the red man is moving. Differently from red man's reference frame, here the time interval between the two figures of the blue man does not correspond to $\mathcal{S}'$-observer's proper time, not being spaced by 1 unit. This is an instrument's limit: to reproduce also $\mathcal{S}'$-observer's proper time more hyperbolas should have been engraved and thus more figures of the blue man could have been present.
In order to know how long the time interval $\Delta\tau$ lasts in the blue man's reference frame, we have to project two subsequent events of the red man's figure along the new time axis t'. From Figure 3.24 we see that the time interval $\Delta t'$ of 1 second measured by the blue man lasts more then 1 s:

$$\Delta t' = (1.06 \pm 0.02)\ \text{s}. \tag{3.24}$$

This measure is in full agreement with the time dilation formula applied for the transformation of a proper time interval of 1 second as measured by an observer travelling at $1/3$ of the speed of light relative to the first observer:

$$\Delta t' = \gamma \Delta\tau = \frac{1}{\sqrt{1 - \left(\frac{1}{3}\right)^2}} \cdot 1 \simeq 1.0607\ \text{s}. \tag{3.25}$$

3.3.3 Length contraction

The second important consequence of Special Relativity is the contraction of the length unit which results in a contraction in the length of the objects from the point of view of the moving observers.
Consider an observer $\mathcal{S}$ who is at rest with respect to the Earth: the red man. He has a ruler with him (hence on the red ruler) that allows to measure lengths in $\mathcal{S}$-observer's reference frame. Close the red man there is a one meter long dog, at rest with respect to him.
There is also another observer $\mathcal{S}'$, the blue man, who is moving away from observer $\mathcal{S}$ with constant speed $v = -c/3$ with respect to him aboard a spaceship. He has also a ruler, namely the blue one. We can say that both the red and the blue ruler are instruments to measure proper lengths in each observer's reference frame.
In Figure 3.25 we used the spacetime globe to depict this situation.

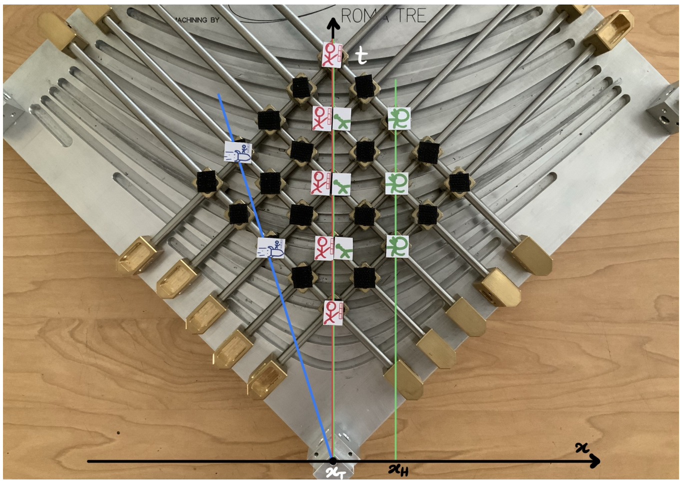


Figure 3.25: A blue observer is moving with respect to a red man. A green dog $L = 1$ m long is standing near the red man. Each observer has his own ruler.

In Figure 3.25 we reproduce the dog at rest with respect to the red man as following: having the animal a physical length is not possible to represent it as a material point. Thus, the dog's tail is located in $x_T = 0$ while its head a little bit further on, in $x_H = 1$. Thus in Figure 3.25 we can see the worldline of both the dog's head and tail as vertical green lines, being the animal at rest in $\mathcal{S}$-observer reference frame.

Then, we can measure dog's length L in the red man's reference system. We take as landmarks to measure the position along x-axis the corner of the die over which lays a figure of the dog (tail or head).

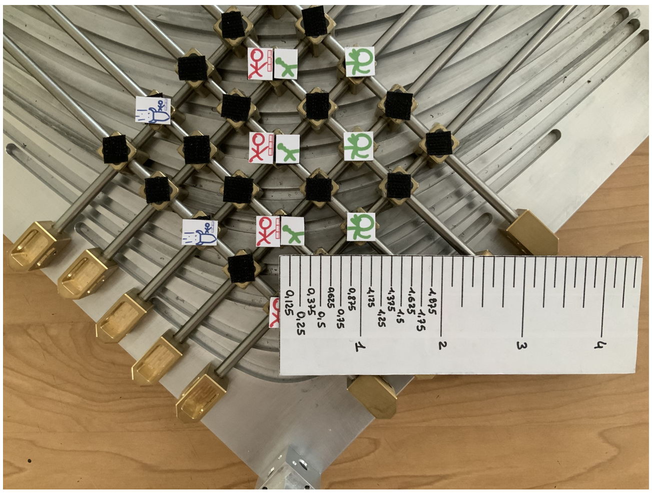


Figure 3.26: Measuring the length L of the green dog in the reference frame of the red man. Using the red ruler $L = 1$ m.

If we had determined the length of the animal through the coordinate of the tail and of the head, we would have to subtracting the head's one to the tail's one. But as we can see from Figure 3.26, our ruler gives directly the measure of the length L that in the red man's reference system turns to be:

$$L = (1.00 \pm 0.02)\ \text{m}. \tag{3.26}$$

By definition, this is the measure of the dog's proper length as the dog in measured in a reference frame at rest with respect to it (the red man's one).
Now we want to measure the dog's length $L = 1$ m according to the blue man, using the blue ruler. We need to change the reference frame and to adopt $\mathcal{S}'$-observer's point of view. Acting on the spacetime globe we arrive at the configuration shown in Figure 3.27.

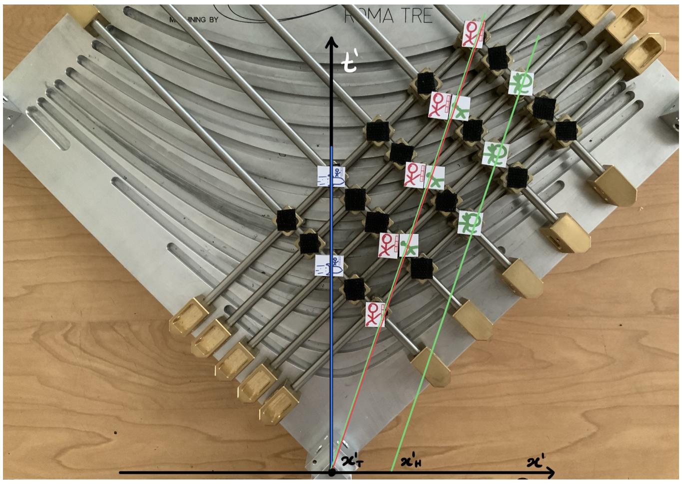


Figure 3.27: Changing reference frame. The blue man's point of view is adopted: the red man and the green dog are now moving with respect to the other observer.

In Figure 3.27 we can see that the dog is now moving with respect to the blue man at the speed v, being inclined the tail's and head's worldline.
How now can we measure a moving object? We have to remember that a length measure has to be performed at one instant of time: this means that the object must be at rest during the whole process. In fact, when we measure the length of the dog, we need to align the tail with one notch of the ruler and the head with another one, to read the value of each notch and to make their difference.
However if after taking the first reading the dog moves, what we read is not the true length of the animal because the body moved during the measurement process.
In the red man's reference frame (Figure 3.26) the measure is performed at one instant of time as the position of the dog's head and tail is taken at the same time. But we know that simultaneity is no longer preserved in Special Relativity: thus in the blue man's reference frame the previous considered position of the dog's head and tail are no more simultaneous

as we can see in Figure 3.27. Then if we consider a certain tail's position, we need to bring back the dog's head at the same time of the tail. This ensures us that the length measure is correctly performed, namely the position of the tail and the head are taken at the same time.

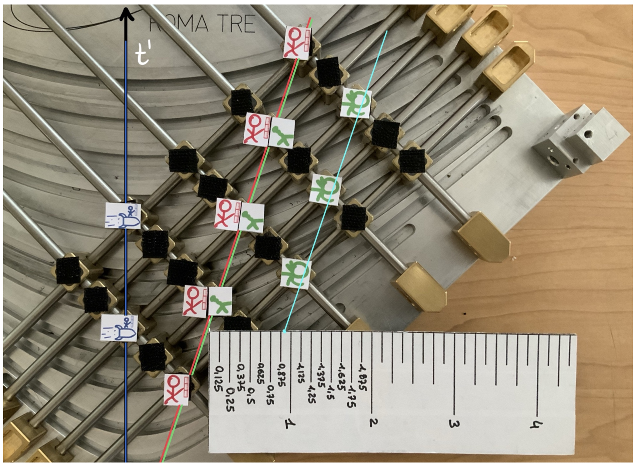


Figure 3.28: Zoom over Figure 3.27 to measure the dog's length in the blue man's reference frame. The light-green line is the dog's head worldline: it allows to measure simultaneously the dog's head and tail. The dog's length is less than 1 meter: $L' = 0.94$ m.

In Figure 3.28 we highlight the worldline of the dog's head in light-green. We consider the first image of the dog's tail whose position is identified by the lower corner of the die. The corresponding head's position is determined by the intersection between the worldline of the head and a straight line perpendicular to t'-axis (represented in Figure 3.28 directly by the ruler) at the head's time coordinate.
From Figure 3.28 we see that now the dog's length L' lays between 0.875 m and 1 m:

$$L' = (0.94 \pm 0.02)\ \text{m}. \tag{3.27}$$

This value is in agreement with the length contraction formula applied for the transformation of a proper length of 1 meter as measured by an observer travelling at $1/3$ of the speed of light relative to the first observer:

$$L' = \frac{L}{\gamma} = 1 \cdot \sqrt{1 - \left(\frac{1}{3}\right)^2} \simeq 0.9428\ \text{m}. \tag{3.28}$$

3.3.4 Addition of velocities

One of the most remarkable achievements of Special Relativity is the correction to the Galilean formula of the addition of velocities.

The problem is settled in this way: we consider an observer $\mathcal{S}$ who is at rest with respect to the Earth: the red man. There is also another observer $\mathcal{S}'$, the blue man, who is moving away from observer $\mathcal{S}$ with constant speed $u = c/2$ with respect to him aboard a spaceship. The blue man has a gun and at certain time he shoots: in $\mathcal{S}'$-observer's reference frame, the projectile has speed $v' = c/2$. We want to know what is the speed of the projectile the red man measures.
In order to solve this problem with the spacetime globe is it is better to proceed with an inverse reasoning, namely starting from the blue man's reference system. In Figure 3.29 we used the spacetime globe to depict this situation.

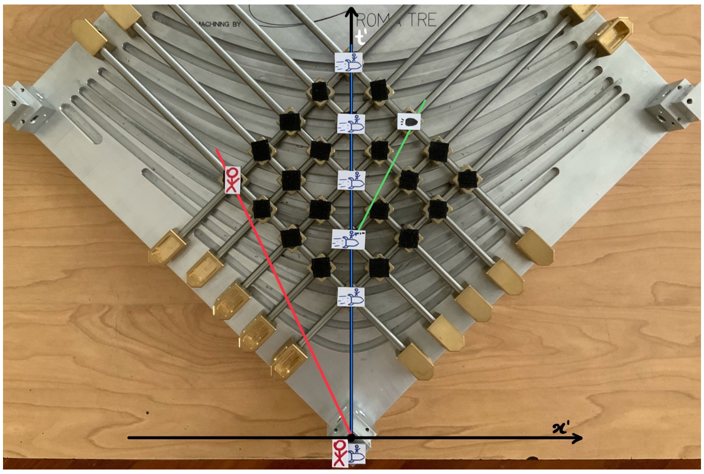


Figure 3.29: The blue man in a spaceship shoots with a gun: the bullet (green line) is moving at $v' = c/2$. The blue man sees a red man moving at $u' = -c/2$.

In Figure 3.29 we see that the blue man (the "moving" observer $\mathcal{S}'$) is at rest as his worldline is a vertical straight line. Then according to the principle of relativity, the red man at rest on the Earth is seen as moving at speed $u' = -c/2$. Finally the green worldline defines the moving projectile at speed $v' = c/2$ with respect to him.
Now we want to know the speed v of the projectile from the point of view of the red man. According to Galilean addition, we should expect to have

$$v = v' + u = \frac{c}{2} + \frac{c}{2} = c, \tag{3.29}$$

namely the speed of light.
As a consequence, since in spacetime diagrams the slope of a worldline is inversely proportional to the speed of the represented body (Eq. (3.18)), in the red man's reference frame one should expect to see the projectile's worldline inclined as the light's one.

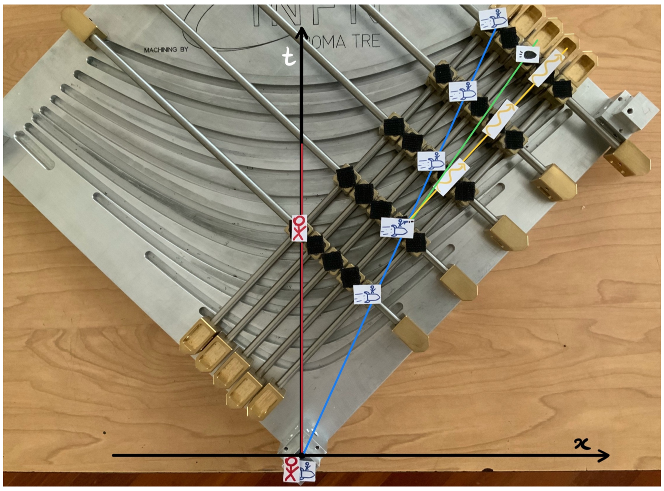


Figure 3.30: Changing the reference frame, according to the red man the blue man is moving at $u = c/2$ while the bullet has a speed less than the light's one (yellow line).

In Figure 3.30 we have applied a Lorentz transformation to boost into the $\mathcal{S}$-observer at rest on the Earth (red man). The red man sees the blue man travelling on the spaceship at speed $u = -u' = c/2$. We see also the projectile's worldline (green line) and the light's worldline represented by yellow wavy rays (yellow line).
If Galilean addition had been correct, in Figure 3.30 we would have obtained the green line (projectile's worldline) parallel to the yellow one (light worldline).
Instead we see in Figure 3.30 that the green worldline is more inclined then the yellow worldline: this means that projectile's speed is less that light one, thus showing that Galileo's formula is wrong in relativistic situations.
The correct value for v is given by:

$$v = \frac{u + v'}{1 + \frac{u \cdot v'}{c^2}} = \frac{1/2 + 1/2}{1 + \frac{1}{2} \cdot \frac{1}{2}} \cdot c = \frac{4}{5} c = 0.8\, c. \tag{3.30}$$

Here a quantitative analysis is more difficult to carry out as a high level of precision is requested. A solution is to measure the change Δx in x-direction and the change Δt in t-direction, obtaining v as their ratio.
From Figure 3.30 we get:

$$\Delta x = (2.13 \pm 0.02)\ \text{m} \qquad \Delta t = (2.56 \pm 0.02)\ \text{s}. \tag{3.31}$$

Then the speed v of the bullet as measured by the red man is (with the correct c value)

$$v = (0.83 \pm 0.01)\, c\ \text{m/s}, \tag{3.32}$$

where the uncertainty σ_v over v is obtained as:

$$\sigma_v = 0.02 \cdot v \cdot \sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta t^2}}. \tag{3.33}$$

There is a little difference with the theoretic result due to the precision required for this measure which is not possible to obtain with the spacetime globe.

3.3.5 Light speed invariance

We know that Lorentz transformations preserve light speed. We can show it similarly to what we have demonstrated so far in paragraph 3.3.4.

We consider the same situation presented in the description of the relativistic addition formula (paragraph 3.3.4) but the blue man does not shoot with the gun: he turns on a light bulb.

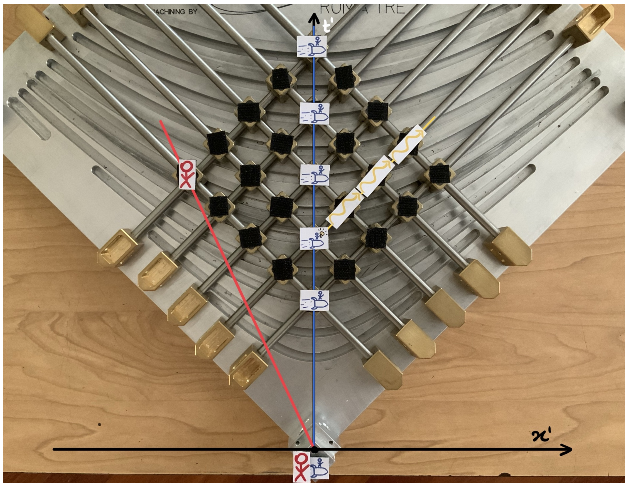


Figure 3.31: Light (yellow arrows) propagates from a light bulb turn on by the blue observer inside a spaceship. A red man on the Earth is seen as moving according to the blue man at speed u'.

As before, we start from the point of view of the moving observer. In Figure 3.31 we see the blue man inside a spaceship with a light bulb. At certain time t he switches on the light which propagates away from him a speed c along the worldline depicted by the yellow wavy rays. As they represent the propagation of the light, they lay over a straight line inclined by an angle of 45° (that can be easily identified from the direction of the bars). Finally in Figure 3.31 we have also the observer on the Earth, namely the red man, that the blue observer sees moving with speed u'.

Now we want to know the light speed with respect to the red man.

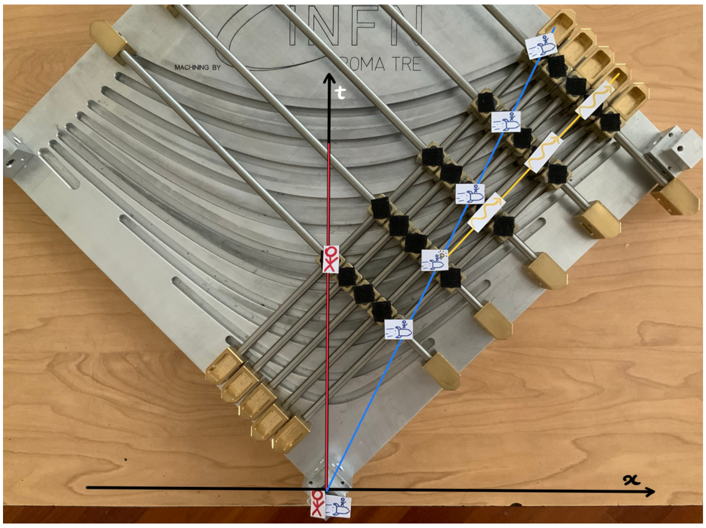


Figure 3.32: Changing reference frame, according to the red man the blue observer is now moving with speed $u = -u'$ while the light still propagates at speed c (the slope of yellow worldline does not change).

After having applied a Lorentz transformation, we see in Figure 3.32 that the red observer is at rest, while the spaceship is now moving at speed $u = -u'$.
As far as light is concerned, in Figure 3.32 its worldline has the same slop as in Figure 3.31. This means that its speed in the red man's reference system is the same as in the blue man's one, either the observer is at rest or moving.

3.3.6 Doppler effect

Another phenomenon, that was not investigated in the original videos of the spacetime globe, is the longitudinal Doppler effect. The transverse can not be shown with the instrument as a bi-dimensional motion can not be represented.
The situation we are going to study is the following: we consider the observer $\mathcal{S}$ (red man) who is at rest with respect to the Earth with a clock (red clock). The second observer $\mathcal{S}'$ (blue man) is moving away from observer $\mathcal{S}$ with constant speed $v = c/3$ with respect to him aboard a spaceship. He has a light bulb that he turns on just the time $\Delta t'$ needed to emit a single complete wave.
In Figure 3.33 we used the spacetime globe to depict this situation again starting from the moving observer's reference frame.

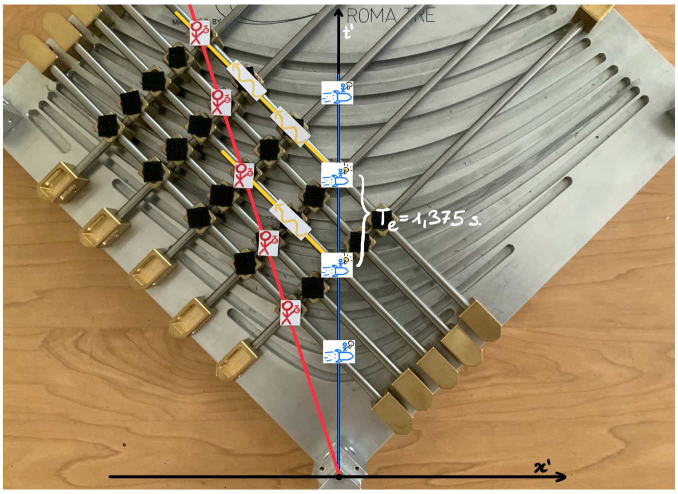


Figure 3.33: A blue man inside a spaceship turns on a light bulb just the time T_e for one complete wave's period. A red man is seen as moving with speed v' by the blue man.

In Figure 3.33 we see that the blue man (the "moving" observer $\mathcal{S}'$) is at rest as his worldline is a vertical straight line. Then according to the principle of relativity, the red man at rest on the Earth is moving with respect to the blue man at speed $v' = -c/3$. In the figure we see also two yellow wavy rays: they represent the beginning part and the end part of the wave propagating away from the blue observer. These two events happen in the same place (the position of the bulb) at two different subsequent instants of time t'_b and t'_e. From the beginning of the propagation t'_b, a time interval $\Delta t'$ elapses before the end part of the wave begins to propagate at time t'_e. Between these two events and the corresponding worldlines we should image to have on the spacetime globe infinite other yellow worldlines representing the subsequent parts of wave included between the beginning and the end part.

Thus we can consider the time interval $\Delta t' = t'_e - t'_b$ as the source's period T_e as measured by the emitter (blue man) with his blue clock.

Taking the time measure as we have done in Section 3.3.2 we get that:

$$T_e = (1.38 \pm 0.02)\ s. \tag{3.34}$$

This light propagates away from the emitter and it is then collected by another observer, the red man, moving at speed $v' = -c/3$.

In Figure 3.33 we see the two light's worldlines intercepting the red man's one in two different events. Indeed they correspond to the event in which the red man receives respectively the wave's beginning and end part.

Now we want to determine the source's period as measured by the red man, namely the time interval between the receiving of the wave's beginning and end part.

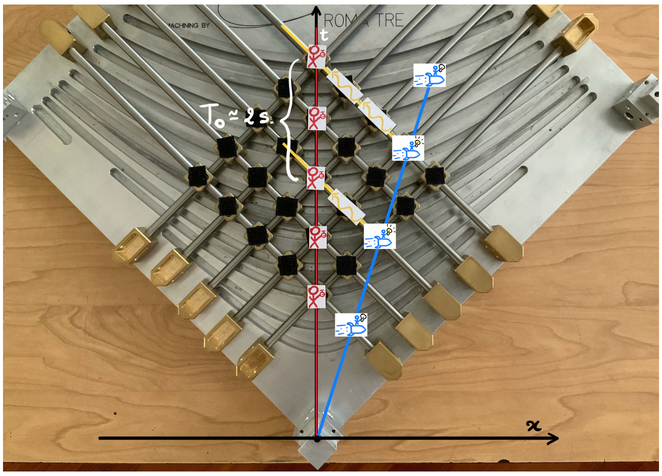


Figure 3.34: Changing the reference frame, according to the red man the blue man is moving with speed $v = -v'$. Light intercepts the red man in two subsequent instants of time, thus determining wave's period T_o.

In Figure 3.34 we have applied a Lorentz transformation to boost into the red man's perspective. Now we can see that the blue man is travelling away from the red one as the distance with respect to him increases with time: this tells us that we are dealing with the Doppler effect from a receding source.
To measure the time duration of wave's period in the red man's reference frame, we need to consider the two time instants when the light catches to the observer.
Thus from Figure 3.34, we can measure the time interval Δt that correspond to the source's time period T_o according to the observer at rest:

$$T_o = (2.00 \pm 0.02)\ \text{s}. \tag{3.35}$$

This agrees with the relativistic Doppler effect formula for a receding source moving at speed $c/3$ within the experimental uncertainty:

$$T_o = \gamma\,(1 + \beta)\,T_e \Longrightarrow T_o = (1.95 \pm 0.03)\ \text{s}, \tag{3.36}$$

where $\sigma_{T_o} = \gamma(1 + \beta)\sigma_{T_e}$.
The spacetime globe can show easily also the Doppler effect for an approaching source (see Section 3.4). The reconstruction with the spacetime globe is the same but in this case we have to turn upside down the instrument as we need a negative time axis to describe an approaching moving body.

3.3.7 Mass invariance

In this last section we want to show another use of the spacetime globe as a energy-momentum diagram (taking inspiration from Saletan [67]). The total energy E and the momentum $\boldsymbol{p}$ of a body constitutes the four-momentum $p^\mu = (E/c, \boldsymbol{p})$: changing reference systems, the four-momentum transforms via Lorentz transformations. As a consequence as far as the transformation of the four-vector p^μ is concerned, its geometrical locus is still a hyperbola, representing the surface of constant mass m, being $m^2 = E^2 - p^2$.
In this configuration the spacetime globe does not provide a visualisation of the change of a reference frame as worldlines can not be represented, being an energy-momentum diagram: thus it does not contain any explicit information about the position of the observers. The aim is different and it is to show the conservation of the quantity $m = \sqrt{E^2 - p^2}$ and the existence of the energy at rest E_0.
We set the problem in this way: on the Earth an observer sees a particle (call it *spacetimon*) of mass $m = 3.75$ MeV/c^2 approaching towards him along a straight path with a certain speed v. Instead from the particle's point of view it is at rest and it sees the observer moving towards it at speed $v' = -v$.
In Figure 3.35 we prepare the spacetime globe to figure the point of view of the observer at rest on the Earth. The vertical axis represents the particle's energy E measured in MeV while, identifying the x-direction as the one of the spacetimon's movement, the horizontal axis p_x, the x-component of the momentum $\boldsymbol{p}$, measured in MeV/c^2.
The observer on the Earth is not depicted in Figure 3.35 but we know to be in his reference frame with the particle (represented by the little image) having a non-zero component of its momentum.

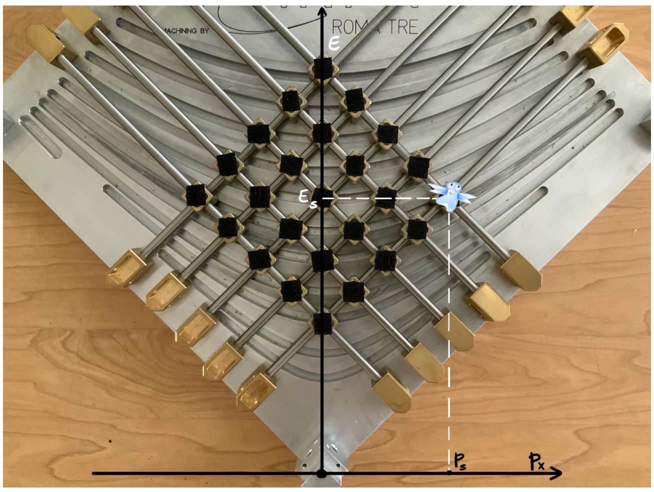


Figure 3.35: In the reference frame of the Earth (not drawn) a particle has x-component of the momentum p_s and total energy E_s. Credits for the particle's image: CERN.

We can simply measure the value p_s of the spacetimon's p_x momentum, obtaining

$$p_s = (2.00 \pm 0.02) \text{ MeV/c}. \tag{3.37}$$

In the same way the total amount of its energy E_s is

$$E_s = (4.25 \pm 0.02) \text{ MeV}. \tag{3.38}$$

Thus it is easy to determine the spacetimon's mass in the reference frame of the observer at rest:

$$m = \sqrt{E_s^2 - p_s^2} = (3.75 \pm 0.02) \text{ MeV/c}^2, \tag{3.39}$$

where:

$$\sigma_m = \frac{0.02}{m}\sqrt{E_s^2 + p_s^2}. \tag{3.40}$$

Now we want to determine the spacetimon's mass in the reference frame of the particle itself. As we have always done, we need to act on the bars of the instrument: however here we do not have a visual guide that allows us to understand when we have to stop. We only know that this happens when the momentum of the particle is zero. Indeed in the particle's reference frame it sees at itself as stationary and then having $v' = 0 \rightarrow p_s' = 0$. As a consequence we need to shift the dice on the spacetime globe until the spacetimon's figure reaches the position along the E'-axis where its momentum is zero.
This reasoning justifies the Figure 3.36 where the spacetimon's point of view as been adopted.

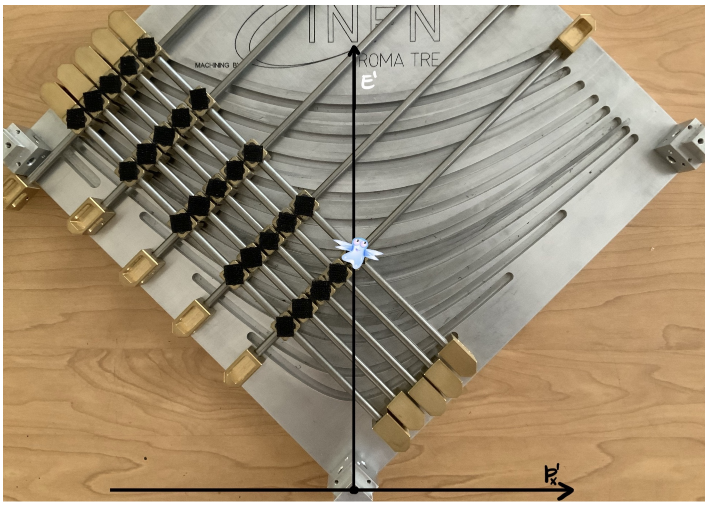


Figure 3.36: Changing the reference frame, from the spacetimon's point of view, it is at rest $p_s' = 0$ but has still energy $E_s' \neq 0$. Credits for the particle's image: CERN.

Thus in Figure 3.36 we are into the spacetimon's reference frame: from the particle's point of view it is at rest and then its momentum $p'_s = 0$.
As far as the amount of the particle's energy E'_s is concerned, it is not equal to zero but instead

$$E'_s = (3.75 \pm 0.02)\ \text{MeV}. \qquad (3.41)$$

This result has two important consequences: first it shows that E'_s is not zero. A body at rest has not zero energy but there is an amount of energy. Then this energy is exactly the energy at rest $m \cdot c^2$ associated to the particle's mass.
The spacetime globe is thus able to show the effect of the changing of reference frames over the total energy amount of a body, showing the conservation of quantity m. But this experience can be thought also in another way: when we move the configuration of Figure 3.35 into the one of Figure 3.36, we can think of braking the spacetimon. In this way, we see moment by moment how speed modifies the particle's amount of energy and thus of the mass, actually showing that mass does not change with speed.

3.4 Digital Education: Python simulations

We have seen in the last sections that the spacetime globe well allows to simply reproduce all the phenomena expected from the Theory of Special Relativity. However it is not easy to build this instrument because it requires a fully equipped mechanical laboratory. Moreover since 2020 because of emergency care due to CoronaVirus epidemic, schools were forced to rethink teaching and learning in terms of digital education, at first called *DAD* - Didactic At Distance - and nowadays *DDI* - Digital Didactic Integrated.
For this reason we tried to think how the experiences with the spacetime globe could be easily carried out also under digital education. The easiest way was to code graphical simulations reproducing the movement one can perform with the spacetime globe.
We choose to use Python language to develop our codes for its versatility and its practicality to create graphs. Moreover Python allows us to create dynamical plot in the form of an animation. One can see the worldlines of the different observer moving on the graphs, thus going from the one observer's reference frame to the another one observer's frame, exactly in the same way we perform with the spacetime globe.

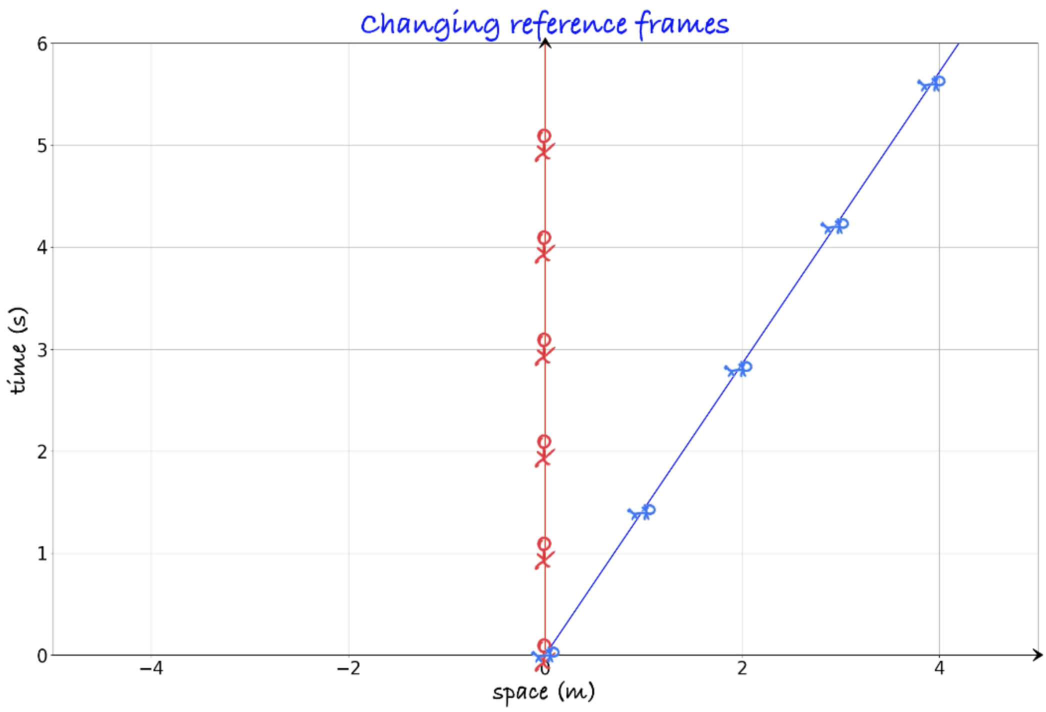


Figure 3.37: Screenshot from one of our Python simulations.

We develop 13 different simulations:

1. Change between reference frames: this simulation shows only the change from one reference frame to another one, just rotating two different worldlines in the same way of Figure B.1. In addition there is a second simulation like this one but having a layer of hyperbolas showing how the figures of the two observers move;

2. Loss of simultaneity: similar to Section 3.3.1, this simulation shows that Special Relativity does not preserve simultaneity of events;

3. Time dilation: similar to Section 3.3.2, this simulation shows that in Special Relativity time intervals change according to the observers' state of motion. By the end of the simulation it is performed also a quantitative result as it is shown the numerical value of the time interval in the moving observer's reference frame;

4. Length contraction: similar to Section 3.3.3, this simulation shows that Special Relativity does not preserve the length measures. By the end of the simulation it is performed also a quantitative result as it is shown the numerical value of the length of an object in the moving observer's reference frame;

5. Addition of velocities: similar to Section 3.3.4, this simulation shows the relativistic

addition of velocities. By the end of the simulation it is performed also a quantitative result as it is shown the numerical value of the interested speed;

6. Light speed invariance: similar to Section 3.3.5, this simulation shows the invariance of light speed;

7. Superluminal motion: this simulation shows one incongruities of superluminal motion. This simulation is not included in our previous description as it is not possible to carried out with the spacetime globe;

8. Doppler effect: similar to Section 3.3.6, this simulation shows the relativistic longitudinal Doppler effect from a receding source. By the end of the simulation it is performed also a quantitative result as it is shown the numerical value of the time which the beginning and end part of the wave reach the observer at;

9. Doppler effect (redshift): this simulation is the same as the previous one but it is referred to the visible spectrum. During the shift from one reference frame to the other one, the light changes its colour towards redder ones in order to explain the redshift phenomenon;

10. Doppler effect (blueshift): this simulation shows the Doppler effect from an approaching source applied for the visible spectrum. It does not only perform the quantitative analysis of the change in the period of the emitting source but it shows also light changing its colour towards bluer ones in order to explain the blueshift phenomenon. This simulation is not included in our previous description but it can be simply carried out with the spacetime globe (see for reference the comment by the end of Section 3.3.6);

11. Mass invariance: similar to Section 3.3.7, this simulation shows the invariance of mass. Throughout all the simulation, it is shown the numerical decreasing of the particle's momentum as well as of its energy. At the same time it is also shown the value of the mass which is seen to be constant;

12. A collision problem: this last simulation is an application of the relativistic kinematic (shown in Appendix B.2.6). It shows a collision problem between two muons in two different reference frames: the center-of-momentum frame and the target one. The simulations is made up of two graphs, the first being a spacetime diagrams and the second one an energy-momentum diagram. As shown in Appendix B.2 the resolution is given by a Poincaré transformation.

The parameters of each simulation, as for instance the speed of the moving observer or the length of the object, can be changed manually on the code. It can be also modified in

order to be able of setting parameters at run-time through a simple **input** function.
However as it is not sure that teachers or students are able to deal with Python codes, we think of them to be used in two different formats:

- for users who are skilled with computing and know Python language, the simulations can be directly launched from terminal (or different applications). The advantage is that teachers as well as students can change parameters like the speeds, time interval, length of the object... However simulations can not been stopped nor rewound: if necessary the simulations have to be reloaded each time;
- for common users we decided to export our simulations as a videos which allow a more friendly employ. The parameter are thus fixed and can not be changed. However the videos can be re-watched as many times as one wants; they can be paused at any moment to point out some specific feature if necessary or rewound.

Obviously for able costumers, the two formats can be mixed: for instance users can modify one code to create several videos changing each time one or more parameters and then use directly their videos.
These simulations-videos could be an alternative to the instrument itself whenever it can not be built, even if the impact the instrument gives is not the same as with the codes since the peculiar laboratorial feature is lost. Indeed the most important aspect of the instrument is that one can experience Special Relativity hands-on, whit an instrument as if it was an experiment of Classical Physics.
Moreover there already exist some online resources used with a didactic purpose like for instance the *OpenRelativity* project ([48]) or *Real Time Relativity* game ([47]) that help students to visualise the effects of Special Relativity through interactive simulations where the gamer is able to move at speeds comparable to the light's one.
We considered that students are always more involved into a digital reality and these videos would enter into this trend. But this is not our primary aim. These simulations could be a very useful instruments to integrate a laboratorial lesson with the instrument (if possible): students are able to revise the activity watching the videos or to explore new phenomena as homework.
In Appendix B.3 we report a short description of one of the developed codes to let understand the idea behind our simulations.

3.4.1 Superluminal motion

We want now describe one of our simulations: we choose the one showing the effect of superluminal motion as this simulation can not be realised with the spacetime globe. Indeed figures over dice laying on a superluminal worldline during the change of reference frames move along hyperbolas $x^2 - c^2t^2 = k^2$. These tracks are not present on our space-time globe.

Now consider the following problem: we have an observer $\mathcal{S}$ who is at rest with respect to the Earth, namely the red man. There is also another observer $\mathcal{S}'$, the blue cat, who is moving away from observer $\mathcal{S}$ with constant speed $v = 0.7\,c$ with respect to him. Finally we consider a third observer a green man aboard a spaceship who is moving too with respect to the red man with speed $u = 2.4\,c$.

Figure 3.38 shows the initial setting of our simulation depicting this situation.

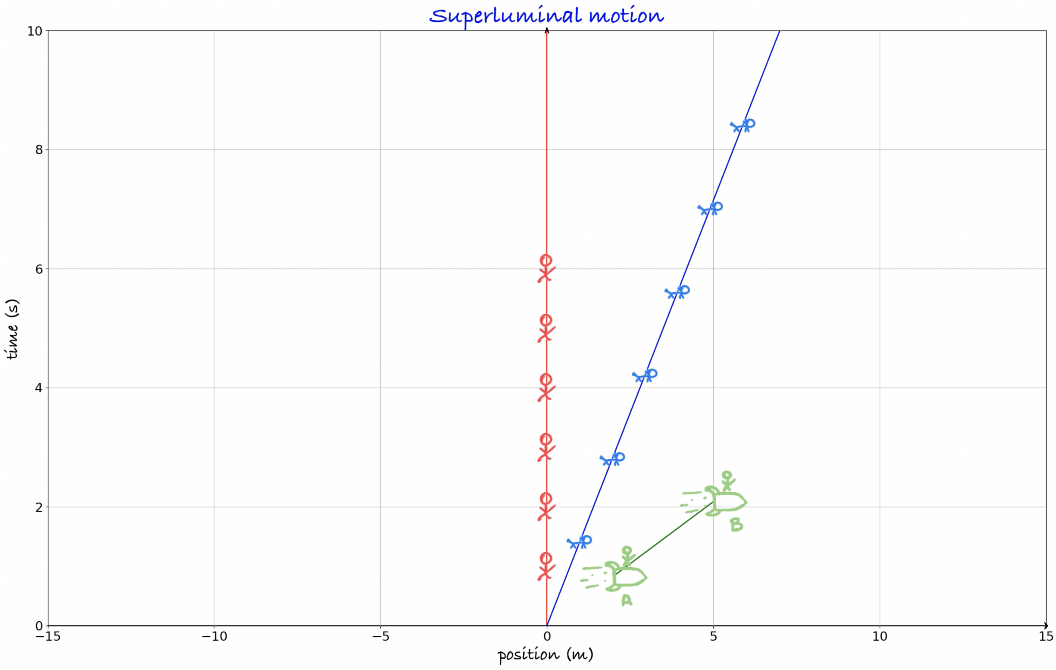
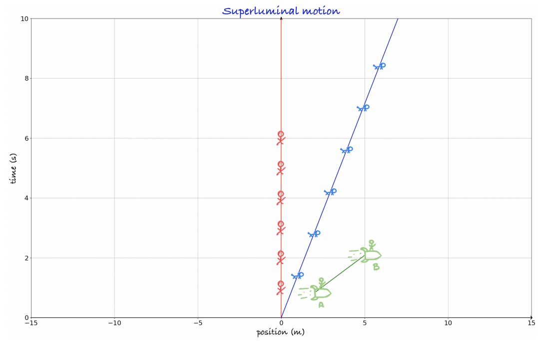


Figure 3.38: A blue and a green observer are moving with respect to a red man. The green one has a superluminal speed.

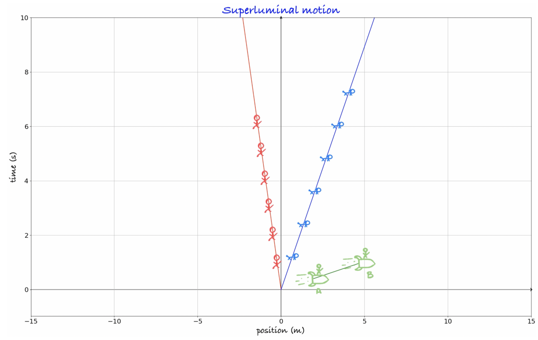
In Figure 3.38 we see that the red man (observer $\mathcal{S}$) is at rest as his worldline is a vertical straight line $x = 0$. In the same way the blue cat is moving as well as the green man being their worldline an inclined line: the former's one equation is $t = (1/v)x$ while the latter's one is $t = (1/u)x$ where $u > c$.

However we see that it is not represented all the green man's worldline but only a segment included between two events A and B. We can interpret them as the departure of the spaceship (event A) and its arrival (event B). Thus the green spaceship leaves at time t_A from position x_A and arrives at x_B by the time t_B travelling at speed $u > c$.

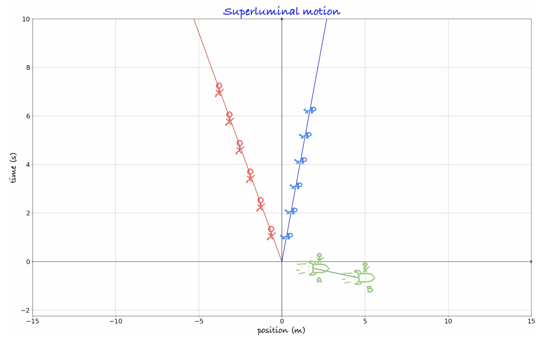
Now we want to observer these events from the blue cat's reference frame. If we evolve the simulation we will see the following images:



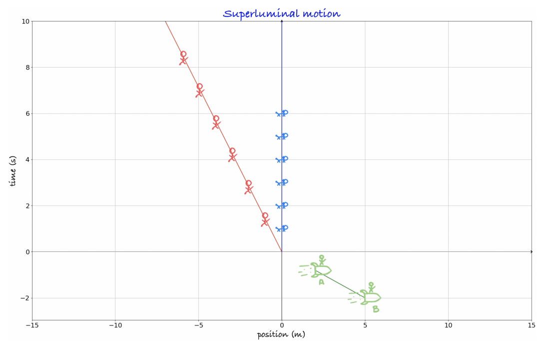
(a) Start of the simulation.



(b) First intermediate step of the simulation.



(c) Another intermediate step of the simulation.



(d) End of the simulation.

Figure 3.39: Transition from the red man's perspective to the blue cat's perspective in the Python simulation of a superluminal trip.

The final situation is shown in Figure 3.40:

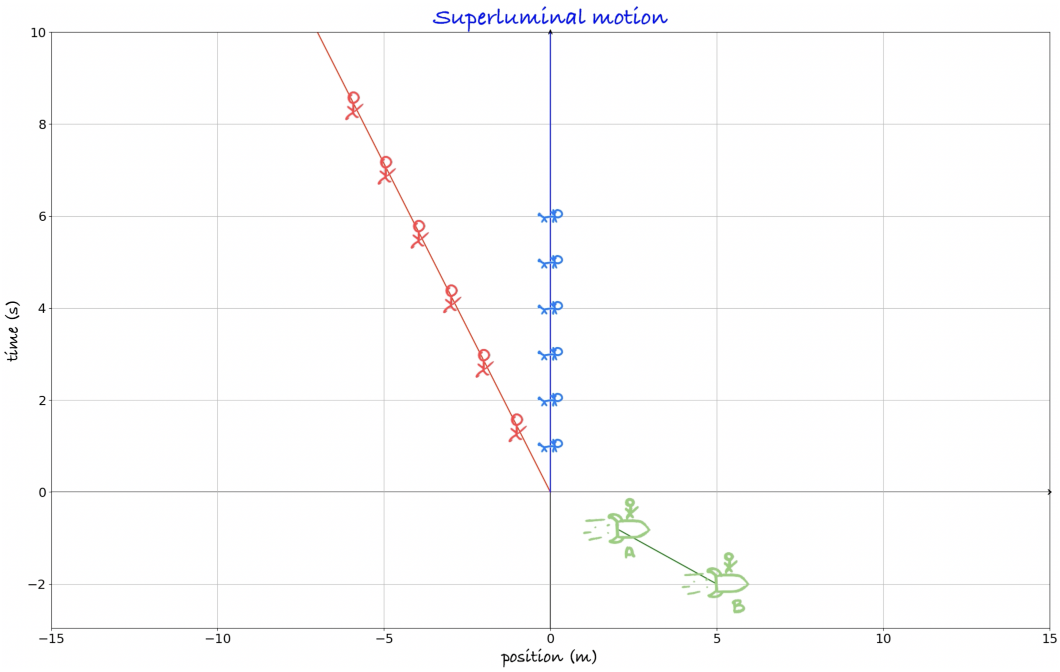


Figure 3.40: According to the blue cat the trip of the green man ends before having started.

In $\mathcal{S}'$-observer's reference frame, the cat is at rest (its worldline is a vertical straight line) while the red man is moving with speed $-v$. The green man is moving with respect to the blue cat but we see that something has changed: from the blue cat's point of view the green man aboard the spaceship arrives before leaving.
Indeed the trip starts at time t'_A from position x'_A and ends at time t'_B from position x'_B but

$$t'_B < t'_A. \tag{3.42}$$

From the blue cat's point of view, event B "the green spaceship arrives at point x'_B" happens before the event A "the green spaceship leaves from x'_A" as in its time event B is more distant then event A (Eq. (3.42)). This means that event B happens before event A.
This simulations led us to explore one of the effects of trips at superluminal speeds, namely the reversal of cause-and-effect link. Indeed an effect (arrival of the observer) comes before the cause (leaving of the observer), allowing us to observe spaceship arriving before having left.

Chapter 4

School experimentation

The spacetime globe provides a complete description of the phenomenology embodied inside the Theory of Special Relativity. However as we want it to be a didactic tool we needed to structure a didactic project allowing us to use this instrument during high-school lessons. We are going to describe how it was carried on.

4.1 Introductory questionnaire

Before working on the spacetime globe and its construction we investigated the teachers' urgencies from a laboratorial point of view in order to validate our idea of working on Special Relativity.

During the first Italian lockdown for the emergency care due to CoronaVirus epidemic we elaborated two questionnaires (see Appendix C). The first one was about the learning-teaching process during the first period of the digital didactic (March-June 2020): questions on which instruments the teachers used, which were the preferred way to test students' knowledge and so on were asked.

In the second questionnaire we asked to identify which were the necessities on laboratory teaching: we wanted to know which themes and type of laboratory should be developed in order to guide create useful laboratorial experiences.

Figure 4.1: Diffusion of our first questionnaire: the darker the red the higher the number of answers from that region.

By the end of the scholastic year 2019-2020 we submitted these questionnaires to a group of high school teachers of Lazio area that usually collaborates with our Department in our didactic activities. Then the questionnaires were spread also in other regions thanks to a circulation through the LS-OSA project [102] of the Science Department of Roma Tre University. The first questionnaire was answered[1] by 90 teachers while the second one by 69: actually not all the teachers answering to the former gave answers also to the latter. Among the teachers that answered to first questionnaire, an amount of 53 has a degree in Mathematics, 33 in Physics, 3 are engineers and one teacher has a degree in Forestry. There is a clear majority of mathematics in our sample (Figure 4.2).

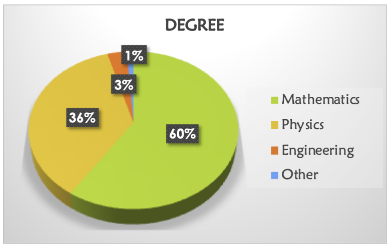


Figure 4.2: Degrees of our teachers' sample.

The 89% of these teachers teaches Physics at least in one class of a high school oriented to scientific studies (*Liceo Scientifico*, *Liceo Scientifico con opzione Scienze Applicate*) while the remaining part is arranged between those ones oriented to classic studies (*Liceo Classico*), linguistic studies (*Liceo Linguistico*) or other schools with a not-specified address.
As for the purpose of this work the first questionnaire is not relevant, we are going to describe the one concerning the development of laboratorial activities.
The questionnaire was structured with 23 questions. Firstly we asked the teachers in which year of a high school curriculum they thought there was a greater urgency from the laboratorial point of view.

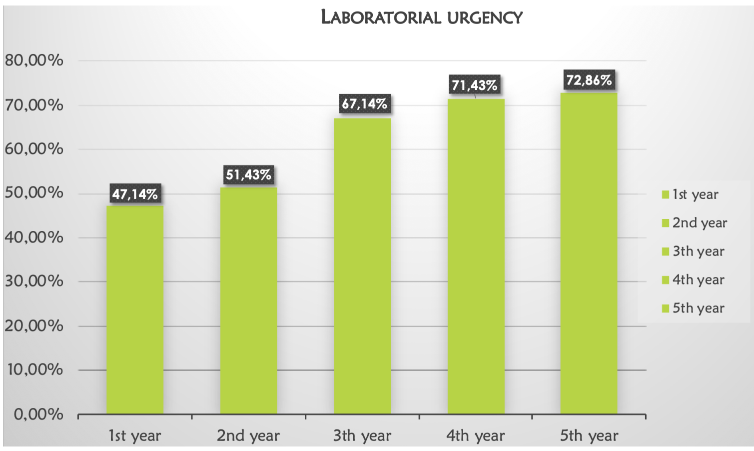


Figure 4.3: Answer to the first question: which year do you feel more a laboratorial urgency for?

[1]Data are updated to the 14th of August 2020.

In the histogram in Figure 4.3 the percentage of each beam refers to the to the number of teachers choosing the year of that beam with respect to the total amount of answering teachers. We see that they need the most to develop laboratorial activities for the last two years with an inverse trend (last year more urgency, first year less urgency).
Then we asked teachers for each biennium and for the last year:

- to assign a degree of importance for each scholastic treated argument of Physics to the development of laboratorial activities between *useless*, *not so important*, *uninteresting*, *enough important* and *fundamental*;
- to suggest other laboratorial activities;
- to tell us in which year of the considered time period they would have performed a given laboratorial activity (for the last year this question there was not);
- if they would have moved some laboratorial activity towards other years excluded from the considered time period.

Then there were other 8 questions about the nature of the laboratorial activities (as quantitative or qualitative experiment) and necessary material to carry on an experiment.
Finally we asked teachers to establish a ranking among all the possible laboratorial activities about the didactic arguments of Physics: the most important one, the second most important and the third one.
We find that teachers' first choice is to develop laboratories about "electromagnetic induction" (12.5%). The second most important laboratory is on "Special Relativity" (14.1%) and finally their third choice is *ex equo* "electromagnetic induction" and "Special Relativity" (11.1%). However we can see that the same argument has been chosen more than once by teachers. So we decide to give a clearer understanding of these data establishing a ranking that overlooks the teachers' one. For each argument we sum the number of teachers selecting it independently if it is their first, second or third choice.

Argument	Number of votes	Percentage
Special Relativity	20	10%
Electromagnetic Induction	19	9.5%
Maxwell's equations	16	8%
Quantum Mechanics	14	7%
Energy Kind of forces	13	6.5%

Table 4.1: Global ranking of teachers' whole choices. Not all the arguments are shown but only the first ones. For the complete one see Figure C.1 in Appendix C.

Table 4.1 contains the main result of the second questionnaire: overlooking from the order of importance given to each arguments, we found that teachers' greatest interest is towards the development of laboratorial activities about Special Relativity as well as electromagnetic induction.

4.2 Teaching-Learning Units

The data of the first questionnaire let us to understand that teachers could really be interested in laboratorial activities on Special Relativity. This was the first reason to have started developing our proposal.
While the instrument was designing we made up of the Teaching-Learning Units (UDA[2]): our project was initially structured with 10 units with a total amount of 30 hours addressed to the fifth year of high schools oriented to scientific studies (see Appendix C). They were thought to fulfil the requirements of the current laws in relation to didactic of Physics for High Schools ([43]) and to the final exam of Physics ([44]).
They can be gathered in 3 groups:

Group 1 - introduction (10 h): this first part contains a brief recap (UDA 1) about prerequisite of Classical Mechanics on Galilean transformations, together with an introduction to Special Relativity (the two postulates), to spacetime diagrams (UDA 2) and to Lorentz transformations (UDA 3).

Group 2 - phenomenology (6 h): this second group describes the main consequences of Special Relativity, namely loss of simultaneity (UDA 4), time dilation (UDA 5) and length contraction (UDA 6).

Group 3 - relativistic invariants (12 h) : the last group contains relativistic addition of velocities (UDA 7), the definition of interval (UDA 8), the light cone (UDA 9) and finally the resolution of twins paradox (UDA 10).

Going into the details of each Learning Unit:

UDA 1 - **Crisis of Classical Physics**: this UDA is a prerequisite to all the other units and it is based on the knowledge, the ability and competences students should have obtained in the second biennium while studying Galilean Relativity. In particular this learning unit contains a brief review of Galilean Relativity: it recalls the Galilean law of addition of velocities and it highlights the problem of incompatibility with the constancy of the speed of light when studying relative motions.

[2]Italian acronym for Unità Didattiche d'Apprendimento.

UDA 2 - **Principles of Special Relativity and spacetime diagrams**: in this UDA the principles of Special Relativity are presented, followed by Minkowski spacetime diagrams. It explains the meaning of a worldline with particular focus over the one of a moving observer and of an one at rest. This unit is fundamental as all our treatise of Special Relativity is based on spacetime diagrams.

UDA 3 - **Lorentz transformation**: this UDA explains in a spacetime diagram the idea of changing reference frames as adopting different points of view. In particular it is highlighted the change from one observer at rest to another one in uniformly motion with respect to him. Together with the previous unit it shows the connection between a reference frame and the spacetime globe. Then the principles of Special Relativity are presented again to introduce the Lorentz transformations between reference frames as the correct ones, fulfilling their requirements. Finally implications over the idea of absolute time, the non-relativistic limit and the constancy of light speed under Lorentz transformations are analysed. In this unit, the spacetime globe is used.

UDA 4 - **Simultaneity**: this unit deals with concept of simultaneity. It is highlighted its dependence upon the finite value of light speed and its non-relativistic limit. By the end it is described the process of synchronisation. In this unit, the spacetime globe is used.

UDA 5 - **Phenomenology - time dilation**: this unit shows the first consequence of Lorentz transformations, namely time dilation. It focuses in particular on the measurement process which is a key point in Einstein original article ([103]). The classic argumentation of muons' lifetime is presented as experimental evidence to support the phenomenon. In this unit, the spacetime globe is used.

UDA 6 - **Phenomenology - length contraction**: this unit contains the second consequence of Lorentz transformations, namely length contraction. As in the previous unit it is provided a wide space to the analyses of the measurement process. It is also mentioned the human perception of the phenomenon, namely how a moving observer looks at the "contracted" objects. In this unit, the spacetime globe is used.

UDA 7 - **Relativistic addition of velocities**: in this unit the correct relativistic formula for the addition of velocities is presented. The role of light speed is analysed, together with the limit towards high relativistic and low velocity.

Finally the problem of the acceleration and force transformation is introduced in order to explain the origin of the misunderstanding with the "relativistic mass". In this unit, the spacetime globe is used.

UDA 8 - **Intervals**: this unit focuses over the problem of determining the "true" time interval and length (namely the proper one) through an analogy with the Euclidean length and the rotation of segments in the Cartesian plane. Then proper time and proper length are presented as relativistic invariants. Finally it is introduced the interval as the distance in spacetime. In this unit, the spacetime globe is used.

UDA 9 - **Light cone**: in this unit different types of events are classified according to the sign of the interval. Past, present and future are described from a physical point of view starting from the definition of light cone. A particular emphasis is given to the physical meaning of present in relation to that one of simultaneity, due to the finite value of light speed. Finally causality and cause-effect relationship between different events are discussed.

UDA 10 - **Twin paradox**: the last unit deals with the twin paradox, with the complete description and solution through calculus of the trip's duration. In this unit, the spacetime globe is used.

4.3 Second introductory questionnaire

The learning units described in the previous section constituted our first didactic proposal for teaching Special Relativity with the spacetime globe. It deals with all the themes relevant from a didactic point of view and that are included in our current scholastic legislation in matter of the teaching of Physics. However it misses the description of the mass invariance as initially we did not figure how spacetime globe could be used to talk about energy. To understand the feasibility of this didactic proposal we needed to show it to teachers in order to have their feedback. The proposal of use the spacetime globe as a didactic tool was presented for the first time by the end of December 2020 to a group of fifty teachers during an online meeting of the training course "Fisicamente" for in-service high-school teachers organised by our Department in the context of the "Piano Lauree Scientifiche" national project. Together with the instrument the whole set of Learning Units were presented.

By the end of our talk teachers were asked to answer a questionnaire (see Appendix C) with 18 questions from which we received replies by more or less an half of the participants (22 answers). The sample is nearly fairly split up in teachers with a degree in Physics

(45.5%) and Mathematics (50%), with only one engineer: only one of them was teaching in a high school oriented to classical studies while the other ones (99.45%) in ones oriented to scientific studies.
Even though all the teachers appreciated our proposal and recognised its potential in order to achieve the learning purposes, a small part of them did not know (2 teachers) or did not want (1 teacher) to use it in his/her didactic at school.
Moreover[3] teachers thought that the use of a physical instrument can help the acquisition of the fundamental concepts (68.2%). The greatest part of them (77.3%) considered a strength of the proposed syllabus this different approach based on the spacetime diagrams and on the idea of the point of view.
However the weaknesses[3] were evident: our project required someone who has the instrument (45.5%) as it is not easily reproducible (it may cost thousands of euros). At the same time the use of graphical simulations as an alternative to the instrument could lead to a "failure" in the performance of the project (22.7%). It also demanded too many hours (18.2%).
We then asked if they were going to experiment our didactic trail (totally or in part) that year; the distribution of the answer is shown in Figure 4.4.

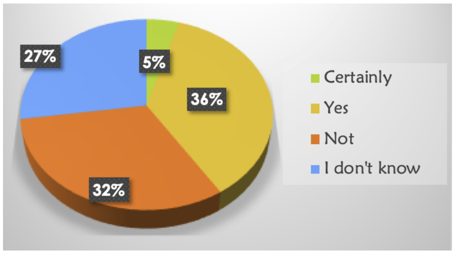


Figure 4.4: Distribution to question 11: are you going to experiment this didactic trail this year?

A number of eight teachers positively answered to the question, even if later on during the year not so many teachers joined our experimentation at school. On the contrary six teachers were in doubt whether to participate or not at the experimentation while seven teachers decided not to take part. An half of them (6 teachers) did not have a group of students of the fifth year while the others did not have time or were worried about how the scholastic year would have evolved[4]. As the teachers who answered negatively to the previous question about the feasibility of our project in their school also answered negatively to this question, it is possible that their response was valid for both the question.
We then asked teachers to have a forward-look, namely if they were going to use our didactic trial in the future. All the answers were positive for different reasons (some of them

[3]More then one answer was allowed to this question.
[4]It was the s.y. 2020/2021, the one after the first Italian lockdown due to CoronaVirus epidemic.

gave the same answers as to the previous question): teachers were interested in because of its originality and its potentiality in letting understand some very abstract concepts. At the same time it offered a different approach to Minkowski diagrams, even if one teacher thought that they were difficult, going too much deeper into the analysis.
An half of the teachers thought that our project could be proposed also in the last year of high schools oriented to classical studies (with some appropriate adjustments) while the other ones did not agree.
Overall teachers were interested in our proposal and in particular in the spacetime globe, thinking it could really help the didactic of Special Relativity, maybe also developing a vertical curriculum during the triennium (second biennium + last year).

4.4 Pilot experimentation

Starting from the result of the second submitted questionnaire in the first months of 2021, we began thinking of how really an experimentation could be carried on in high schools keeping in mind the emergency care with a compromise between the total thought length of the project and the necessary arguments to be treated.
We endorsed teachers' suggestions about length of our project. In order to structure a pilot experimentation that was well suited to teachers' necessities, we decided to focus over the themes of greatest interest from a scholastic point of view. This led us to structure two lessons two hours long.

First lecture

The first lecture begins with a briefly recap about relativistic aspects of the motion in Classical Dynamics through some videos showing flying airplanes from the inside of another flying airplane. We start introducing the idea of a point of view in the description of the motion near to the one of a reference frame. Then the Principle of Relativity is presented in the form of a covariance for the laws of Dynamics under Galilean transformations. Then a critical argument is introduced, namely the reference frame upon which the value of light's speed is defined leading to the necessity of a new principle of relativity and of new laws of transformations. Then Einstein's principles of Special Relativity are introduced together with Lorentz transformations as the right laws of transformation between inertial reference frames. Forthwith we present the spacetime globe as an instrument reproducing a Minkowski diagram: we explain what is a spacetime diagram, how observers are identified and how we can understand observers' state of motion looking to worldlines. The idea of point of view is shown comparing two spacetime diagrams depicting the state of motion of two different observers according to both of them with their representation through the spacetime globe. The mechanical movements from one point of view to an-

other one is explained to be a Lorentz transformation between the two reference frames without demonstrating its mathematical foundation.
Finally two consequences of Lorentz transformation are presented: loss of simultaneity (as in Section 3.3.1) and time dilation (as in Section 3.3.2) followed by the classical argumentation about muons' lifetime and examples given from twin astronauts. These descriptions were accompanied by the spacetime globe in the same way as shown in Section 3.3.

Second lecture

The second lecture starts with a recap of the main contents of the previous one. Then all the other relativistic phenomena are faced: using the spacetime globe as in Section 3.3 we investigate the length contraction (as in Section 3.3.3) with the Ladder paradox in one of its version with a train and a tunnel and the relativistic addition of velocities (as in Section 3.3.4). Finally as a consequence of this result, it is shown that the light's speed is preserved under Lorentz transformations (as in Section 3.3.5). Moreover we add a description of the visual effect of the length contraction (Lampa-Terrell-Penrose effect).

From the given description of our lessons with reference to Besson's classification [19] (see Section 1.5), we can understand that our approach is not a kinematic-geometric one as we actually do not introduce the invariant quantities and is not based upon the features of spacetime. This is also clear by the pattern of our Teaching-Learning Units: they have still the structure of the kinematic-algebraic approach following strictly the structure of Italian high school books. However it presents the main topics of Special Relativity through the spacetime diagrams using the spacetime globe, avoiding the introduction of light clocks and the resulting calculations. This is actually an aware choice as the critical period the Italian school was facing due to the pandemic epidemic caused many delay in the scholastic programs so that teachers could have been less inclined towards full new experimentation (as many of them told us in the final comments of the second questionnaire). Thus we decided just to introduce the spacetime globe within the normal Italian scholastic structure of the lectures about Special Relativity. Indeed we also decided not to include the description of the mass invariance with the spacetime globe as it could have required additional time.
However, despite the critical issues of the scholastic years some teachers decided to join our proposal. During spring 2021, we received three requests coming from three different schools of Rome: one teacher from the high school Cannizzaro, two teachers from the high school Virgilio and one teacher from the high school Democrito. Our experimentation involved an amount of five different groups of students, one coming from the first school, two from the second and two from the third. The lectures were not held in the same way: indeed three groups followed the classes in presence while the other ones remotely (show-

ing in a screen the live image of the spacetime globe). For the same previous reasons, four of these five groups already attended scholastic lectures on Relativity. Thus, for most of them our experimentation should be considered as an integration and a review.
Moreover as a central idea throughout all the lectures, we stressed the concept that Special Relativity deals with how events appear to different observers depending on their state of motion. Independently from Dimitriadi and Halkia ([13]), we substituted the idea of changing the reference frame with that one of adopting a different point of view.
We finally structured two questionnaires for the students following our lectures whose complete texts can be found in a drive folder (see Appendix C) together with their answers:

- a pre-test was given at the beginning of the first class; it is made up of twelve questions about Classical Mechanics concepts and Galilean Relativity: it is organised with seven multiple-choice questions and five open questions. In addition to those questions we also included other 3 questions about students' previous knowledge of Special Relativity. Four questions of the multiple-choice ones were taken from the Relativity Concept Inventory ([104]). We also looked to the Force Concept Inventory ([105]);

- a post-test was given at the end of the second class. It consists of thirteen questions about the main Special Relativity concepts, namely the ones investigated during the lectures. It is organised with nine multiple-choice questions and four open questions. Six of the nine multiple-choice questions were taken from the Relativity Concept Inventory ([104]).

4.4.1 Pre- and Post-test

Students were asked to choice a fictional name to identify their responses to the pre-test and to use it also for the post-test. Taking into account that some of the students were not present to both the lectures, we received a total amount of 95 answers to the pre-test with respect to the 85 answers to the post-test. A number of 77 students answered to both the tests.
In order to understand if our questionnaires were well calibrated we carried out two different analyses. The first one regards the multiple-choice questions: following Ding and Beichner [106] and Aslanides and Savage [104], we used classical test theory which provides a set of statistical indexes that estimate the discrimination and consistency of a questionnaire. *Discrimination* is the capability of a test to quantify the understanding of topic. *Consistency* refers to the ability of each question to measure the same broad understanding. The analysed indexes can be gathered into:

- indexes who measure a feature of a single question:
  - the *difficulty* P_i for each question

$$P_i = \frac{N_i}{N}, \tag{4.1}$$

defined as the ratio between the number N_i of the correct answers to the i-th question and the total amount N of the answers to that question. It is a measure of the questions' easiness: the greater the difficulty, the lower the easiness;

  - the *discrimination index* D_i for each question

$$D_i = \frac{N_H - N_B}{N/4}, \tag{4.2}$$

that takes a normalised difference between the correct answer to a question i in the top quartile N_H and the bottom quartile N_B of students. It is a measure of the questions' capability of distinguishing students with high results from those with low results;

  - the *point biserial coefficient* r_{pbi} for each question

$$r_{pbi} = \frac{\langle X_{r,i} \rangle - \langle X_{w,i} \rangle}{\sigma_x} \sqrt{P_i(1 - P_i)}. \tag{4.3}$$

Here, $\langle X_{r,i} \rangle$ is the mean value of the total score for those students who correctly answered to question i, $\langle X_{w,i} \rangle$ is the mean value of the total score for those students who incorrectly answered to question i, σ_x is the standard deviation of total scores while P_i is the value of difficulty index for question i. This index measures the correlation between each item score and the total score for the inventory.

However, in order to get an overall view about these three indexes we evaluate also their mean value (as in [104]): thus we have also a mean difficulty index, a mean discrimination index and a mean point biserial coefficient. The error associated is given by the standard error of the mean

$$\sigma_{\bar{y}} = \sqrt{\frac{\sum_i (y_i - \bar{y})^2}{N(N-1)}}, \tag{4.4}$$

where y denotes one generic index, y_i its values and $\bar{y}$ its mean value.
This also allows us to give also a statistical uncertainty to the values of each index for each

question, namely the standard deviation σ_y, obtained from Eq. (4.4) as

$$\sigma_y = \sqrt{N}\sigma_{\bar{y}} = \sqrt{\frac{\sum_i (y_i - \bar{y})^2}{N-1}}. \tag{4.5}$$

Indeed this might be not the correct uncertainty as we should know the real distribution of the answers among the different questions.

- indexes who evaluate the test as a whole:
  - the *Kuder-Richardson reliability index* r_{test} or KR-20 index

    $$r_{test} = \frac{K}{K-1}\left(1 - \frac{\sum_i P_i(1-P_i)}{\sigma_x^2}\right). \tag{4.6}$$

    Here, K is the number of questions, P_i is the difficulty value for the i-th question and σ_x is the standard deviation of total scores. It measures how the questions of the test are correlated each other, namely the internal consistency of the test. As we associated to the value of P_i an uncertainty, we can give to Kuder-Richardson reliability index an uncertainty too (see Appendix C.1):

    $$\sigma_{r_{test}} = \frac{K}{K-1}\frac{\sigma_P}{\sigma_x^2}\sqrt{\sum_i (1 - 2(P_i - P_i^2))(P_i - P_i^2)^2}; \tag{4.7}$$

  - the *Ferguson's delta* δ

    $$\delta = \frac{N^2 - \sum_i f_i^2}{N^2 - N^2/(K+1)}. \tag{4.8}$$

    Here, N is the total number of the student answering to the test, K is the number of the questions and f_i is the number of students with total score i. It is a measure of the distribution of the student's scores over all the possible value. Greater the broadness, better is the capability of the test to discriminate among students at different levels.

As far as the open questions are concerned, the answers have been gathered according to three epistemological profiles: everyday, meta-scientific and scientific:

- everyday: this kind of answer reflects the creation of situational meanings derived from everyday contexts or common knowledges and sense;
- meta-scientific: the student does not explain on the basis of a functioning model (microscopic/macroscopic) the causal relationships of the physics parameters involved. The student merges common sense to some scientific notions taken from scientific

context (lectures, books, mass media...). Thus spontaneous schemes are framed into scholastic reasoning.

- scientific: the student proposes a model (qualitative and/or quantitative) based on a cause/effect relationship or provides explanatory hypotheses introducing models that can be visualised at a theoretical level.

They summarise three different ways of answering strategies corresponding at three different levels of capability in dealing with problems. This kind of analysis allows us to distinguish between common knowledge and scientific knowledge (Besson [68]) plus a meta-scientific knowledge.

4.4.2 Pre-test analysis

Our pre-test is made up of 12 questions, being 7 of them a multiple-choice ones and the remaining 5 open ones.
In particular:

- the first two questions are about the detection of absolute motion: one is a closed question and the other open;
- the following two questions are closed ones about the understanding of the path followed by a ball in a moving train according to different observers;
- the next three questions are about the relativity of the motion in Classical Dynamics thinking on relative velocities: the first two are open ones while the other closed;
- the following two questions are about absolute time in Classical Dynamics. One is a closed question and the following open;
- the following two questions are closed ones about simultaneity in Classical Dynamics;
- the last question is an open one about the existence of a maximum speed.

In addition to those questions there are also other 3 questions about students' previous knowledge of Special Relativity.
We separately analysed multiple-choice questions and open ones.

Multiple-choice questions statistical analysis

As far as the seven multiple-choice questions are concerned, we analysed the indexes described in Section 4.4. For all the indexes, the desired values are taken from Ding and

Beichner [106].
We show two tables: in the first table, Table 4.2, we display the values for all the first three indexes (P_i, D_i and r_{pbi}). The associated errors are always obtained from Eq. (4.5).

Item	Difficulty index	Discrimination index	Point biserial coefficient
Question 1	0.7 ± 0.2	0.2 ± 0.1	0.44 ± 0.06
Question 3	0.3 ± 0.2	0.4 ± 0.1	0.54 ± 0.06
Question 4	0.3 ± 0.2	0.4 ± 0.1	0.40 ± 0.06
Question 7	0.6 ± 0.2	0.4 ± 0.1	0.44 ± 0.06
Question 8	0.5 ± 0.2	0.4 ± 0.1	0.47 ± 0.06
Question 10	0.6 ± 0.2	0.2 ± 0.1	0.50 ± 0.06
Question 11	0.9 ± 0.2	0.1 ± 0.1	0.34 ± 0.06

Table 4.2: Value of the difficulty index, discrimination index and point biserial coefficient. Sample size $N = 95$ students. Errors are calculated from Eq. (4.5).

The second table is Table 4.3 where we summarise the mean value of each index of Table 4.2 together with the value of the reliability index and the Ferguson's delta.

Test statistics	Pre-test values	Desired values
Mean difficulty index	0.56 ± 0.08	$[0.3, 0.9]$
Mean discrimination index	0.31 ± 0.04	> 0.3
Mean point biserial coefficient	0.45 ± 0.02	> 0.2
Reliability index	0.35 ± 0.05	> 0.7
Ferguson's delta	0.91	> 0.9

Table 4.3: Pre-test statistics. Sample size $N = 95$ students. Errors are calculated from Eq. (4.4). The desired values are taken from Ding and Beichner [106].

At a very first glance we see that not all the values of the indexes are well suiting their range of validity. Only the value of the mean difficulty index and the mean point biserial coefficient are inside their desired range while the mean discrimination index and Ferguson's delta are borderline. In order to perform a complete analysis, in Figure 4.5 we plot the value of difficulty index, the discrimination index and the point biserial coefficient index for each questions at once with the lower desired bound.

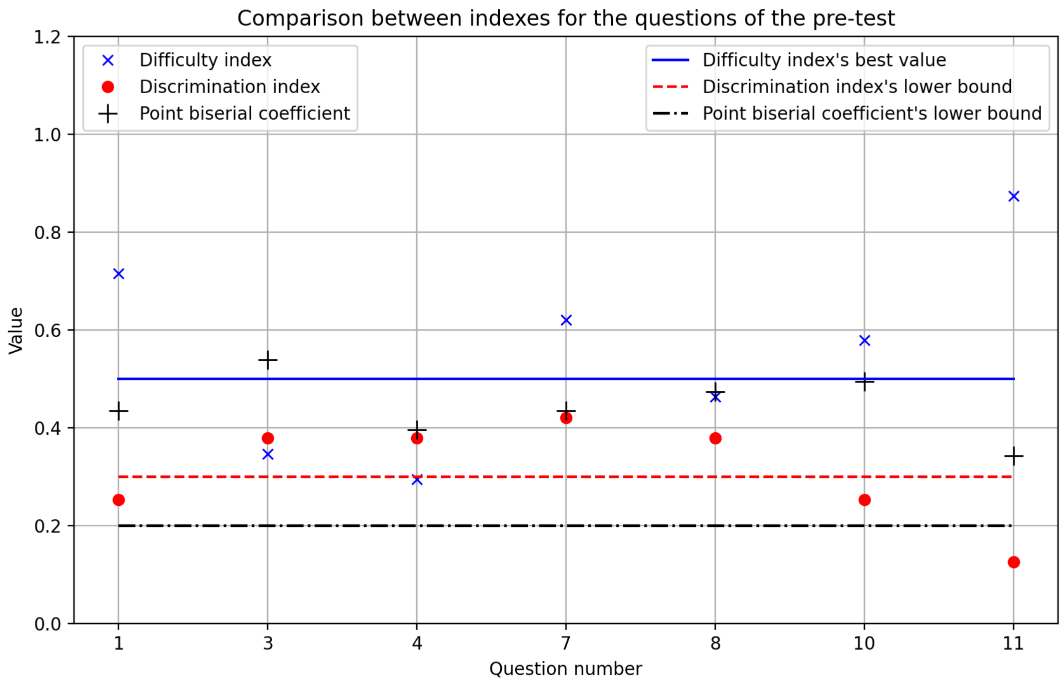


Figure 4.5: Comparison between indexes difficulty (blue ×), discrimination (red ∘) and point biserial coefficient (black +). The related lower bound is the blue solid line (-), the red dashed line (–) and the black dash-dot line (-.).

From the mean difficulty index (4.3) it seems that the difficulty of the pre-test is acceptable as a value of $\langle P \rangle = 0.51 \pm 0.08$ indicates that on average the questions are not either too difficult or easy. However this is not true for all the items of the inventory. Figure 4.5 suggests that questions 3, 4 and 8 may be too difficult, questions 1 and 7 a little bit easier than the other ones while question 11 too much easy. As quoted by Ding and Beichner [106] a reasonable value for the difficulty index should be 0.5 for all the items.

Questions 3 and 4 regard the path followed by a ball thrown out of a moving train (Figure 4.6).

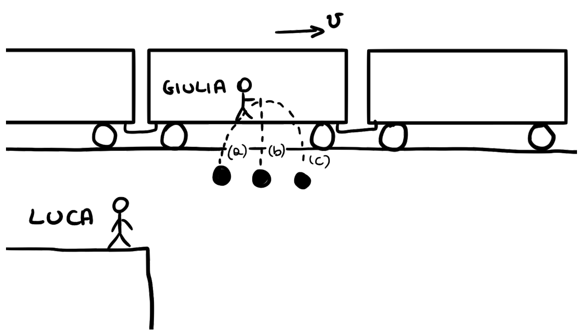


Figure 4.6: Which is the path of the ball according to Luca (question 3) and Giulia (question 4)?

The contents of these questions are treated during the high-school Physics courses. They are a bit tricky questions but students should have sufficient abilities to correctly answer. Question 3 had random answers being equally distributed among the three choices.
Answers to question 4 are in proportion of ~2/3 wrong to ~1/3 right: we guess that students may be misled by thinking of following the ball. As the train goes away from the object, while it is falling, it may appear to Giulia that path (a) is the correct one. We address that for both the questions the idea of trowing the object out of the train rather then inside the train could have deceived the students. Question 8 (the only other one with a low difficulty index) will be discussed in the next section.
An additional consideration regards questions 10 and 11.

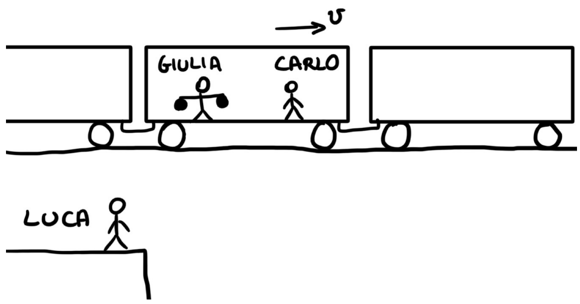


Figure 4.7: In what order do the balls hit the ground according to Luca (question 10) and Carlo (question 11)?

Students seem quite confident that inside the train (question 11) the balls hit the ground at the same time: this is confirmed by the high value of the difficulty index. But it is not the same for question 10, namely adopting the point of view of the external observer Luca. The difficulty index decreases showing that for many students simultaneity is not preserved: as for question 4, ~1/3 of the students thinks that the balls do not hit the ground at the same time.
Question 1 was taken by an online document of a Rome's high school ([107, pg. 807]) and was about the chance of detecting absolute motion: students easily answered that aboard a moving car one can not detect motion even if their justifications to the answers do not show a scientific reasoning.
To evaluate the discrimination index, following [106], we defined as internal criterion to gather the students their score to our multiple-choice test. Thus we created four groups: the top quartile included students with score $\geq 6/7$, two middle groups students with score $[5/7; 6/7)$ and $[3/7; 4/7]$ and finally the bottom quartile students with score $\leq 2/7$.
The mean value of the discrimination index (Table 4.3) $\langle D \rangle = 0.31 \pm 0.04$ is just above the desired value of 0.3. A low value of the mean discrimination index may suggest that on average our items are thinly able to distinguish the high-achieving students from the

low-achieving ones. From Figure 4.5 we see that the values of the discrimination index of the questions are equally distributed over and under the lower bound, close to it. Ding and Beichner [106] suggested that the reason may consist in having misunderstood the request or in an extreme value of the difficulty (too high or too low). The dependence on the difficulty may be consistent within our analysis especially for questions 1, 3 and 4 as well as for the couple 10-11.
The last index to analyse single items is the point biserial coefficient. From Table 4.3, we see that its mean value is acceptable being over the lower bound threshold. It shows that each item is consistent with the other ones in the test. The test is internally coherent in investigating concepts of Classical Physics.
However, the Kuder-Richardson reliability index (Table 4.5) is completely out of range, suggesting that the test may not have a strong internal reliability. As suggested by Aslanides [25], this means that the items of our test are not well connected.
The Ferguson's delta value is just above the threshold one. The test has an acceptable potentiality in discriminating among students at different levels, even if the whole analysis suggests that our items should be reviewed, mainly trying to level out the difficulty.

Open questions analysis

The five open questions asked to explain whether it was possible or not that something happened. They are related to relativity in Classical Dynamics. We now present for each question a short description of the analysis: the titles given to each question are not the same of the test. They just summarise the questions' content: to read the full questions, see Appendix C.

Q. 2: try to explain whether it is possible or not to be aware of the own state of motion being inside a moving car with no glasses.
This item is related to question 1: in that question students had to say if it was possible or not to be aware of the own state of motion being inside a moving car with no glasses. In question 2 students were asked to motivate their choice.
From our analysis we have to exclude 15 answers (~16% of the total answers) corresponding to not given or meaningless answers. Among the remaining 80 students, most of them (~66%) gave common answers related to sensory experience (seeing, hearing, perceiving); we can gather them into four different groups:

- ~30% of the answers concerns the possibility of perceiving the motion through acceleration or the presence or the absence of some kind of movement within the car. This group lacks of cause/effect relationship between acceleration and the motion inside a non-inertial reference frame, namely the chain acceleration-fictitious force-movement. The students' answers show that they ascribe directly to the (non-inertial) motion within the car the main source of detection of its state of motion without mentioning inertial acceleration. Furthermore this reasoning is not related in any way to the principle of inertia.

- ~21% of the answers simply states that as they are not able to see outside, it is simply impossible to be aware of the car's state of motion.
- ~30% this group is a sort of evolution of the previous one: yet students are not able to detect the motion because of the absence of glasses inside the car but they explicitly wrote about the lack of external points of reference. This detail is a precursory notion of reference frame.
- ~19% the answers of this final group refers to some kind of "reference frame" (external or internal), even if this idea is not always used appropriately. Either they think not having a reference frame or, as they move within a reference frame, they do not perceive the motion.

Another ~23% of the students gave an answer that has still some features related to common sense but also scientific aspects. The detection depends upon the state of motion of the car, namely if its motion is accelerated or not, and the sensory effect upon the body. Two students thought to perform an experiment but these answers should have been completed with more details as for instance adding that if the car was moving with constant speed, even an experiment would have given no result. One student just used the principle of inertia without giving a real explanation about it and finally another answer is related (in some not clear way) to Mach's principle. This item is particular as the student wrote "*There are not points of reference (Mach's principle)*". This let us think that also the previous common answers about the absence of points of reference could be linked to some of the students' understanding of Mach's principle (3 groups of students over the five constituting the examined sample did deal with Mach's principle in their lessons). This idea is supported also by some answers to question 6 (addressing a similar topic), containing references to this principle.

Finally a ~11% of the students shows some kind of scientific reasoning for the question: they distinguished between an accelerating car or not, some of them explaining the effect of acceleration in terms of forces. Indeed these answers do not differ a lots with respect to the previous ones (meta-scientific) but we choose to classify them as scientific because of their use of a more appropriate vocabulary and because they gave more precise explanations.

None of the answers to this question refers to Galilean principle of Relativity: this is however not surprising at all. As seen in Section 1.3, some authors inferred about the failed full grasps of Classical Relativity: most of the students still relies upon sensory experience or uses a reasoning arising from common experience (the lack of the effects of non-inertial forces). Independently from considering possible to detect uniform motion (question 1), they do prefer to use a common or dynamical explanation rather than a law and this agrees with the research of Ramadas et al. [12], stressing that Galilean principle is just *a "cliche" to remember.*

Q. 5: explain why if you are in a moving train and another one passes near you at the same speed but in the opposite direction, it seems to have a higher speed.
As before we invalided 12 answers (~13% of the total answers), thus remaining with a sample of 83 answers. Among the remaining students, most of them (~71%) gave a common answer that can be expressed as simply "Speeds are summed" (or doubled). A little part of them (~39%) also specified that this is due to the fact that the two trains are moving in opposite directions.

As far as the meta-scientific answers are concerned (~29%), we can identify different aspect: the greater part of them (~71%) referred to the presence of reference frames, while other 3 students thought in terms of vector addition of velocities, one student cited the *Dialogue Concerning the Two Chief World Systems* (maybe remembering Galilean Relativity) while the last one referred to the presence of relative motion. Finally other two answers correctly identified in the composition of velocities the origin of this phenomenon and the subsequent sum of their modules.

No truly scientific answers were found among these replies as, regardless of the kind of answers given, there is the general tendency to treat this phenomenon as an apparent one. Indeed the ~52% used expressions as *it seems*, *I see* and *it appears*. One student explicitly wrote "*one get the illusion that*" while another one wrote that the doubling of the speed "*is not true*". The major part of the students completely neglects that this effect can be measured and it is not just an illusion: students are implicitly assuming the existence of an unique "real" (then absolute [13]) reference frame according to which the speed of the second train would not be high. This result is quite surprising if we consider that for instance the 81% of this cluster of students stated it was impossible to detect uniform motion (question 1). As Panse et al. [11] observed, this is a recurring misconception about reference frames.

Q. 7: what do we learn about relative and absolute motion from the experience of misunderstanding whether it is our train which is departing or the one next to us?

From the answer to this question we need to exclude 34 items (more o less one-third of the total answers), thus shortening our sample to 61 answers.

The common sense answers (~26%) involve explicitly eye's dependence of the motion, namely the fact that our eyes are used to follow the moving object, as well as the impossibility of detecting the motion and a change in the perception of time and of the spatial coordinates. Finally a consistent group of students (the half of them) found a common way to express the reason of the questioned phenomenon that is "it is a matter of relativity".

Meta-scientific answers (~72%) show that students well understood that there is something observer-dependent. We can identify some clusters:

- students referring to the perception and observation of the motion which is observer-dependent;
- students referring to the motion which is observer-dependent;
- students referring to the Mach's principle to use a third reference frame to establish who is really moving.

Only one answer can be categorised as scientific: it well contextualises the phenomenon, using appropriate words.

This question was in same way different with respect to the other ones as it was not addressed to find out students' knowledge or ability but rather their competences. Indeed we can not identify a "correct answer", namely some scholastic references in the textbooks, while instead students are asked to apply their knowledges and abilities to a real-life situation. The answers are towards the right understanding of the phenomenon but perhaps it is necessary to go further, trying not to overlook the sensitive experiences ("to see the train moving") but to support them scientifically, thinking in terms of experiments. Indeed students tended to consider the perception of motion or motion itself as depending on the adopted reference frame: they did not mention that there are some physical parameters that defines it and that can be measured in order to obtain information about it.

Q. 9: try to explain if the taken time of an object to fall down inside a moving train is the same as measured from outside.

This item is related to question 8: in that question students have to say if an observer outside a moving train, seeing at an object falling down inside, measures a greater, equal or smaller time with respect to an observer inside the train. We obtained that ~49% of the students agree that it requires always the same time. A 20% thinks that the falling time is less aboard the moving train then outside and another ~19% that is grater. Finally the remaining students either do not know what to answer (~10%) or think that it depends on the speed of the train (~2%).

In question 9 students were asked to motivate their choice. From the analysis we exclude 25 items (~26%), being them answers like "I don't know" or similar.
Among the remaining 70 items, one half is completely common sense answers related to air resistance, intuition or general sentence as "time does not change", actually not replying to the question. A consistent part of the common sense answers (nearly ~51%) are linked to the concept of velocity: independently from the given answer, they ascribed it to the fact that the speed of the falling object is summed to the one of the train.
Among the meta-scientific answers (~47%) we can find references (~33%) to time dilation with a clearly misunderstanding of the question, even if someone of students did not give the "correct" answer to the previous question (time is shorter). We can find another group (~15%) similar to this one mentioning Special Relativity but contextualising their answers: the time interval is the same, even if there is a small variation according to Special Relativity that could had been a higher effect if the train would have moved faster. Then another cluster (~27%) again answered that time is invariant but they detailed more with a scientific-like language. The last group (~24%) agreed that time is greater as the trajectory is wider because of the train movement from the external point of view. The laying misconception that in projectile-motion the time of falling depends on the x-motion is present in this group as well as in the common sense answers.

Finally we identified two scientific answers (~3%) that correctly linked the time of falling to the vertical motion which is determined uniquely by the gravitational acceleration and is not influenced by the speed of the train. However it lacks the more general frame of time invariance in Classical Mechanics.

Q. 12: if a car has a very performing engine, being $v = at$, waiting for as much time as required, we can reach each desired speed. Is it true or not?
This is the last question of the pre-test (excluding the three following ones exploring students' previous knowledges of Special Relativity): we remove 22 answers stating "I don't know", 6 other answers stating "It is correct" and two answers stating "It is not correct".

Thus we remained with a sample of 65 (~68%) replies.

Between the common answers we identified two groups equally distributed:

- a cluster of 9 answers (~53%) reported that the correctness or not of the question depended on the limits of the engine itself;
- another cluster of 8 answers (~47%) reported that the question was true/wrong due the absence/presence of the friction. One reply identified that air friction increases with speed, a more scholastic feature.

The remaining 47 answers are all meta-scientific (~64%). They are more o less equally divided into two major groups and another small one:

- the first group of 23 items (~49%) replied that the reasoning is wrong due to the existence of a limit speed, namely the light's one. Most of the answers seems to be truly scholastic: it would be interesting to investigate whether or not this conception is actually already known before instruction in order to understand if it is a common knowledge;
- the second group of 19 items (~40%) replied that, according to the formula $v = at$, the reasoning is correct, being the speed proportional to time and to a constant acceleration. It is interesting an answer among the other: "*Actually, the speed of light can not be exceeded but if the car has an infinitely powerful engine, I can say that it can also exceed it*". It can be considered a prototype of this kind of answers: students seemed to considered the law as a kind of truth, not actually questioning it in relation to the real world. Nevertheless it shows a correct understanding of the formula.
- the last group of 5 items (~11%) replied that, being the car a massive body, it can not reach the speed of light.

Among the answers to this questions it was not possible to find a complete scientific ones. Firstly no one contextualised the replay into the Classical Mechanics frame. Actually some items inside the second group of meta-scientific ones have a more appropriate scientific language but a complete answer should include both the reasoning on the inertia effect of the acceleration and the consequent energy supply for the car.

General questions about Special Relativity

By the end of the pre-test students were asked to answers to three questions about Special Relativity, namely what they knew about it, where they heard about Special Relativity and

where they looked for information about it.
The 40% of the students did not replay to the first question, saying that they knew nothing or less. Another half of the students (~52%) gave answers that can be clustered into different groups (which has to be not considered as rigid categorises as sometimes there is a crossover between two of them):

- it is a theory developed by Einstein;
- it is a theory dealing with inertial frames;
- it is a theory based on two principles;
- it is a theory that started from or overtook Classical Mechanics;
- it is a theory dealing with space and time and their nature according to different observers (contraction, dilation and so on);
- it is a theory trying to solve the conflict between Classical Mechanics and the Theory of Electromagnetism;
- it is a theory that does not deal with gravitation which is treated by General Relativity.

There is then another small group of answers (4) that in some way list arguments from more then two different categories. However all these group of answers are pointing out to different true aspects of Special Relativity and perhaps we argue that they can be considered as what remained etched in students' memory after lectures on Special Relativity[5].
Finally a group of five students gave answers which are more complete then the other ones: Special Relativity is a theory born in the context of the conflict between Electromagnetism and Mechanics and, starting from two principles, it is able to describe the phenomena of reality at high speeds according to different inertial frames and that reduces to Classical Mechanics at low speeds.

Then students were asked to tell us where they heard about Special Relativity before attending lectures at school (if they did). Over 95 answering students, a quarter of them never heard about it before school while the remaining 75% did. They could choose between social networks (Facebook, Instagram, Twitter...), YouTube videos, newspaper, science books for general public and television shows. Figure 4.8 depicts the distribution of their answers.

[5] As we said, our groups already attended lessons on Special Relativity.

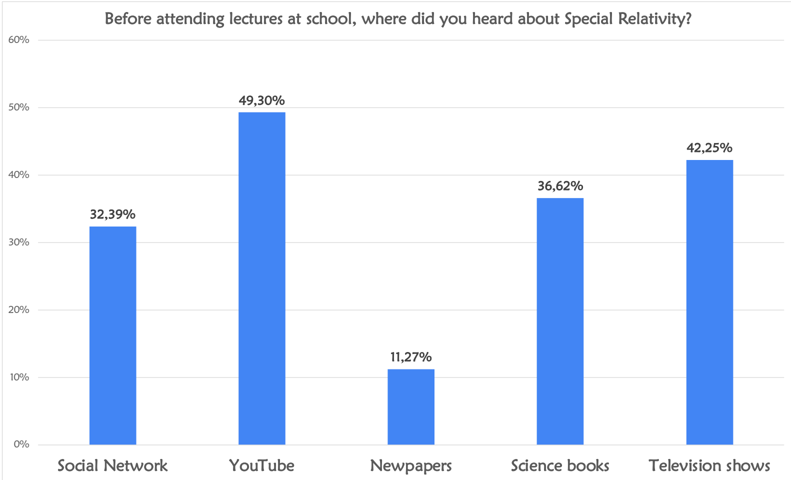


Figure 4.8: Distribution of students' source of information about Special Relativity.

In the histogram in Figure 4.8 we report the percentage with respect to the total amount of students without the ones who never heard about Special Relativity. The sum of each bins is not normalised to 100% as more choices were allowed, thus almost all the students received information from more then one source.

In the last question students were asked to answer if they searched for more information about Special Relativity on their own. Differently from the previous question, now the total amount of students who did not search for more information are the ~39% of the total 95. The remaining ~61% of the students had to choose between social networks, web pages on Internet, YouTube, newspaper, science book for general public and television shows. In Figure 4.9 is shown the distribution of their answers.

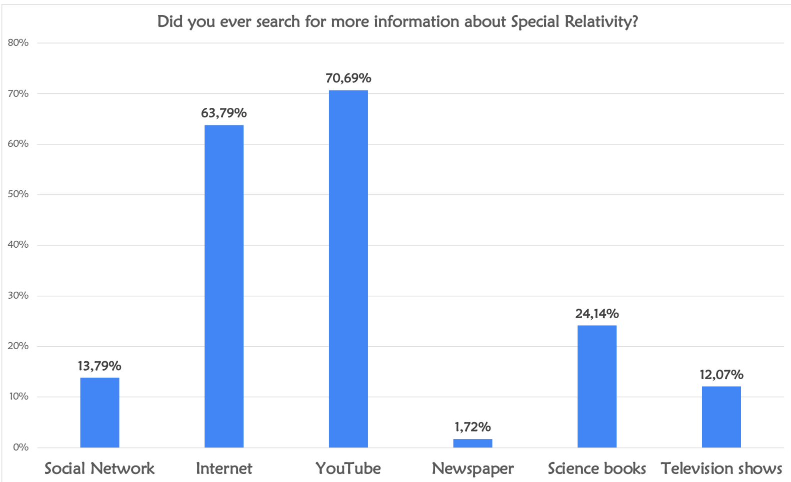


Figure 4.9: Distribution of students' own researches about Special Relativity.

As we can see, the preferred channel of information is still YouTube rather then the surfing on web pages, even for personal researches. In the chapter ***Conclusion*** we will discuss more in detail the consequence of this information.
Finally we make a comparison between this question and the previous one; we saw that:

- a ~15% of the students neither heard about Special Relativity nor searched for more information;
- a ~6% of students who never heard about Special Relativity searched for more information: maybe scholastic lectures intrigued them;
- a ~19% of the students who new about Special Relativity did not continue searching for more information;
- finally the remaining ~58% of the students already heard about Special Relativity and kept on looking for more information.

This splitting suggest that each one of them may correspond to a certain degree of interest in Special Relativity and thus each time we have a didactic action the students' responses may vary according to it, namely more o less one out of 7 students probably will follow a lecture with less attention.

4.4.3 Post-test analysis

Our post-test is made up of 13 questions, 9 of them are a multiple-choice ones and the remaining 4 open ones.
In particular:

- the first two questions are closed questions about the loss of simultaneity;
- the following four questions are about time dilation: two of them are multiple-choice questions while the other two are open questions;
- the following five questions are about length contraction: three of them are closed questions while the other three are open ones;
- the last two questions are multiple-choice ones about relativistic velocity-addition.

We separately analysed multiple-choice questions and open ones.

Multiple-choice questions statistical analysis

As far as the nine multiple-choice questions are concerned, we analysed the indexes described in Section 4.4. Once again, the desired values are taken from Ding and Beichner [106].
As before, we use two table: in the first one, Table 4.4, we display the values for all the first three indexes (P_i, D_i and r_{pbi}). The associated errors are always obtained from Eq. (4.5).

Item	Difficulty index	Discrimination index	Point biserial coefficient
Question 1	0.9 ± 0.2	0.8 ± 0.1	0.3 ± 0.1
Question 2	0.9 ± 0.2	0.7 ± 0.1	0.2 ± 0.1
Question 4	0.5 ± 0.2	0.8 ± 0.1	0.4 ± 0.2
Question 6	0.6 ± 0.2	0.9 ± 0.1	0.5 ± 0.2
Question 8	0.5 ± 0.2	0.9 ± 0.1	0.5 ± 0.2
Question 10	0.7 ± 0.2	1.0 ± 0.1	0.6 ± 0.2
Question 11	0.6 ± 0.2	0.9 ± 0.1	0.6 ± 0.2
Question 12	0.5 ± 0.2	0.7 ± 0.1	0.4 ± 0.2
Question 13	0.6 ± 0.2	1.1 ± 0.1	0.7 ± 0.2

Table 4.4: Value of the difficulty index, discrimination index and point biserial coefficient. Sample size $N = 85$ students. Errors are calculated from Eq. (4.5).

In the second table (Table 4.5) we summarise the mean values for each index of Table 4.4 together with the value of the reliability index and the Ferguson's delta.

Test statistics	Post-test values	Desired values
Mean difficulty index	0.66 ± 0.05	$[0.3, 0.9]$
Mean discrimination index	0.90 ± 0.04	> 0.3
Mean point biserial coefficient	0.47 ± 0.05	> 0.2
Reliability index	0.59 ± 0.02	> 0.7
Ferguson's delta	0.94	> 0.9

Table 4.5: Post-test statistics. Sample size $N = 85$ students. Errors are calculated from Eq. (4.4). The desired values are taken from Ding and Beichner [106].

In Figure 4.10 we plot the values of difficulty index, the discrimination index and the point biserial coefficient index for each questions at once with the lower desired bound to have an easier comparison.

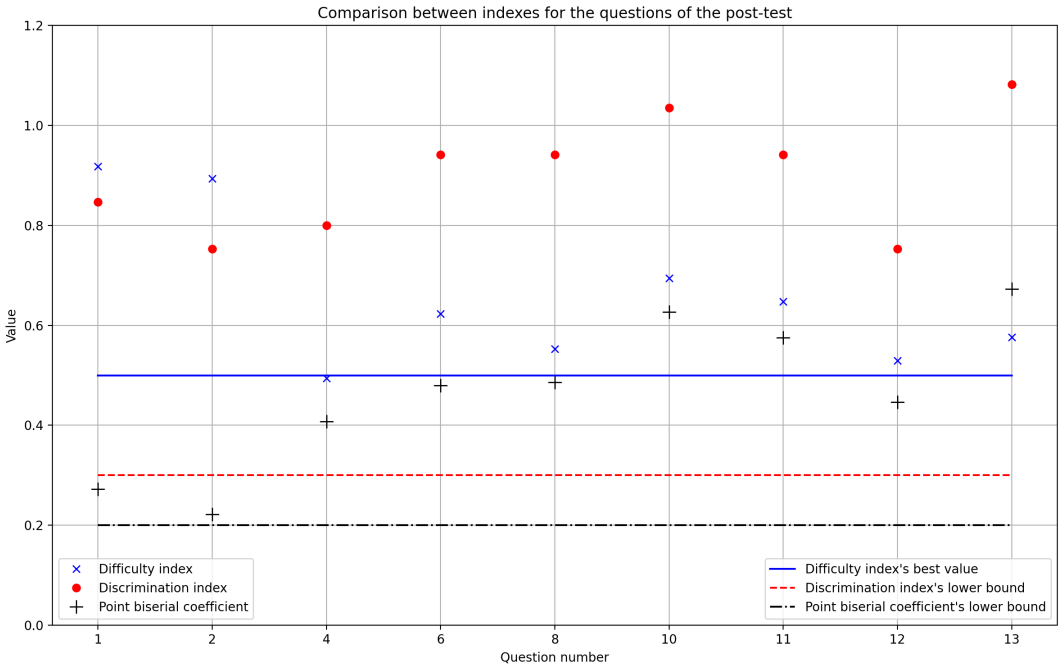


Figure 4.10: Comparison between indexes difficulty (blue ×), discrimination (red ∘) and point biserial coefficient (black +). The related lower bound is the blue solid line (-), the red dashed line (–) and the black dash-dot line (-.).

From Table 4.5 we see that our average difficulty value over the entire test is $\langle P \rangle = 0.66 \pm 0.05$, suggesting that questions are rather easy than difficult.

Then to evaluate the discrimination index we again defined as internal criterion to gather the students their score to our multiple-choice test. Thus we created four groups: the top quartile for students with score $[8/9; 9/9]$, two middle groups for students with score $[6/9; 7/9]$ and $[4/9; 5/9]$ and finally the bottom quartile for students with score $[2/9; 3/9]$. The mean value of the discrimination index (Table 4.5) is $\langle D \rangle = 0.87 \pm 0.04$ well above over the desired value of 0.3, thus suggesting that the questionnaire is able to discriminate high-achieving students from low-achieving ones.

From the comparison between difficulty and discrimination indexes we can conclude that actually questions from the third to ninth well discriminate top quartile students from low ones: the mean value of difficulty index for this subset is $\langle P_{[3 \div 9]} \rangle = 0.59 \pm 0.02$. This shows that actually question 3 to 9 are neither too easy nor too difficult with respect to standard value of 0.5 (blue solid line): thus the high value of D index (with respect to accepted value in red dashed line) indicates a good discrimination. As for questions 1 and 2, we can see from Figure 4.10 that we may expect a lower discrimination index due to their high difficulty index, actually indicating a low difficulty for these questions with respect

to the other ones. For these reasons, a revision of questions 1 and 2 may be considered.
The mean value of point biserial coefficient (Table 4.5) is $\langle r_{pbi} \rangle = 0.47 \pm 0.05$ above the desired value of 0.2: this means that each item is reliable in its request.
However as we can see from Fig. 4.10 also for the r_{pbi} value questions 1 and 2 are too close to the lower bound (black dash-dot line) with respect to the other items. It indicates that these two first questions do not test the material at the same level of the other ones.
Finally we evaluate the entire test with Kuder-Richardson reliability index (r_{test}) and Ferguson's delta (δ). From Table 4.5 we see that the first index is the only one that has a value under the desired one, namely $r_{test} = 0.59$ (threshold = 0.7). We can infer that this is due to the fact that we selected a group of items from the Relativity Concept Inventory which, if considered as a whole, is reliable ([104]). As a consequence our restricted subset may not be well connected. It is necessary to review items with too low discrimination index and point biserial coefficient.
The value of the Ferguson's delta (Table 4.5) is greater than the lower bound, thus providing a good discrimination among students.

Open questions analysis

The four open questions are two about time dilation and two about length contraction. We now present for each question a short description of the analysis: the following titles given to each questions are not those of the test.

Q. 3: how would you summarise time dilation phenomenon?
From our analysis we have to exclude 27 answers (~32% of the total answers) corresponding to a not given or meaningless answers and other 13 (~15% of the total answers), copied from Internet.
Among the remaining 45 students, more then one half of them (~61%) gave common answers and we can distinguish different alternative thoughts:

- ~19% of students simply stated that "*the greater is the speed, the slower the time flows*" which can be regarded as a very popular way to summarise Special Relativity;
- ~44% of students stated that time depends on the reference frame which is very similar to the previous one but the answers contain an additional element, namely the reference frames. However there is not any statement whether they are moving or at rest. It seems to be an intrinsic property of the reference frames;
- ~33% of the students looked to time dilation as a property only of the moving reference frame, thus stating that time gets dilated only according to the moving observer.

We also noticed that one student among the previous ones referred to human perception of time as changing (*time's perception changes* [...]) perhaps forgetting to consider that physical time can be only measured. Another one alluded to time dilation as an apparent phenomenon, depending on the reference frame adopted (*time* [...] *seems to be greater* [...]). One student also mentioned that it is a property of a body to modify time, dilating it: maybe he got confused with General Relativity and with masses' property of bending spacetime. Finally we report a third answer stating that *in different reference frames time can dilate*: it is no clear what this chance depends on.

Another ~18% of the total students gave an answer that has still some features related to common sense but also scientific aspects. An half of the students linked time dilation to the principles of Special Relativity (in particular the constancy of light speed), however not really explaining the meaning of the phenomenon itself. Another student started from loss of simultaneity to explain that again time *perception* is no more universal. Other two students wrote about measuring time and the result one obtains according to different observers but with some vagueness (one of them did not explicit how it varies while the other referred only to high speed). The last answer is out of these kind of replies: the student still alluded to a difference between local time (maybe the one of the observer we are referring to) and absolute time because of the presence of the term v/c in Lorentz transformation. Similar to some consideration of Scherr et al. [29], it seems that this student tries to justify, actually mixing, his belief with some feature of Special Relativity, revealing that, as outlined in literature, Lorentz transformations are not an operative tool.

Finally a ~23% of the student showed a scientific reasoning for the question: in each reference system time flows at the same way but when one observer measures the time of the other reference system, he will register a different value. Some of the students highlighted explicitly the need of comparing the measure of the two different reference frames. However from a linguistic point of view the answers are not correctly expressed: sometimes students got the point but did not express it in an Italian good lexical form.

Q. 5: explain if we can perceive the time dilation phenomenon.
This item is related to question 4: in this question students have to say if it is true or not that being aboard a moving train one can see from his own clock a different value with respect to a clock at rest. The answers were given perhaps randomly as over 85 students an half of them replied that it was true, the other half it was false. In question 5 students were asked to explain why the previous one was true or false. In analysing this question, we take into account also the answer given to question 4.

As before we need to invalid 15 answers (~18% of the total answers), the grates part of which (~73%) thought that time dilation could be observed being aboard the moving train.

A small part (~11%) of the remaining sample of 70 answers replied in a common way, stating correctly that it was false as times flows equally. However these students did not give an answer in the previous open question 3 (or copied from Internet or common one): even thought they correctly answered to question 4 (except one), this may not be considerable to infer a correct understanding of time dilation phenomenon. Maybe just one student among these could be excluded from this last consideration as his answer to question 3 has scientific feature and it is coherent with question 4 and 5. However his explanation to this question is poor from scientific contents.
Furthermore some of these students still referred to the perception of time as changing while the measurement of it would remain the same. Actually the only student who gave the incorrect answer to question 4 (it is true to measure time dilation) stated that it is due to the *different time perception.*

The greatest part of the answers (~79%) is a meta-scientific one and we can recognise different kind of reasoning:

- about ~60% of these students clearly did not understand the phenomenon: relating question 3, 4 and 5 it emerges that these students believe that time dilation is a local phenomenon namely it is a property of the reference frame. They ascribe the truthfulness of the awareness of time flowing slowly to time dilation phenomenon itself: thus being aboard a moving train one can perceive time dilation without taking into account the relativity principle. The answers to all these three questions show a consistency and coherency into students' reasoning, even thought not scientific;
- the other ~40% of the students with meta scientific answers correctly replied to question 4 (it is not possible to perceive time dilation phenomenon) while their explanations refer to the idea that there is not a change of reference frame or equivalently the observer is always in the same reference frame. It seems to be a sort of learnt statement, a kind of "magic formula" that can be used in relativistic situation.

Finally the remaining 10% of the students shows a scientific reasoning connecting question 3, 4 (correctly falsely answered) and 5. They stressed the relativity nature of time dilation stating that the time difference arises only from the comparison between clocks of different reference frame. In only one answer there is a kind of reference to the covariance principle asserting that *in every inertial frame the measure of a physics event must give the same result.* Even if there is a confusion with the meaning of *event* as the full correct answer should have involved a mention to the result of an experiment, not a measure of an event, this is the only answer referring to the relativity principle.

Q. 7: how would you summarise the length contraction phenomenon?

From the answer to this question we need to exclude 28 items (more o less one-third of the total answer) while other 16 answers (~19%) were copied from Internet, thus shortening our sample to 41 answers.

The common sense answers (~29%) are less in number with respect to the ones related with question 3. As one could expect length contraction was summarised as "*the greater the speed, the shorter is the length of a moving object*". This assertion has some different declination according to what gets contracted. Indeed students referred to length or lengths, space or spaces and also to the moving body itself. Moreover similarly to time dilation, also length contraction seems to be an intrinsic property of the reference frame: a half of these students stated that the phenomenon depends on the status of a reference frame, namely if it is moving or not. Indeed we noticed the students used verbs has *it is*, *it becomes*, *it results* contracted as if the phenomenon is an external property to be ascribed to the body or to the reference frame.

We noticed also that the common answers are the half of ones of question 3 while the meaningless answers and the one copied from Internet increased: we could infer that perhaps there is a link with the less common knowledge of this phenomenon, leading to a decrease of this kind of replies and an increase of the other ones.

The number of the meta-scientific answers (~51%) also increased with respect to the ones of question 3. We can distinguish different addresses for these replies:

- a small percentage of students (~14%) used Lorentz transformations as an explanation to the phenomenon;
- the ~29% of these students made reference to the visible effect of rotation for a body moving at high speed (Lampa-Terrell-Penrose effect), perhaps getting confused by it. They used verbs as *to see*, *to be*, *to appear*, thus perceiving the phenomenon as an illusion (as some students expressly stated) due to this rotation;
- the remaining 57% of students referred their answer again to a visual phenomenon, not linked to the previous rotation. They simply stated that an observer in relative motion with respect to an object will *see* it contracted, shorter then if it was at rest. They tended to use the same verbs as the students of the previous grouping.

What emerges is a tendency towards the consideration of length contraction sometimes as an apparent effect due to Penrose rotation, sometimes as a visual effect, not a quantitative one. The latter position could be inferred to be influenced by the former: this is partially true. Indeed during the experimentation as I noticed from a first look to students' answers referring to the previous rotation, thinking that it might be a source of misconception, I decided to remove this part of my lecture. Thus not all the groups knew about Penrose rotation but in the same way some of them stated that length contraction is a visual effect.

The remaining ~20% of the answers can be addressed as scientific ones and focuses over different measurement of a length in two different reference frames in relative motion, even thought without explaining the measure's process and in which direction the length gets contracted.
Finally as a check of the given answers, we compared these replies with the one to question 11 in which, similarly to fourth, we asked if being aboard a train moving at high speed we would have seen the length of the carriage contracted. We found that ~66% of students correctly replied to this question and that almost all the incorrect answers were given by the students who did not answered to this seventh question or copied from Internet. This is to infer that maybe that ~66% of students grasps the relative feature of length contraction phenomenon.

Q. 9: explain the result of an exercise over length contraction.
This item is related to question 8: in that question students had to indicate the length of a table in a moving train according to an external observer between L, γL and L/γ, being L the proper length. In question 9 students were asked to motivate their choice.
As before we need to invalid 37 answers (~44% of the total answers) given by students, almost all of them having replied incorrectly to question 8. Our sample is now of 48 answers. This shows that students, regardless whether they answered to question 8 correctly (~36%) or not[6] (~64%), are not able to motivate their choice. This may points out a not full understanding of the phenomenon.

Among the remaining 48 students, there are only two common sense answers stating that inside the moving reference frame (the train) length are shortened.
All the other (~96%) are meta-scientific ones and we can identify different groups of answers:

- one ~24% of the answers showed an internal contradiction: students choose an incorrect answer to question 8 (according to the external observer, length are not contracted but dilated $L' = \gamma L$) but they motivated this with length contraction. Maybe students got confused with relativistic formulae;

[6]Only two students stated that the measured length is L, thus these wrong answers are related to $L' = \gamma L$.

- another ~33% choose the correct answer to question 8 except for two of them: they motivated it simply because of "length contraction". Moreover two students among them stated that the measured length is shortened with respect to one *real*, thus implicitly inferring the existence of a privileged reference frame;
- ~17% of students used mathematical reasoning: being the length contracted, the only possible answer is $L' = L/\gamma$ as in this way $L' < L$;
- two students answered correctly to question 8 and argued about the two extremities of the table occupying two different spatial positions (they used improperly the word *space*) and that as a consequence the light coming from them would have reached the external observer in two different time;
- the last ~22% of students referred to the presence of motion, each one of them stressing different features of it, never however writing a complete reasoning: the 40% of them stated that there is a motion while another 40% added that the measured length is along the direction of the speed. The last 20% of them indicated that L/γ is the correct formula as it prevents from considering the two extremities of the table at different instants of time, thus not measuring two simultaneous events. They are the only answers referring in same way to the requirement of measuring distances as difference of spatial coordinate taken at the same instant of time.

Remaining questions.
Other four questions have not been analysed so far, namely question 1 and 2, 12 and 13. The first two are about simultaneity for the same event from the perspective of two different observers: inside a train moving with respect the platform at high speed, Giulia lets two balls fall. Students were asked to say if the two balls hitting the ground of the carriage are simultaneous events with respect two different observers, one inside the carriage and one on the platform. With respect to the latter, the ~91% of students stated that events are not simultaneous (it was not asked to specify the order of falling) while with respect to the former, the ~89% of students agreed on the simultaneity of events. Analysing the replies we saw that over the total students, most of all (~85%) correctly answered to both the questions. Only a low percentage of them seems to believe in the existence of an absolute simultaneity of the events (~4%) or in the completely absence of it (~5%): due to the lack of open questions about this topic it is not possible to infer any significant conclusions on students' reasoning about simultaneity.
Questions 12 and 13 involve relativistic addition of velocities: in topic 12 students were asked to follow the step of the first Einstein's thought experiments, namely what would have happened if someone was able to run close to a light beam. In question 13 students

were asked to determine the speed of a light beam emitted from a source moving at relativistic speed from the point of view of an observer at rest with respect the source itself. About question 12, the ~53% of students answered that it was not possible to see the light beam being at rest while another ~35% thought it was. The remaining ~12% did not know what to answer. About the last question, most of the students (~59%) correctly answered that the speed of light would have measured to be always c. Beyond a ~6% of students that did not how to answer, the remaining part is equally split between $c - v$ (~17%) and $c + v$ (~18%). However comparing the replies to the two questions we noticed that only the ~37% of the students correctly answered to both them. For instance among the students that gave the correct reply to question 12, one third of them (~31%) gave a wrong one to topic 13 which addresses the same concept of the previous one (light speed is constant in every reference frame). In the same way among the students who said in question 12 that it was possible to "see a light beam at rest" almost an half of them (~47%) gave the correct answer to topic 13. This high percentage with respect to the former correlation might be due to the different kind of questions, namely the fact that the second one is more "scholastic"[7] then the first one which requires a deeper reasoning. These correlations could infer that maybe it is not learnt at all that light has always the same speed and thus the second postulate has not been grasped.

[7]Question 13 is very common among problems in textbooks.

Conclusion

In this thesis we have introduced the *spacetime globe*, an instrument which allows to represent interactive Minkowski diagrams. Its most important and peculiar feature is that with mechanical actions we can manually switch from different reference frames, being them at rest or moving. Using the instrument, the description of various relativistic phenomena from the point of view of different observers can be achieved without the complex geometrical prescriptions of Minkowski diagrams, rather with simple movements of the components of the spacetime globe.
The use of Minkowski diagrams is well documented in literature as useful didactic instruments for higher instruction but in our opinion they can be adapted efficiently also for secondary schools. Thus we began structuring a pilot experimentation involving some high schools of Rome with the aim of investigating if the spacetime globe could be an effective didactic tool. Even if the emergency care due to CoronaVirus pandemic did not allow many teacher to participate to our investigation, we received a positive feedback from a questionnaire we submitted to them. We worked in the direction they suggested, reducing the amount of hours of our intervention and creating simple animations with Python. These allows teachers to have an instrument similar to the spacetime globe but easier to use, without the need to share the instrument between many students or coming to our University to use it. These simulations may be also employed during the curricular courses of Special Relativity and to provide students a tool for their homework or let them to autonomously investigate more advanced phenomena as Doppler's effect.
We organised two lectures on the main topics of Special Relativity (loss of simultaneity, time dilation, length contraction, relativistic addition of velocities) using the spacetime globe to show students the relativistic phenomena. Our lessons were held both remotely and in presence; we asked students, that already have attended instruction on Special Relativity, to answer a pre-test and a post-test at the beginning and at the end of the class, respectively. The former contains open and closed questions on Classical Mechanics while

the latter contains open and closed questions on Special Relativity.
The items of the pre-test were not taken from a validated one and this could be the reason of its non-consistency. Nevertheless some items (as the two first) were taken from an exercise book wrote by a high-school teacher, thus it should not be beyond their capabilities. The items of the post-test were taken from the Relativity Concept Inventory ([104]) which was thought for higher instruction and needed revisions as the author wrote. The result of the questionnaires have been analysed and mainly showed that there is a slightly greater difficulty in Classical Mechanics: if we consider only the closed questions for both the questionnaires, normalising for the total number of closed questions and of the students, it comes out that the 64% of students well answered to items about Classical Mechanics while the 72% to the ones about Special Relativity.

From the analysis of questionnaires it clearly emerges that students even after instruction still preferentially adopt a meta-scientific reasoning, combining their common knowledge with scholastic one. It has been difficult to find a really scientific answer. This is particularly evident if we consider the relativity principle: Pietrocola and Zylbersztajn [27] addressed that in Special Relativity students ascribe the equivalence of inertial reference frames to the non-presence of pseudo-forces. But we found that this tendency is also present as far as Galileo principle of relativity is concerned (question 2 of the pre-test). Moreover students do not fully understand the invariance of light speed (question 12 of the post-test). This let us think that students do not use principles as operational tools since they prefer to explain phenomena referring to dynamical-kinematic effects[8]. Reasoning in terms of principles requires a functional understanding of the topics which appears not to be present. It seems also that students are still learning by contents and they are not able to apply their knowledge in non-ordinary different contexts. This agrees also with the general tendency of relaying upon sensory experience as first source of scientific knowledge.

The negative result arising from the statistical analysis of the pre-test shows that some concepts of Classical Mechanics are not accommodated at all (as for instance the problems about trajectory in question 3-4). We noticed the tendency of regarding as apparent the phenomena occurring in moving reference frames that infers implicitly the presence of a privileged reference frame. This reasoning is particular evident from answers to questions 5 and 7 of the pre-test, also from the same students who qualitatively well reasoned through a meta-scientific arguments. More o less a quarter of the students (~23%) still believed in the existence of an absolute reference frame (question 8).

[8] For instance aboard an uniformly moving car I can not detect its motion because of the lack of pseudo forces.

It would be interesting to submit the questionnaire to high-school groups of students that have just receive instruction upon this topic to understand if the major problem consists in the high difficulty of the test or because students have forgotten some contents.
As a research target we should take seriously into account that, as it emerges from Table 4.1, among the most important topics for teachers there are curricular arguments of basic Physics (as the forces or energy) and not of Modern Physics. In recent years from the Institutions there has been a great boost in dealing with Modern and Contemporary Physics but, as it emerges from the pre-test, some standard arguments are not still well understood. We need to go on developing in parallel the other aspects of Physics in order to provide a solid basis for the understanding of the more advanced ones.
Students are strongly committed to the concepts of Classical Mechanics (even if they are not aware of it): perhaps it could be helpful to rethink its teaching in the relativistic key. If from the point of view of history of science it is important to see how theories evolve, it may be useful for students throughout all the secondary instruction to have some important concepts anticipated as that one of space and time.

About Special Relativity we have learnt that students still look to time dilation as an intrinsic property of the reference frame (in particular of the moving one), thus not arising from the comparison between clocks of different reference frames. Some students refer to it in a classical way, namely that "the greater is the speed, the slower the time flows". In this way one half of them thinks that time dilation phenomenon can be observed in the “moving” frame, again confirming that they do not use principle of relativity and its consequences to reason also about relativistic phenomena.
As far as length contraction is concerned, it is more evident a general tendency toward meta-scientific answer. Indeed even if most of the students well point out the existence of two different reference frames to explain the phenomenon, they however think about “seeing" an object contracting rather then measuring a contracted length, a confusion that Panse et al. [11] already found concerning Classical Mechanics: *It was also found that they tend to confuse measuring with viewing.* As shown in literature, objects seem to be shorter, thus is only an optical illusion. We have thought that this was to be ascribed to the introduction of the Penrose rotation (briefly mentioned during the lectures but then removed for the last two groups) but we have seen answers like the previous ones also from students to whom we did not describe Penrose rotation. Probably this is telling us that at this level of instruction students are not ready to qualitative treat Penrose rotation even if maybe it is more enjoyable for students to understand what actually a relativistic observer would see. We thus address that students’ misconception of thinking on length contraction as a visible phenomenon is more strongly linked with their commitments to sensory experi-

ence rather then on time dilation. Indeed from the used verbs (*it results, it is, it becomes, it is contracted*) they still think that lengths are properties of the body, showing a mechanistic view of reality ([17]).
No open questions were submitted to students about simultaneity: from the closed questions we found that almost all the student answered correctly. The absence of clinical interviews with students did not help our investigation about this topic and in general about all the contents of the pre- and post-test. However during instruction while dealing with Ladder paradox in its version with a moving train and a tunnel (pg. 136), a student understood that the trick was to notice that in the reference frame of the train the two events determining the paradox are not simultaneous[9]. Maybe a guided open question with the formulation of this paradox should be added to the post-test in order to investigate students' solutions.
A common feedback from the answers to the questionnaire is a not-always suitable use of language. When referred to length contraction, students equally used terms as *length*, *lengths*, *space* and *spaces*: I think that it is important to spend time in increasing the scientific vocabulary of students, pointing out the real meaning of the various term adopted in Physics in order to improve their qualitative reasoning on natural phenomena. For instance, to clearly grasp the meaning of length contraction phenomena, students must recognise the difference between *to measure* and *to see* which actually has not been well understood. It is necessary a scientific thought on language, namely a scientific, rigorous analysis of our words and of their meaning. This will be helpful for students not only to understand the meaning of first postulate of Special Relativity but also to deduce its consequences and it will be part of their complete development. Indeed this emergency care due to CoronaVirus epidemic shows how scientific reasoning makes the difference. This actually does not mean to understand Physics but to be able to apply a method to critically analyse the large amount of information we receive every day.

From the general questions over Special Relativity (Figures 4.8-4.9 in Section 4.4.2) we know that three-quarters of the students before instructions searched for information about Special Relativity. This could be one of the first source of misconceptions, even before instruction, as the famous one that "all is relative" we observed in some answers. It emerges that students prefer to look to videos, they need to see rather then to read about this theory. This result agrees with the outcomes of Kettle's review ([108]) on the didactic use of videos in secondary school Physics teaching. She pointed out that both students and teachers watch videos, using them in different ways: even if their didactic use is strongly

[9]In this case, two guillotines at the extremities of the tunnel cutting the rear and the front part of the train according to an observer inside the train.

recommended in order to improve learning, they are not effective help resource for students. Our suggest is to give more importance to this aspect that the scholastic practice has to embrace in order to intercept the learning structure of the current students which is chaining. The traditional approach to Special Relativity, which is currently still adopted by Italian high-schools, is no more suitable: as Selçuk [22] pointed out, it is necessary to differentiate the teaching strategies in order to achieve a better understanding of Special Relativity which is considered difficult, extremely mathematically, thus only for gifted students [41].

In this way, the spacetime globe as a didactic proposal can face these current issues being an instrument which allows a direct and concrete contact with relativistic situations where students are asked to imagine and visualise phenomena far from everyday life. This difficulty is even more accentuated dealing with students who need a visual approach towards reality.

Thus the spacetime globe would be a support for the algebraic formulation of Special Relativity through the graphical representation on diagrams whose primary strength is to allow students to see the abstract concept of changing the reference frame, one of the most important source of misunderstanding for students on Special Relativity. In order to differentiate the didactic instruments, in our lectures we did not conceive a purely kinematic-geometric approach to Special Relativity but rather we wanted to make use of Minkowski spacetime diagrams as a privileged way in teaching Special Relativity. Similar to [13], yet independently, during the instruction we preferred to think in terms of observers and not of reference frames, substituting the idea of changing the reference frame with that one of adopting a different point of view. But, as Scherr et al. [8] highlighted, this could lead to identify the concept of observer with that one of reference frame, eventually resulting in associating observers at rest with respect to each other with different reference frame. However we did not investigate this possibility nor from the answers to questionnaires it arises such a consideration.

Another important strength is that the spacetime globe allows to directly see the *relativistic* nature of the investigated phenomena as we can very quickly and easily switch from two reference frames in relative motion. For instance, understanding the relativity of time dilation has been addressed ([22, 25]) to help the misconceptions and from the answers to the post-test we found that this reasoning was explicitly pointed out by some students.

Our experimentation is still at his beginning and we can not give a full answer for the research question “is the spacetime globe an effective didactic tool?” Indeed starting from the sample analysed, students before our experimentation already attended instruction on Special Relativity and thus we were not completely able to verify the efficacy of our

approach nor we submitted our tests to groups of students not following our lectures in order to compare the result. However our prior result consisted in structuring a proposal of a didactic strategy to teach Special Relativity using the spacetime globe and our pilot experimentation allowed us to reconfirm some of alternative conceptions of students on Classical Mechanics and Special Relativity. As reported in literature, the latter misconceptions are related to former ones. As students before our intervention have already attended curricular classes on Special Relativity, in agree with Dimitriadi and Halkia [13] we may consider that the tradition approach used in high schools seems to reinforce the these alternative conceptions.
Our pilot experimentation also allowed us to understand how to modify the structure of our intervention, for instance highlighting the meaning of reference frame, avoiding to tell about Penrose rotation. The global result of the questionnaires give indications about rethinking them, especially the pre-test. Moreover we did not explicitly conceive our questionnaire to reveal the understanding of the Minkowski spacetime diagrams through the spacetime globe: surely we need to investigate what students grasp about them from the use of the instrument with addressed questions.

A final outcome from our project is a small book in Italian we wrote about Special Relativity. It is addressed to high-school teachers and contains the complete description of the relativistic contents of the learning teaching units presented in Section 4.2. About the topics it deals with, they are the same as the ones contained in Italian textbooks (relativistic phenomena, Lorentz transformations, light cone and intervals) but it differs by the approach we used. Indeed this work supports and guides teachers (as well as students) in the study of the relativistic phenomena using the spacetime globe (or our Python simulations) through the Minkowski spacetime diagrams. The contents of this book are the topics presented in this thesis (Section 3.3) explained with a more appropriate didactic language. There is also the descriptions of light cone and intervals using Minkowski spacetime diagrams. It will take part in our future experimentation with teachers whenever it is set to be carried out autonomously.

About teaching

WE started our work asking what learning is; however of the same importance for these times there is another question:

what is really teaching?

Two years ago I bumped into two different posts of two friends of mine, Michele and Sofia, being both high-school students but differently aged, that are the following (translated in English).
Sofia's thought:

> *Unfortunately you count on one hand those teachers who have the courage to change the direction of our system and to suggest something alternative which might interest young persons. They follow to the letter the usual programme which they have known by heart for years. Banality.*

Michele's thought:

> *A few days ago a teacher, testing the whole students and finding hindrances due to the difficulties of each of them with their own instrument*[10]*, said: "this distance learning is not working at all". This made me think a lot. The introduction of distance learning has put a lot to the test each one of us but above all it has brought out some fundamental elements of education. Indeed the world of teaching has been split into two parts: many teachers have adjusted their didactic to the circumstances, putting the relationship and the enrichment of the students in first place. Instead other teachers, continuing their own way with blinders, have the didactic unchanged, showing more and more their mania for evaluation at a time when there is no grading meeting that matters. This may seem like a stupid attack by a kid towards teachers or worse, a justification of the students but it is*

[10] It was an online course during the lockdown period.

> *not meant to be either, indeed it is simply a thought about the period that we are living.*

Asking them to explain a little more deeply their thoughts, both of them told me about teachers that, near to instructional targets, have other objectives.
Sofia found a good teacher of Mathematics who was interested in students' life, in their troubles and worries. This teacher involved his students in a theatre project concerning not only Math but also themes covering their growth. He was able to put the students at ease as he did not consider himself as in a superior position of power but he was close to them. He knew his students and he considered their opinions as relevant, even if there were more then thirty years apart. He did not underestimate his students.
In the same way Michele told about the different approaches teachers had during online classes. He reported about some of them who were able to overcome the digital hindrance changing their targets, shifting from the instructional ones to the more important educational and emotive ones. They began not to focus only over the didactic evaluation of contents but also on establishing a relationship with their students. For instance, the teacher of History and Geography was interested in giving not only a numerical mark for their homework but also to each student a different, personal indication on how to grow up and improve.
These two examples highlight a key point of teaching: students and school as preparation for life. Students as unique and special person who is completely different one another. Then our question about teaching is intimately tied with:

what is really teaching with respect to students?

My teacher of Didactic of Physics Orietta Proietti, whom I am infinitely grateful to, was used to say that *we do not have to create scientist: that is the aim of university instruction. We have to create a scientific culture, never forgetting that we are dealing with persons.* Teaching can not be defined if we do not have clearly pointed out its role with respect to students, in their own life.
We do not have to forget the *educational* role of school: *education* from Latin words *e + dūcĕre*, literally to take out something. Thus if I have to take out something from students, I need to know what is inside them otherwise I would never be able of intercepting my students. Teaching needs to start from what is already inside a student. But how can I know it, if I do not spend time listening, not talking to, my students, if I do not build a relationship, a dialogue with them?
We know that adolescence is a complex yet wonderful period of life characterised by new ways of thinking: students aim to build autonomy from the family with a growing desire for self-determination and decision-making capacity. This implies that adolescents begin

building their own identity, deciding what is good or bad according to themselves, establishing and developing who they will be tomorrow. There is a desire to show who they are, that they have a role in the world too and in the peer group, that they can and they know how to decide independently. They begin to face questions like: what is the real criterion to choose? What is good for me? What does it hurt me? What does it define me? In brief, who am I?

We can not ignore these questions: we can not think that a good teaching does not take into account these thoughts in the hearts of students. Because this will affect their learning process. Spending our students half a day at schools, teachers can not disregard the personal development of each student. Institutions strongly count on students and each restart begins always from school, that it is from students. Then teaching needs to restart from students too. As different persons, they have different dreams and objectives and they have to be taken out from the interior world of the students. This is the main role of teaching in order to reveal the real person each student is. We can not think of delegating nor postponing this process as there will not be time to be carried out, once students leave the school.

Here, I do not want to deny the importance each subject has as well as its formal teaching, with the achievement of the requested knowledge, abilities and competences: to paraphrase, *these you ought to have done, without neglecting the others*[11]. What is import is to define the suitable objectives according to each course the students are going to follow in order to face reality. For instance scientific courses are the most appropriate as they could teach to develop critical thinking and reasoning which is the scientific method. Learning to think, learning to create their own opinion, autonomously, not learning how to solve an exercise. There are the university courses that teach specifically the Divine Comedy or the Physics of Black Holes. But the growth process of a students happens within the scholastic time: it is at school that we train the adults of tomorrow. It is not far to assume that the way school acted on us results in what we are today and the Italian scholastic system aims towards the passing mark (the mark 6). Are our students really "enough" according to our subjects? Is this a subtle message we are leaving? From this point of view, I do not have a full positive memory of my high-school period. I remember some classmates of mine who were constantly angry with teachers because of the evaluation they gave them, thus reducing school to a matter of a number (the mark).

If school lacks in something, if it is failing, it is perhaps because it has forgotten the educational role. To start again from school is to start from students. Let us begin to build a relationship of trust with them: this means listening to them, talking to them about their problems, their worries, the particular beauty they have within them. Let the students

[11] Mt 23,23.

become *our* students: if they can not trust us and know that we are open towards a deep listening of them, they will not listen to us. Listening starts from dialogue and dialogue is the knowledge of the other. How can a student listen to a teacher who does not know who he/she is? I heard about a teacher who the first day after summer holiday had a lecture...
As the thoughts of Sofia and Michele tell us, students are more willing to listen to a teacher who listen to them. Students need someone they can really relay upon, someone who believe in themselves according to who they are, not to who we would like them to be. Because this prevents them from showing off their true beauty. Students will believe in themselves if someone have already believe in them: this will be an authentic source of self-efficacy, thus allowing students to positively respond to challenges of lives. Students would make the difference if teachers themselves made the difference.
Once I was talking with Roberta Bolzanello, a very good high-school teacher of Physics, who told me that the more the students grow up, the more they get *tired of life*. For this reason she prefers to work with students of the first year who are *purer* from this tiredness. It was an epiphany: they get tired of life[12] because nothing really fascinates them anymore. But what I think is important is the loss of passion which is primarily addressed towards themselves. Without being passionate to the beauty they have within themselves, it is normal for them to get tired of life. It is normal that school has to struggle with students. Otherwise there would not be all this attention towards teaching. But if they are aware that life calls them to set in motion the proper beauty within them, they will be faithful to the promise of beauty that they are carrying within them.
The real educational problem of school is that students have no perception of this beauty. Teachers are given this task: to truly believe in the beauty of the students entrusted to them and above all to communicate it to them as a precious treasure.
This is the educational "bet" of the school: to bet on students and on their beauty. Students need to feel that someone values them, that they think of them as someone beautiful! *The teacher must be a lover of the beauty inherent in students*
I am strongly persuaded that if teachers let them know they are there for them, that they want to know them, to know who they are, also teaching will change.
This is teaching according to me.

[12] I can prove it is true: my friends who are a group of high-school students strongly agree with this sentence.

Appendix **A**

Appendix to Chapter 2

A.1 A philosophical debate over the postulates

Here we want to examine from an epistemological point of view the two principles of Special Relativity, not with regard to their content but to their attribute "principle".
In his work of 1905, Einstein wrote:

> Wir wollen diese *Vermutung* (deren Inhalt im folgenden "Prinzip der Relativität" genannt werden wird) zur *Voraussetzung* erheben und außerdem die mit ihm nur scheinbar unverträgliche Voraussetzung einführen, daß sich das Licht im leeren Raume stets mit einer bestimmten, vom Bewegungszustande des emittierenden Körpers unabhängigen Geschwindigkeit V fortpflanze.

This test is usually translated in Italian as:

> Assumeremo questa *congettura* (il contenuto della quale nel seguito sarà chiamato "principio di relatività") come *postulato*, e oltre a questo introdurremo il postulato con questo solo apparentemente incompatibile, che la luce nello spazio vuoto si propaghi sempre con una velocità determinata V, indipendente dallo stato di moto dei corpi emittenti ([103]).

And in English as:

> We will raise this *conjecture* (the purport of which will hereafter be called the "Principle of Relativity") to the status of a *postulate*, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity V which is independent of the state of motion of the emitting body ([109]).

Let us now focus over the words highlighted in italic:

- congettura → conjecture → **Vermutung** = hypothesis, supposition
- postulato → postulate → **Voraussetzung** = premise, presupposition

Thus Einstein's words would sound as:

> We will raise this *conjecture* (the purport of which will hereafter be called the "Principle of Relativity") to the status of a *premise* [...]

The difference is remarkable: in the Italian translation as well as in the English one, the word *Voraussetzung* used by Einstein is translated as "postulate", not as premise: this status of postulate is attributed both to the Principle of Relativity and the Principle of constancy of light's speed.
What is a postulate? Why did Einstein distinguish from conjecture?

A.1.1 Postulates in Mathematics

In the tradition of Mathematics, a postulate has a well defined role within a theory as it is "*a preposition that, without being demonstrate, it is assumed, or required to be to the interlocutor, as a basis of a demonstration of a theory*"[1].
For instance, the five postulates formulated by Euclid are statements taken as true without the need of being demonstrated: thus we can consider them as a self-evident, self-consistent, existing in themselves.
They do not need to be demonstrated as they are the necessary instruments to found all the Euclidean geometry and to create the statements it contains: they are the "first principles" above which nothing else can exists.
However as well as in any philosophical discussion, there are different schools of thought: about the nature of postulates, of geometry in particular, Poincaré discussed[2].
They can not be considered as *synthetic a priori judgements*[3], namely some assertions always true pre-existing to our intellect as, if they were of this nature, we might not even be able to conceive their opposite propositions. But this is not true as we know that there exists non-Euclidean geometry which arises exactly from the negation of the V postulate.
Them if the postulates are not statements existing *a priori*, what are they?
In the same way we can not consider postulates as experimental truths because geometry will result to be as an experimental science, no more an exact science but continuously subjected to revision.

[1]Treccani online Encyclopedia, "Postulato" - my translation from Italian.
[2]For further information: *La scienza e le ipotesi* [110].
[3]Even if Kant did consider in this way.

Thus Poincaré considered the postulates a *conventions*, namely no more than a *definitions in disguise*[4]:

> thus postulates can remain rigorously true, even if the experimental laws, that led to their use, are approximate.[5]

Einstein first criticised this view as it would make impossible to use geometry to make predictions and assertions about the behaviour of real objects. He preferred to turn geometry into a physical science where the truth of the axioms was established by the agreement with reality.

A.1.2 Postulates in Physics

Now we have to see how to consider postulates in Physics, that are called "principles": we need to start from the fact that differently from Mathematics, Physics is set as an experimental science, thus deductive and not inductive.
Poincaré [110] pointed out that the English taught Mechanics as an experimental science while in the European continent it is taught as *a priori* deductive science, saying nevertheless that the English were right.
Mechanics and more in general all Physics does not start from statements, deriving then by induction new ones that, according to the principle of induction itself, are necessarily true propositions. Physics, *deducing* its laws from the observation of reality, namely from the experiments, is a deductive science.
The truth it states as well as the contents themselves (space, time etc.) can not stay *a priori* over our intellect as entities pre-existing to Physics.
As a consequence

> an experimental law is always subject to revision; we can expect to have it substituted with another one more precise.[6]

Thus we have to be careful whenever a law of Physics is raise to a "principle" as this will give it an *a priori* truth that can not be falsified. This will prevent us from contradicting a physical law if a condition was found able of denying it, preventing the progress of science. Indeed, as Popper reminded us, science *progresses from an issue to an issue*:[7]: a scientific theory arises as an attempt to solve a problem and it is only because it lifts up other problems regarding itself that we can understand whether it is true or not. If then a law is a *a*

[4] *Definizioni travestite.*
[5] [110] - my translation from Italian.
[6] *Ibidem*
[7] [80] - my translation from Italian.

priori principle, problems can not contradict it and prove it to be wrong if necessary.
For instance writing his *Principia*, Newton himself considered the principle of inertia as an acquired yet experimented truth, according to the works of Galileo and to the Keplero's laws. Newton was confident in stating it because a principle would have been contradicted only by some peculiar and strange facts that were never found.
Also Poincaré excluded the possibility of undermining the principle of inertia, replacing it with a new more exact law, simply because we could never subject the principle of inertia to a decisive test, being it impossible to obtain this proof.[8]
Thus

> this law, experimentally verified in same peculiar cases, can be fearlessly extended to general cases since we know that in such general cases experience can neither confirm nor contradict it.[9]

Therefore, regardless of the peculiar case of the principle of inertia, Poincaré's statement can be extended to all the principles (or laws) of Physics: having established experience as the *only source of truth*, we must accept the fact that the truth of a law is limited to the number of particular cases in which it was verified.
Then, since a law appears to be correct for a large (finite but large) number of cases, it is *generalised*: in circumstances similar to an event, one can infer that an analogous fact will occur.
On the basis of this logical process, which is necessary to be able of making predictions, an inductive character is often erroneously attributed to Physics. However, it can not be accepted precisely because it undermines the same nature of Physics which is and remains the science that studies Nature and consequently it requires a continuous dialogue with it.
Thus is it sufficient the process of generalisation to let us state that a law of Physics is true?
We have to keep in mind that, as Hertz quoted, *which is derived from experience can again be annulled by experience.*[10]
The experience allows us to derive a law which is true, namely mirroring the nature of the relations between physical entities. Nevertheless it is also the *proof* to which we submit our law, that is what enables us to determine whether is it true or false but only for a limited number of cases that are not sufficient to be able of stating that generally the law in question is true or false.
Experience is necessary as a basis for constructing the laws of a physical theory but it can never completely contradict it: so what is the role of these principles?

[8]We should rewind the whole Universe to the first moments of its life [110, pg. 100].
[9][110] - my translation from Italian.
[10]*Ibidem*

They become *definitions*, arising from experience, established as principles and considered absolute truths by our intellect. It is important to highlight that formulating these "primitive" statements (meaning that they come before a theory) is a necessary step: in a formal system we have to start from some assumptions that will be never questioned. The *Gödel's incompleteness* theorem of mathematical logic infers that in each formal theory there is an assertion that can not be demonstrated within the theory itself, nor it can be demonstrated its negation. Thus in order to develop a theory, there is always a starting point which will remain indemonstrable, a statement that can not be proved, yielding the theory to be *incomplete*.
However, Physics has an additional instrument: experience itself.

The nature of postulates of Special Relativity

Coming back to the postulates of Special Relativity, the translation of the word "Voraussetzung" as "postulate" has some important epistemological inferences as it establishes it as a statement absolutely true.
Giving to the assertion "light speed is constant within each inertial reference frames" the role of a postulate, it means that it will be always true.
Einstein was criticised especially by empiricist physicists about his work, being it merely speculative even if the two postulates had an experimental basis.
As Einstein pointed out during a conference in 1921:

> Considering the specific nature of the theory of relativity, I care to highlight that the origin of this theory is not speculative but its discovery is completely due to the desire of adopting, as best as possible, the physical theory to the observed facts.[11]

Einstein's two principles of Special Relativity have experimental deductive nature, not a *a priori* one: some past experiments led us to think that light speed might be invariant and thus the scientist decided to use this feature as a ground for his theory, noticing that it led also to further confirmation of other phenomena. In the same way the fact that many experiments of Electromagnetism and Optics were not influenced by the movement of the Earth was not considered as a proof of the principle of inertia but rather an argument in favour of its truth. As a consequence, its validity is limited to the discovery of a counterevidence, as usually science progresses: numerous arguments were found in support of the aether but they were not sufficient to prove its existence.
The principles of Special Relativity are not a simple result of the intellect: Einstein always emphasised the role of intuition but also of deductive thinking in formulating principles.

[11] [50] inside [70] - my translation from Italian.

As this process may lead to different theories, what is important is to find new predictions that let one of them prevail over the others.
As a consequence using "principle" or "postulate" without explaining its experimental origin can mislead.
The word used by Einstein *Voraussetzung* has a more correct translation in "premise", "presupposition", leading to consider its non-formal certainty. Indeed, we can not fail to consider that Special Relativity was born as a theory articulated around risky predictions, of doubtful certainty of which perhaps more denials than confirmations were expected.
As Einstein himself pointed out:

> The theoretic scientist is not to be envied. Because Nature, or more exactly the experiment, is a relentless judge and not many kind in her work. She never says "Yes" to a theory: in the most favourable cases she answers: "Maybe"; but in the overwhelming majority of the cases, she says only: "No". When an experiment agrees with a theory, according to the nature, it means "Maybe"; if it does not, it means "No". Probably one day each theory will receive its "No"; for most of them this happens immediately after being formulated.[12]

Einstein did not define nor he attributed a character of unique truth to physical laws: thus scientific truths can not be absolute such that they remain unchanged over time.
Then that Einstein's crucial passage of his work (pg. 172) from *Vermutung*→*Voraussetzung* must be interpreted as the transition from a *supposition*, namely something that is not sure (thus a conjecture, an hypothesis) to a *presupposition*, namely something that is supposed before something else, that is a condition necessary to something else.
The postulates of Special Relativity therefore constitute necessary presuppositions to derive from them all the formal organism functional to validate the empirical observations and their own foundation, that is, their justification that lies always and only in the experimental observations of the old phenomena as well as of the new ones.

The prospective of the Falsifiability

The strength of Special Relativity seems to lay in the existence of a very rich but especially various (i.e. of different fields of Physics) phenomenology supporting it, thus constituting a valid description of the physical world.
It is a theory that has a greater amount of empirical information with respect to the previous ones and a greater explanatory power: it is able to explain phenomena with greater precision, unifying problems that were previously impossible to solve.

[12]Text written by Einstein on November the 11th of 1922 in a book dedicated to the memory of professor Heike Kamerlingh Onnes - my translation from Italian.

It was then confirmed by subsequent proofs, comparing the expected facts with the observations which constitutes, according to Popper, the criterion that allows us to make one theory preferable over the others ([80]).
However the greater is the amount of information of a theory, namely the amount of assertions and previsions it contains, the lower is the probability (in mathematical sense) that all of them can be satisfied simultaneously[13].
As a consequence the greater the content of a theory, the greater the probability of being falsifiable (as it has a minor probability of being true): thus it has an higher degree of being controllable as it can be submitted to stricter controls. Only if they have been overcome, one can infer that it is a satisfying theory.
This epistemological thought is called *falsifiability*: only a theory that can be questioned is considered as interesting and then taken into account. Only that theory standing beyond all the tests which it has been submitted to, is believed as a true one, even if it can still be falsified. This is opposed to *verificationism*, a philosophical doctrine maintaining that a theory can be accepted only if it can be verified or confirmed statistically by a large number of experimental proofs.
The Theory of Special Relativity moves towards a higher informative content but a less logical probability than the previous theories it unifies (Newton and Maxwell's). Consequently, as it has exposed more easily refutable predictions, it has been subjected to stricter controls from which, however, it was the winner.
Experiments as Hafaele and Keating's one or Bertozzi's one as well as the application of Special Relativity to particles' motion within the modern accelerators constitute demonstrations of that predictions, nevertheless being highly unlikely compared to the previous theories.
Indeed a theory is expected to be *independently controllable* [80]: not only it must be able to give the solution to the unsolved problems but also it is supposed to bring new consequences to scientific knowledge that are obviously controllable. This is an important feature to highlight as otherwise ad-hoc theories could be developed without taking a step forward in the scientific progress, namely without letting science be closer to the truth. Indeed without any new predictions (since they have been proven) one would never be led to believe that one theory is closer to the truth with respect to another one.
Actually the presence of new predictions results in new crucial experiments that allow to consider a theory better than another one which is suitable to save known phenomena as Lorentz-Fitzgerald's theory, an attempt to preserve Newtonian Mechanics taking into

[13]For instance if a theory infers that a = "On Friday it will rain" and b = "On Saturday it will be sunny", the theory stating that ab = "On Friday it will rain and on Saturday it will be sunny" has more logical content (it says more information) but the probability of truth of this sentence is smaller than the probability that each one of the two assertions a and b occurs independently of one another $p(ab) < p(a), p(b)$ [80].

account the new result obtained from Michelson-Morley's experiment.
The *empirical success* ([80]) too is a fundamental requisite a theory is not able to leave out of consideration. As far as Special Relativity is concerned, the observed facts totally agree with the theoretical contents of the theory and thus with the two principles conceiving it: this circumstance could lead someone to think about them as proven truth.
On one side there is the effectiveness of the theory in the view of the scientific progress while on the other side there is the truth of the principles it states: is the first condition enough to make the Theory of Special Relativity "true", thus being able to confirm the principles?

Subjectivistic and Objectivistic thought of truth

The experimental evidences of the phenomena (as loss of simultaneity, time dilation) predicted by the theory itself as consequences of the content of the principles can not represent a new and different confirm of the principles themselves. For instance if one is able to observe the phenomenon of time dilation, having previously considered as true the principle of constancy of light speed, then one can not state that this principle is consequently necessary valid.
From the point of view of mathematical logic, if the principles $\mathcal{P}$ imply some phenomena f, then the assertion:

$$\frac{\mathcal{P} \to f, f}{\mathcal{P}} \tag{A.1}$$

is <u>false</u>. If the assertion $\mathcal{P}$ implies f and f is a true assertion, then it is false to state that $\mathcal{P}$ is true.
Then stating that the principles of Special Relativity are true since its experimental proofs have been demonstrated true is a fallacious reasoning from the logical point of view. This is the so-called *affirming the consequent*, a formal fallacy in which from the statement of an effect one infers the existence of a cause.[14]
The only one assertion which is necessarily true is instead:

$$\frac{\mathcal{P} \to f, \neg f}{\neg \mathcal{P}} \tag{A.2}$$

If the statement $\mathcal{P}$ implies f and f is a false one ($\neg f$), then also $\mathcal{P}$ is false: it is the *modus tollens*.[15]
Then the only conclusion one can infer is that if the experiments are not explained by a theory ($\neg f$), then necessarily this theory is not the right one.
For instance the time dilation is a "new" phenomenon but it exists regardless of a new

[14] "If it rains, then the road is wet." The assertion "Since the road is wet, then it rains" is false.
[15] "If it rains, then the road is wet". The assertion "Since the road is not wet, then it does not rain" is true.

theory: it is true that we know about it thanks to a theory that predicts it and then the experiments carried out to detect it were guided by the theory. Probably without Special Relativity we would have never thought to perform such experiments since they were thought to control (in Popper's meaning) this theory. But this is not enough to validate it. Indeed if we accept the whole corpus of Special Relativity as a true theory since it allows us to explain some theoretical and experimental inconsistencies, we consider as true all its parts, the postulates as well as their consequences[16]. It is thus clear that, as we previously have considered true those postulates, then time dilation for instance finds an evident explanation by Special Relativity as it exists already within that theory. But as we already seen, this is not enough to let us consider the principles as true as phenomena pre-exist to the theory that tries to explain them.

Indeed with wrong premises, anything can be inferred: if a is a wrong statement, from it one can infers true as well as wrong assertions. Postulates could be also wrong but actually they would continue to confirm the phenomena and to be confirmed by them.

The truth of the principles of Special Relativity might appear as a *law of non-contradiction.* Since reality does not prove it wrong, namely the experiments agree with the postulates, then they imply some kind of truth: in this way, law of non-contradiction becomes a parameter to accept or refuse a theory.

This kind of thought is called *subjectivistism* of scientific truth.

However, the subjectivistism is not a useful criterion of truth as one could also demonstrate the non-contradiction of a certain system which is wrong. But if an *experimentum crucis* is found contradicting this theory, one would determine its falsity.

This idea is validated by Physics itself which proceeded in this way throughout its history: the theory of aether was considered a true one, then it was demonstrated to be wrong. We can refer also to other examples as the idea of the *vis viva* or of the *caloric fluid.*

Non-contradiction, the truth of a theory *through the verifiability*[17], can not prove its truth. Instead Einstein seems to suggest (pg. 177) that it can not exist a clear thought with respect to the theoretical contents. He seems to agree with an *objectivistic* point of view by which one can only says that:

> we are looking for truth but, once we have found it, we can not know to have found it; we do not have any criterion of truth but nevertheless we are guided by the idea of truth as a *regulative principle* and, even if we do not have criteria through which we can recognise truth, there are thing that have the appearance of criteria of progress towards truth.[18]

[16] Loss of simultaneity, time dilation...

[17] [80]

[18] [80] - my translation from Italian.

An example of this criterion is the heart itself of Special Relativity, namely the covariance principle of laws of nature under Lorentz transformations. Einstein himself considered it as a *heuristic aid in search for general laws of nature* [79]. It is a guide to search over for new laws as well as Newton's laws were up to XIX century: Classical Mechanics, by having supplied us with the motions of celestial bodies in a surprising levels of details, is a *considerable measure of truth* [79] but it can not hold electrodynamic and optical phenomena separated.

Popper explained this concept with an example: let us consider a mountain which is always surrounded by clouds. An excursionist who attempts to climb the mountain will never be sure to have reached its peak since, due to the clouds, he will never distinguish the main peak from the shorter ones.

Thus the objectivistic point of view is constantly looking for the truth but at the same time it knows of the absence of an effective criterion for the identification of the truth, having refused, being them fallible, the subjectivistic criteria as the theory of coherence (law of non-contradiction), the theory of evidence ("known as true" = "true") and theory of instrumentalism ("what is useful is true").

We have some criteria which route us towards the truth: but at the same time, our questioning of having reached it implies the idea of an objective truth which, however, we have to admit that we may not be able to reach.

The peculiar feature of science is the *critical approach* without experimental proofs to state its own principles: the interest is to test physical theories rather than to confirm them, in order to find some errors leading Physics towards more complete ones.

We are interested not to *establish the certainty or the probability of scientific theories*[19]. It is a different aim: according to science, it is fundamental to behave critically, not dogmatically, looking for conditions which can subject its theories to critical tests, eventually in order to refute them.[20]

The approach of Einstein

Einstein was always careful with his statements, especially with the contents of the scientific assertions with respect to their predictions.

Einstein's open-minded approach did not concern only the Theory of Special Relativity but also the Theory of General Relativity: he was aware that as well as all scientific theories, also his one could have been confuted.

In 1919 during a conference in Vienna which Popper himself attended, Einstein exposed

[19] [80]
[20] [111]

the new theory of gravitation without presenting it as the ultimate one.[21]
Indeed some years before, with respect to the effect of the deflection of a light ray and the possibility of revealing it during an eclipse, Einstein wrote that it would have been possible to verify the "correctness or non-correctness of this deduction"[113].
Similarly, with respect to the effect of the shift of the light spectrum due to the presence of a gravitational field (*gravitational redshift*), Einstein pointed out that "if it did not exists a redshift of the spectral line by a gravitational field, then the Theory of General Relativity would be untenable".[22]
Einstein's attitude was always open to take into account circumstances that could have support as well as confute his theory. This consideration influenced the thought of Popper who will state that he made explicit some implicit key points within Einstein's work.
Einstein was critic with respect to his own Theory of General Relativity: as well as the Theory of Special Relativity, it is a provisional conjecture. It is only a step in the path of science towards a richer and more controllable knowledge.
In a letter of June 1935 addressed to Popper, Einstein stated to agree with falsifiability as the crucial feature of each theory concerning reality.[23] But some roots can be found yet in 1920 ([79]): Einstein well understood that the principle of covariance under Lorentz transformations should be continuously subjected to *critique*, led not by reason itself but by observations:

> if a general law of nature were to be found which did not satisfy this condition, then at least one of the two fundamental assumptions of the theory would gave been disproved ([79]).

This proves that the use of the word *postulate* or *principle* to describe the principles of Special Relativity is ambiguous and does not reflect the profounder awareness Einstein had about them, according also to the epistemological falsificationist interpretation of Popper. The postulates of Special Relativity are open (as it is implied by the assertions of Einstein himself) to the possibility of not stating any absolute, *a priori* effective truth but they are only a way *to adapt the physical theory to the observed facts*: the truth is not *the only one aim of science* [80] but this one has to deal also with *answering to our problems* [80]. As soon as a scientific truth (or better a conjecture about it) sets itself an answer to a really important issue, it becomes relevant to science.

[21] [112]
[22] [113] - my translation from Italian.
[23] [114]

Summarising

Summarising what we have investigated so far, the truths science states (theories as well as postulates or principles) are not eternal, immutable dogma: they are a guide in the path of searching for the real one truth, however without any certain criteria that allows us to determine whether a theory is true or not.

The laws of Physics always appear as hypotheses themselves since it would be enough only one experimentum crucis to contradict a theory, no matter how numerous are the controls we have subjected it to. The verificationism would require an infinite class of controls to prove that a theory is certainly true whilst falsifiability needs only one counterexample to prove that a scientific hypothesis is wrong: *it is enough only one black swan to falsify the statement that "All the swans are white"*.

Indeed our hypotheses can be falsified, namely *confuted by experience*. Consequently we can deem as scientific theories only those systems of assertions that can be subjected to controls aiming to show their falsity, that is those theories having always at least one potential falsifier allowing their control. Thus it is more correct to talk about the falsifiability of a scientific theory instead of its "truth".

Another relevant example concerns the Theory of Thermodynamics in which 1+3 principles (principle 0, first, second and third principle) are defined: indeed the first and the second principles are assumed to be true and it is unlikely that physicists are going to deny it. They are strong statements, assumptions effective to describe reality that has never been contradicted even if there have been many attempts for instance to create perpetual motions: theoretically these principles could be still contradicted by the same reality.

It is interesting to highlight that the old steady-state model of Fred Hoyle hypothesised the "creation" of hydrogen atoms (very slowly) in the empty space, thus contradicting the first principle of the Thermodynamics. But, this is the point, nobody ever accused this theory of being "not-scientific" one, since it speculated about previsions that could have been experimentally proven.

In conclusion, the principles of Special Relativity can not constitute principles in *a priori* and absolute meaning but as well as all the physical laws they are presuppositions open to the possibility of being denied. Using the more correct translation of "premise" gives immediately the physical meaning of these assertions instead of the formal-mathematical one.

A.2 Upon the variation of the inertia of the energy

Consider a reference system $\mathcal{K}$ with a body of mass m that emits two photons each of them with energy ε. Before the emission, the at-rest energy is $E_{0,i} = mc^2$ while after it becomes $E_{0,f} = mc^2 - 2\varepsilon$. Thus after the cooling the body diminishes its mass by $\Delta m = \Delta E_0/c^2 = -2\varepsilon/c^2$.
Now let consider an inertial frame $\mathcal{K}'$ in uniformly relative motion with respect to $\mathcal{K}$ with speed v along the x-direction. The law of transformation for the energy implies that for the body (being it at rest $p_x = 0$):

$$E'_i = \gamma E_{0,i} = \gamma mc^2. \tag{A.3}$$

While after the emission:

$$E'_f = \gamma E_{0,f} = \gamma(mc^2 - 2\varepsilon). \tag{A.4}$$

Then it will seem that in $\mathcal{K}'$ the mass diminishes by:

$$\Delta m = \frac{\Delta E'}{c^2} = \frac{E'_f - E'_i}{c^2} = -\frac{2\gamma\varepsilon}{c^2} = \gamma E_0, \tag{A.5}$$

that is it seems that, since the body in $\mathcal{K}'$ is seen as moving, the lost mass is more then in $\mathcal{K}$ by the γ factor, thus enhancing the idea of a relativistic mass.
However we must remember that in $\mathcal{K}'$ the body is yet moving and then before the emission its momentum is:

$$p'_i = -\gamma\frac{v}{c^2}E_{0,i} = -\gamma mv \tag{A.6}$$

While after the emission:

$$p'_f = -\gamma\frac{v}{c^2}E_{0,f} = -\gamma mv + 2\frac{v}{c^2}\varepsilon \tag{A.7}$$

Then the variation of mass is:

$$\Delta m^2 = \frac{1}{c^2}\left(\Delta E^2 - \Delta p^2\right) = \left(\frac{4\gamma^2\varepsilon^2}{c^2} - \frac{4\gamma^2\varepsilon^2 v^2}{c^4}\right) = \frac{4\gamma^2\varepsilon^2}{c^2}\cdot\frac{1}{\gamma^2} = \frac{4\epsilon^2}{c^2} \tag{A.8}$$

Actually also in the inertial frame $\mathcal{K}'$, the variation in the mass of the body is always $-2\varepsilon/c^2$, showing that the inertia mass of an object or its variation is the same in all the reference frames.

A.3 Metric in Minkowski spacetime

A general Lorentz transformation has components $\Lambda^{\mu}{}_{\nu}$ such that

$$\Lambda = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix}. \tag{A.9}$$

Thus if a^{μ} is a four dimensional vector, its Lorentz-transformed is

$$a'^{\,\mu} = \Lambda^{\mu}{}_{\nu}\, a^{\nu}, \tag{A.10}$$

where the usual convention on the summation over repeated upper and lower indices has been adopted. For instance, the four dimensional position vector $x^{\mu} = (ct, x, y, z)$ is a quadrivector since, when an inertial frame is changed, x^{μ} transforms as $x'^{\,\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$.
Instead the peculiar property of spacetime pseudo-Euclidean geometry is represented by the *metric tensor* or simply *metric* η of elements $\eta_{\mu\nu}$:

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.11}$$

This consideration on the real geometrical nature of spacetime points out what we just touch upon by the beginning of Section 2.3. Spacetime is a four-dimensional space that owns a pseudo-Euclidean geometrical structure with a real space coordinates and a real time coordinate. All the particular properties of the nature of spacetime are inside the metric and there is thus no need to introduce a complex time that remains a mathematical formalism inside an Euclidean four-dimensional space which is useful to derive the same considerations one can get from an analysis of the pseudo-Euclidean spacetime.
For instance in Eq. (2.49) we derived the definition of interval starting from an analogy with the three-dimensional Euclidean space. However we can consider now two events connected by a displacement four-vector $\Delta x^{\mu} = (c\Delta t, \Delta x, \Delta y, \Delta z)$: we know that their distance is the interval[24] $\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$. But it is easy to show that this is obtained from:

$$\Delta s^2 = \eta_{\mu\nu}\Delta x^{\mu}\Delta x^{\nu}. \tag{A.12}$$

[24] It is different from the definition in Eq. (2.49) because of the choice of the signature.

Being Δs the distance between two points (events) in spacetime, Eq. (A.12) defines the squared scalar product between the two four-vectors Δx^μ and Δx^ν.
Thus the length s of a four dimensional vector a^μ in spacetime, that is its norm, is defined as:

Definition 15 (Spacetime distance)**.**

$$s^2 = |\langle a^\mu, a^\mu \rangle|^2 = \eta_{\mu\nu} a^\mu a^\nu = a_\mu a^\mu. \tag{A.13}$$

As we said before, *the interval is invariant under Lorentz transformations* by construction since it is the distance in spacetime and thus, as Euclidean distance is invariant under rotation in the three-dimensional Euclidean space, the interval is invariant under the rotation in the four-dimensional spacetime, namely under Lorentz transformations. Formal demonstration about invariance of s^2 can be found in textbooks on Special Relativity (as [70, 115, 100]) and Lorentz transformations are derived as the only transformation preserving the interval. Now we can add some more constraints about the element of matrix (A.9): given two quadrivectors a^μ and a^ν the infinitesimal displacement is $ds^2 = \eta_{\mu\nu} da^\mu da^\nu$. If we now consider a Lorentz transformation of a^μ and a^ν, the new infinitesimal interval will be $ds'^2 = \eta_{\mu\nu} da'^\mu da'^\nu$. Then because of the invariance of ds^2 we have:

$$\begin{aligned} ds'^2 &= ds^2 \\ \eta_{\mu\nu} da'^\mu da'^\nu &= \eta_{\alpha\beta} da^\alpha da^\beta \\ \eta_{\mu\nu} \Lambda^\mu{}_\alpha da^\alpha \Lambda^\nu{}_\beta da^\beta &= \eta_{\alpha\beta} da^\alpha da^\beta \\ \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta da^\alpha da^\beta &= \eta_{\alpha\beta} da^\alpha da^\beta \\ (\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta - \eta_{\alpha\beta}) da^\alpha da^\beta &= 0. \end{aligned} \tag{A.14}$$

The last relation must be true for any infinitesimal displacement da^α and da^β, then:

$$\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \tag{A.15}$$

That can be rewritten in matrix notation as

$$\Lambda^T \eta \Lambda = \eta, \tag{A.16}$$

which is the non-Euclidean analogue of the orthogonality condition for the rotation matrices in the Euclidean space.

A.4 Implication of length contraction

The length contraction can not be explained admitting a physical deformation of the object because a deformation is described by a stress tensor: according to $\mathcal{S}$-observer who does not measure the road contracted, the stress tensor is zero. But if it is zero in $\mathcal{S}$ reference frame, it is zero in all the other reference frames. Thus observer in $\mathcal{S}'$ would not see any contraction as according also to him the stress tensor would be null. The same reasoning holds if we star considering $\mathcal{S}'$-observer: according to him the stress tensor would be not null and thus also $\mathcal{S}$-observer should see the road restricted.
Moreover the only physical deformation could not explain the contraction of *space* where no atoms are and then no contraction would be possible. This is particularly important when we deal with the muon experiment: if we do not assume that space itself contracts according to the travelling muons, their mean lifetime would not be sufficient to reach Earth.
Another particular evidence that the three-dimensional road the two observers are measuring is different, is given in Figure A.1.

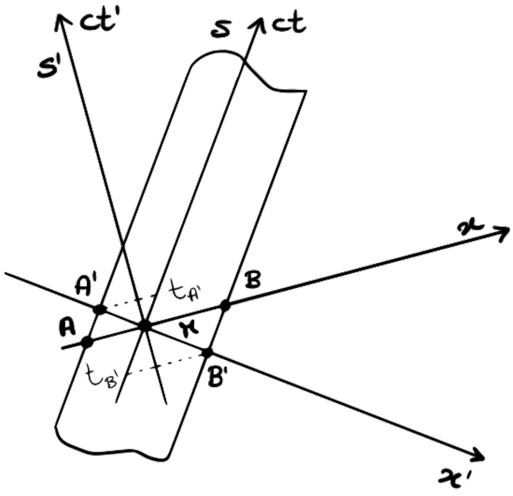


Figure A.1: According to observer $\mathcal{S}'$ the road AB is $A'B'$ which is shorter then AB but it is another three-dimensional object, different from the three-dimensional road AB. It is made up of event A' belonging to $\mathcal{S}$'s future and B' belonging to $\mathcal{S}$'s past.

Consider two observers $\mathcal{S}$ and $\mathcal{S}'$ that meet together at event M. According to observer $\mathcal{S}$, at the time of meeting, the road he measures is the three-dimensional cross-section AB, belonging to the plane of his present as the road AB exists simultaneously at the his present moment.
However observer $\mathcal{S}'$ will see that his three-dimensional plane of present intersect the four-dimensional worldtube of the road in events A' and B' that are then simultane-

ous with respect to him. Because of loss of simultaneity, according to him the three-dimensional road existing at time of the meeting is made up of the events $A'B'$. But as we can see from Figure A.1, event A' belongs to the future of $\mathcal{S}$-observer: the time coordinate of event A' follows the time of the meeting t_M that is the present time of observer $\mathcal{S}$. In the same way, event B' belongs to the past of $\mathcal{S}$-observer as its time coordinate comes before the time of the meeting t_M. According to $\mathcal{S}'$-observer the three-dimensional road at his present consists in a part that existed in $\mathcal{S}$'s past, a part that exists also in $\mathcal{S}$'s present (event M) and a part that will exist in $\mathcal{S}'$ future.

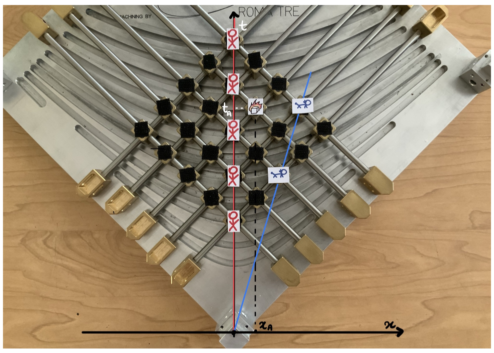
The conclusion is that necessarily the road the observers are measuring must be a *different* three-dimensional objects. This implies that road should be a four-dimensional entity in order to be possible different three-dimensional roads that are the cross-section of the observers' set of simultaneous event (present planes).

Appendix **B**

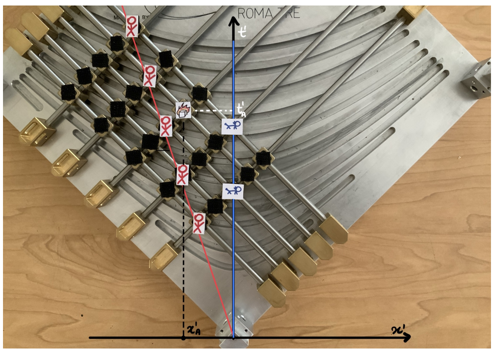
Appendix to Chapter 3

B.1 The Physics behind the spacetime globe

Consider Figures B.1a and B.1b:



(a) $\mathcal{S}$-observer's reference frame.



(b) $\mathcal{S}'$-observer's reference frame.

Figure B.1: Changing reference frames from the red man's one ($\mathcal{S}$-observer) to blue cat's one ($\mathcal{S}'$-observer).

In $\mathcal{S}$-observer's reference frame (Figure B.1a) we label the positions of $\mathcal{S}$-observer (red man) as R_i and the ones of $\mathcal{S}'$-observer (blue cat) as B_j:

- R_i is a collection of events in the form $R_i = (0; i)$. They lay over the line of equation $x_i = 0$;
- B_j is a collection of events in the form $B_j = (j; m \cdot j)$ where m is a real number equal to the inverse of the speed of $\mathcal{S}'$-observer with respect to $\mathcal{S}$-observer (in the case of Figure B.1a $m = 3/c$). They lay over the line of equation $t_i = m\, x_i$.

In the same way, in $\mathcal{S}'$-observer's reference frame the mechanical transformed positions of $\mathcal{S}$-observer (red man) are labelled as R'_k while the ones of $\mathcal{S}'$-observer (blue cat) as B'_h:

- R'_k consists of events in the form $R'_k = (-k; -n \cdot k)$ where n is a real number equal to the opposite inverse of the speed of $\mathcal{S}$-observer with respect to $\mathcal{S}'$-observer (in this case of Figure B.1b $n = -3/c$). They lay over the line of equation:

$$t'_i = n\, x'_i = -\frac{c}{3}\, x'_i; \tag{B.1}$$

- B'_h consists of events in the form $B'_h = (0; h)$. They lay over the line of equation:

$$x'_i = 0. \tag{B.2}$$

Then it easy to see from images in Figure B.1 that for each value of i, R_i lay on the same hyperbola which R'_k events lay on (for each value of k). In the same way for each value of j, B_j events lay on the same hyperbola which B'_h events lay on (for each value of h). Then according to Preposition 1, R_i and R'_k are linked by a Lorentz transformation $\mathcal{L}_1$ as well as B_j and B'_h by transformation $\mathcal{L}_2$.
Moreover we can show that transformations $\mathcal{L}_1$ and $\mathcal{L}_2$ are the same.
Consider P_i, a set of events having coordinate $(x_i, t_i) = (0, i)$ and a boost with speed $v = c/3$: the corresponding Lorentz-transformed set of events P'_i has coordinates:

$$\begin{aligned} x'_i &= \gamma(x_i - vt_i) = -\gamma v\, i = -\frac{c}{3}\gamma\, i \\ t'_i &= \gamma(t_i - vx_i) = \gamma\, i \end{aligned} \tag{B.3}$$

The obtained P'_i set consists in a collection of events laying on a line of equation

$$t'_i = -\frac{1}{v}x'_i = -\frac{c}{3}x'_i, \tag{B.4}$$

with slope $m' = -1/v = -3/c$.
We see that Eq. (B.4) describes the same line of the one consisting of events R'_k in Eq. (B.1).
As a consequence set P'_i coincides with the set R'_k and, being $P_i \equiv R_i$, then R'_k events are the Lorentz-transformed of events R_i with a boost of speed $c/3$.
Indeed as example we consider the second event of set R_i on the spacetime globe (Figure B.1a) and we measure its coordinates using our ruler: $R = P = (0, 3.25)$ with an uncertainty of $\pm\, 0.02$ on both the coordinates.

According to Eq. (B.3) the new coordinates of the transformed event should be[1]:

$$P' \simeq (-3.447 \pm 0.007; 1.15 \pm 0.02), \tag{B.5}$$

where the uncertainty over x' and t' has been obtained multiplying 0.02 respectively by $\gamma/3$ and γ. Now if we measure the position of the second event of set R'_k on the spacetime globe in Figure B.1b, we find that its coordinates are:

$$R' = (-3.44 \pm 0.02; 1.13 \pm 0.02) \tag{B.6}$$

Then comparing the confidence intervals of the measure of x and t between P' and R', we see that they overlap. The two set of values are thus compatible and it is true that the coordinates of event R' are obtained from the Lorentz transformation of the coordinates of event R as they have the same coordinates.
In the same way, if we consider a set of events $Q_j = (j, (3/c)j)$ and a boost with speed $v = c/3$, then the corresponding Lorentz-transformed set of events Q'_j has coordinates:

$$\begin{aligned} x'_j &= \gamma(x_j - vt_j) = 0 \\ t'_j &= \gamma(t_j - vx_j) = \frac{1}{\gamma v} j = \frac{\sqrt{8}}{c} j \end{aligned} \tag{B.7}$$

Once again, being $Q_j \equiv B_j$, and Q'_j the Lorentz-transformed of Q_j, we will have that $Q'_j \equiv B'_h$ if the two lines which the two set of events Q'_j and B'_h lay on coincide.
Eq. (B.7) describes the line $x'_j = 0$ with a spacing of $(\sqrt{8}/c)\, j$ between each event on the t' axis. Then we see that the line described by the set of events Q'_j coincides with the one described by the set of events B'_h, being this one $x' = 0$ (Eq. (B.2)).
This means that $B'_h \equiv Q'_j$ and that B'_h set is the Lorentz-transformed of B_j set with a speed $v = c/3$.
As done before, we can consider as a check one event of the set B_j and the corresponding event of the set B'_h. Consider in Figure B.1a the first image of the blue cat: it has coordinate $B = Q = (1.06, 3.13)$ with an uncertainty of ± 0.02 on both the coordinates. Now according to Eq. (B.7) the transformed event Q' should have coordinates:

$$Q' \simeq (-0.042 \pm 0.03; 3.01 \pm 0.06), \tag{B.8}$$

where the uncertainty over x' and t' has been obtained as $\Delta x' = \Delta x + v\Delta t$ and $\Delta t' = \sqrt{8}\Delta t$ where $\Delta t = \Delta x = 0.02$.

[1] In natural units.

Now if we measure the coordinate of the same events in Figure B.1b, we get that they are:

$$B' = (0 \pm 0.02; 3.06 \pm 0.02) \tag{B.9}$$

Once again the two confidence intervals of the measure of x and t of event Q' and B' overlap and then we can conclude that they are identifying the same event on the spacetime globe. Then B' is the real Lorentz-transformed of event B.
Thus we have that the mechanical movement that moves:

$$B \to B' \tag{B.10}$$

$$R \to R' \tag{B.11}$$

corresponds to the correct Lorentz transformation shifting the perspective from the red man's one to the blue cat's one. It changes the reference frame from the $\mathcal{S}$-observer's one to the $\mathcal{S}'$-observer's one.
This is due to the fact that the movement $B \to B'$ allows us to make blue cat's worldline vertical (Lorentz transformation of B_j set) and also it calculates R'_k, the corresponding events of R_i set as seen from the cat (Lorentz transformation of R_i set).
In a single shot, we can make a Lorentz transformation of the worldline of both the two observers, defining the positions of both the blue cat and red man at each time.
It is important to make a physical consideration: actually the Lorentz transformation is only one, namely that one boosting our prospective into the blue cat's one. We do not have to confuse the determination of the coordinates of the events with the physical problem.
A Lorentz transformation is a change in the prospective of describing a motion. In our context we are performing a single Lorentz transformation as we want to move into the blue cat's reference system. We then have to transform all the events in the Minkowski diagram to determine how they look like from the blue cat's perspective. This consists in:

- transforming the blue cat's events (B_j);
- transforming the red man's events (R_i).

But the boost is always the same, namely the one with v speed equal to the blue cat's speed.
The single boost into the blue cat's perspective has as a consequences that $B \to B'$ and $R \to R'$. Individually the R' and B' coordinates (and in general all the events along the worldline of both the observers) are determined applying formulas in Eq. (3.1) to the coordinates of R and B events.

B.2 A collision problem

B.2.1 Introduction

Physics deals with the phenomenon of a collision between two or more particles through two different descriptions: the one of the center-of-momentum frame (COM frame) and the one of the target frame of reference (LAB frame[2]).
We are used to determine the state of the particles from their four-momentum $p^\mu = (E/c, \boldsymbol{p})$ and the Lorentz transformations define how it changes from one reference system to another. Moreover the use of the invariant mass $M^2 = P^\mu P_\mu$ (here P^μ is the total four-momentum of the system) constitutes an easier way to determine the components of the four-momentum of the particles in the two reference frames.
Now we are going to investigate what happens to the four-position x^μ of the particles.

B.2.2 Problem formulation

Let consider two particles with the same mass m moving towards each other: in the COM frame (Figure B.2) one of them starts from coordinate $x_1 = -L$ whilst the other one from coordinate $x_2 = L$, both with speed v. The particle on the left moves towards right with velocity $\boldsymbol{v}$ while the particle on the right moves towards left with velocity $-\boldsymbol{v}$.

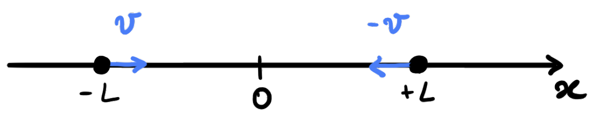


Figure B.2: Schematic illustration of the problem.

To simplify we can suppose that the two particles are created in the coordinate $x = \pm L$ with that velocity.
We want to:

- describe the motion of the two particles in the COM frame and in the LAB one coinciding with the particle on the right;
- draw Minkowski diagrams;
- determine how much does it takes to the particles to collide.

B.2.3 Resolution in the center-of-momentum frame

The description of the phenomenon in the COM frame is very easy and it can be illustrated as in Figure B.3:

[2]It is called also the laboratory reference frame: one of the particle is at rest.

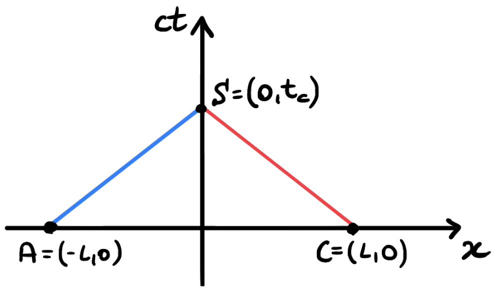


Figure B.3: Minkowski diagram of the problem in the COM frame.

The particle on the left is created in the event $A = (-L; 0)$ whilst the one in the right in the event $C = (L; 0)$. The worldline of the former is $t = (1/v)(x + L)$ (blue line in Figure B.3) whilst the worldline of the latter is $t = -(1/v)(x - L)$ (red line in Figure B.3).
Here the adopted point of view is the one of an observer *external* to the system, who is looking to the phenomenon and registering the components of the four-position of the two particles with respect to his own reference frame. Consequently the creation of the particles happens at the time $t = 0$ of the external observer, in the spatial point $x = \pm L$.
Now let us determine where the collision happens: from Figure B.3 we see that the two worldlines cross at the event S. It gives back the spatiotemporal coordinates of the collision which takes place in $x_S = 0$ (since we are in the COM frame) at time $t_S = t_c$.
As far as the time at which the collision occurs as measured by the external observer is concerned, according to the symmetry of our problem, it is equal to the time one of the two particles needs to travel the distance $d = L$ from the coordinate of the collision, moving with speed v.
Thus it is:

$$t_c = \frac{L}{v}. \tag{B.12}$$

B.2.4 From the COM to the LAB frame

Now we want to draw the Minkowski diagram of this phenomenon in the LAB (or target) frame in which one of the two particles is at rest (our target will be the one on the right) whilst the other one (the projectile) is moving.
Thus the diagram will be:

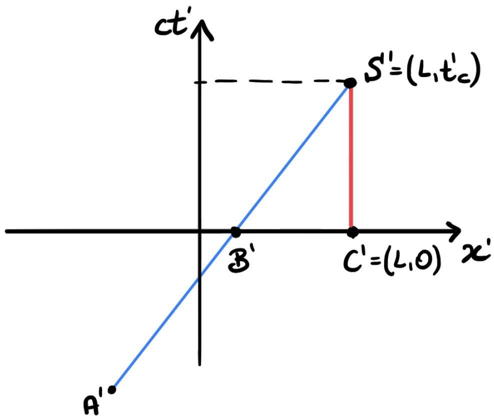


Figure B.4: Minkowski diagram in the LAB frame.

We need to determine the coordinates of the collision event S'.
Thus we need only to find out the right transformation of the four-positions from the COM frame to the LAB frame, since we already know that the four-momentum transforms via Lorentz transformation.
We base on that the target particle is always at rest in the coordinate $x = L$. It is important to highlight this feature of our problem as it is telling us that we can not directly apply Lorentz transformations to determine how the four-positions x^μ are transformed: indeed, taking into account the interpretation of the Lorentz transformations as the change of the point of view, they would yield us $x = 0$ as the position of the particle on the right and not $x_{C'} = L$ as instead it requires the formulation of the problem in the LAB frame. This difference we will see to be relevant.
The description goes through the point of view of the particle on the right, since in the target frame the other one has speed[3]:

$$v' = \frac{2v}{1+v^2}. \tag{B.13}$$

This is exactly the result one gets when the speed of the projectile particle (the one on the left) is determined with respect to the target particle (the one at rest): this proves that we are assuming the point of view of the latter in the description of the kinematics variables.
Thus the features of this reference frame are mixed:

- the physical quantity are defined with respect to the target particle;
- the coordinate are still reported on the Minkowski diagram of the *external* observer.

Thus going from the COM frame to the LAB frame the values of the coordinates are determined with respect to the particle at rest but drawn by an external observer.

[3]It can be easily demonstrated with the invariance of the four-momentum.

In order to find the right transformation, let us consider Figure B.5:

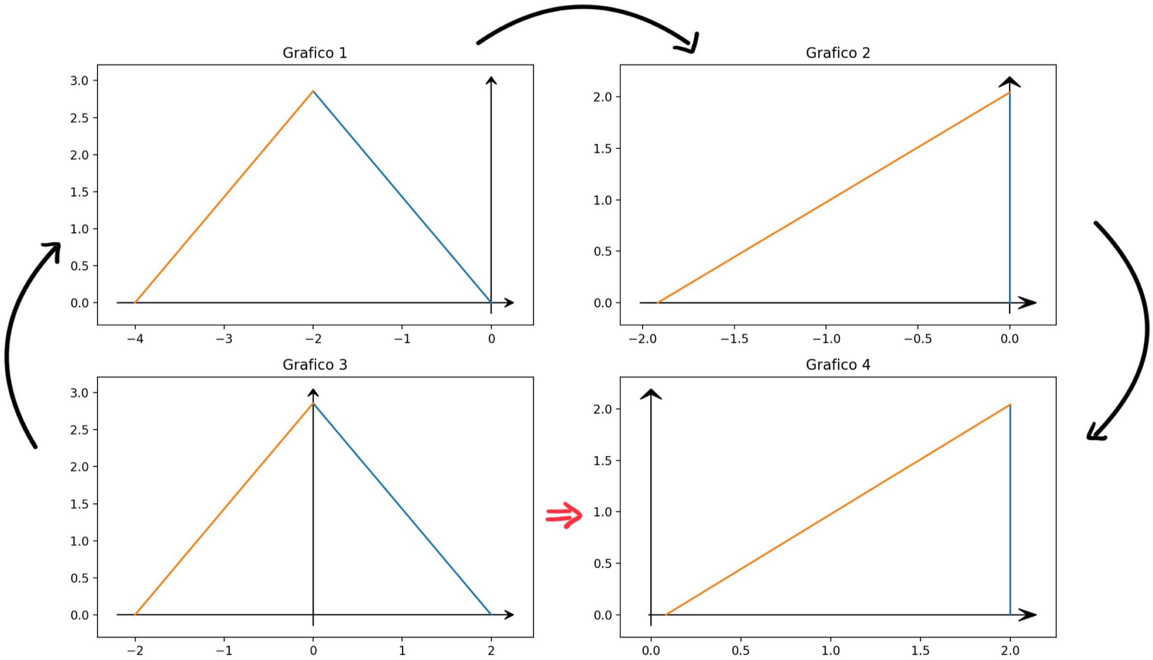


Figure B.5: Composition of the transformations.

Graphic 3 (bottom left) shows the description in the COM frame (the particle "are born" in $x = \pm L = \pm 2$) while Graphic 4 (bottom right) shows the target frame. Thus we are interested in the transformation of the coordinates $3 \to 4$.

Let us consider the reference frame in Graphic 1 (top left) and 2 (top right): the frame 1 is the COM frame translated by the vector $\boldsymbol{a} = (-L; 0)$. In this reference frame the spatial coordinates have been translated by an amount of $-L$: the particle on the right is now moving from the the origin in $x = 0$ whilst the one on the left from $x = -2L$.

Instead the reference frame 2 is the description of the collision from the point of view of the particle on the right: it was obtained rotating in the Minkowski spacetime the worldline of the particle on the right until it was vertical in $x = 0$. The observer describing the event is "riding" the particle on the right. Only in this reference frame it is possible to determine the variables of interest as for instance the duration of the process (from the birth of the particle to the collision) from the point of view of the particle on the right (the one at rest).

In order to go from frame 1 to 2, it is enough to apply a Lorentz transformation with β factor $\beta = v/c$, where v is the speed of the particle.

However we are not interested in the description of Graphic 2: we want to reach the description of Graphic 4 (target frame). But the frame 2 and 4 are linked by a translation of vector $\boldsymbol{b} = -\boldsymbol{a} = (L; 0)$.

Thus the transformation $3 \to 4$ is equivalent to the composition of the transformations

$$3 \to 4 = 3 \to 1 \to 2 \to 4, \tag{B.14}$$

namely going firstly through the frame 1, then through the frame 2 and then back to the frame 4. In this way we can ensure that the spatiotemporal coordinates of the particles are the correct ones and that the diagrammatic representation is the one of the external observer. Otherwise we could not take into account that the spatial translation has an effect also on the time coordinate, since Lorentz transformation mix the spatial coordinate with the temporal one. We will see that the result of the composition of all these transformations is different from the simple Lorentz transformation $3 \to 4$, exactly because one has to take into account the external observer with respect to who we are performing the transformation and the reference frame we are in.
Now we denote as x_i and t_i the coordinates in the i-th reference frame ($c = 1$):

- transformation $3 \to 1$:
$$\begin{cases} x_1 = x_3 - L \\ t_1 = t_3 \end{cases}; \tag{B.15}$$

- transformation $1 \to 2$ (we take into account that the velocity of the boost is negative: then hence on v means only the speed):
$$\begin{cases} x_2 = \gamma(x_1 + vt_1) \\ t_2 = \gamma(t_1 + vx_1) \end{cases}; \tag{B.16}$$

- transformation $2 \to 4$:
$$\begin{cases} x_4 = x_2 + L \\ t_4 = t_2 \end{cases}. \tag{B.17}$$

We are interested in the transformation $3 \to 4$. Rewriting the coordinates of the reference frame 4 with respect to the ones of the reference frame 3, we have:

$$\begin{cases} x_4 = \gamma((x_3 - L) + vt_3) + L \\ \\ t_4 = \gamma(t_3 + v(x_3 - L)) \end{cases}. \tag{B.18}$$

Denoting with (x, t) the coordinates in the COM frame and with (x', t') the ones in the target frame of the particle on the right, we have:

$$\begin{cases} x' = \gamma((x - L) + vt) + L \\ \\ t' = \gamma(t + v(x - L)) \end{cases} . \tag{B.19}$$

These are the correct transformations from the COM frame to the target frame as far as the spatiotemporal coordinates are concerned. They differs from the Lorentz transformation as these ones are taking into account the different position of the observer with respect to who the transformation is defined.
Besides we point out that in order to obtain the correct transformations we can not only add the component of the translation $+L$ to x' exactly because, as we previously stated, this translation has also an effect on the temporal coordinate.
These transformations are useful only to determine the spatiotemporal coordinate of the particles in the new reference frame.
Moreover we can show that these transformations do not effect the energy and the momentum of the particles, meaning that the four-momentum transforms "classically" via Lorentz transformation. Indeed transformation (B.19) can be rewritten as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + r^{\mu}, \tag{B.20}$$

where the matrix $\Lambda^{\mu}{}_{\nu}$ is the matrix of Lorentz transformations while r^{μ} is the vector

$$r^{\mu} = \begin{pmatrix} \gamma\beta L \\ (1 - \gamma) \cdot L \end{pmatrix}. \tag{B.21}$$

The transformations (B.20) are called *Poincaré transformation.*
Let now determine how the four-momentum p^{μ} transforms: since $p^{\mu} = mu^{\mu}$ where $u^{\mu} = dx^{\mu}/ds$ is the four-velocity in the COM frame, the four-momentum p'^{μ} in the target frame is:

$$\begin{aligned} p'^{\mu} &= m\frac{dx'^{\mu}}{ds} = m\frac{d}{ds}\left(\Lambda^{\mu}{}_{\nu}x^{\nu} + r^{\mu}\right) = \\ &= m\frac{d}{ds}\left(\Lambda^{\mu}{}_{\nu}x^{\nu}\right) + m\frac{d}{ds}\left(r^{\mu}\right) = \\ &= m\frac{dx'^{\mu}}{ds} + m\frac{d}{ds}\left(r^{\mu}\right) = \\ &= p'_L{}^{\mu} + m\frac{d}{ds}\left(r^{\mu}\right), \end{aligned} \tag{B.22}$$

where $p'_L{}^{\mu}$ denotes the Lorentz transformation of the four-vector p^{μ}.

Taking into account that $ds = cdt/\gamma$, then:

$$\frac{dr^0}{ds} = \frac{\gamma^4}{c^3}\beta L \boldsymbol{v}\cdot\boldsymbol{a} + \frac{\gamma^2}{\beta^2}a \tag{B.23}$$

$$\frac{dr^i}{ds} = -\frac{\gamma^4}{c^3}L \boldsymbol{v}\cdot\boldsymbol{a}. \tag{B.24}$$

These two terms contain some quantities depending on the acceleration $\boldsymbol{a}$ of the particle with respect to which we are performing the change of the reference frame. But since the particle moves with constant velocity, then $dr^\mu/ds = 0$.
Consequently

$$p'^\mu = p_L'^{\ \mu} \longrightarrow p'^\mu = \Lambda^\mu{}_\nu p^\nu, \tag{B.25}$$

that is the four-momentum under these new transformation always transforms via Lorentz transformations.
The spatial translation of the reference frame, which turns into a quadri-translation (both time and space are translated), has not modified the energetic components of the particles (energy and momentum).
This assumption is probably implicit when the energetic content of the particles is considered, going from the COM frame to the LAB one. Indeed we usually write directly the equations

$$\begin{cases} p_{cdm} = \gamma_{cdm}(p_{lab} + v_{cdm}E_{lab}) \\ \\ E_{cdm} = \gamma_{cdm}(E_{lab} + v_{cdm}p_{lab}) \end{cases} \tag{B.26}$$

as, if the speed is constant, the translation r^μ does not effect the four-momentum of the particle and thus the invariant mass of the system. In accord with the principle of relativity, the energetic descriptions of the collision in the two different reference frames are equivalent.

B.2.5 Resolution in the LAB frame

We have seen that if the quadri-position in the COM frame is x^μ, then the one in the LAB frame is x'^μ, defined by the set of transformations:

$$\begin{cases} x' = \gamma((x - L) + vt) + L \\ \\ t' = \gamma(t + v(x - L)) \end{cases} . \tag{B.27}$$

The inverse transformations are:

$$\begin{cases} x = \gamma((x' - L) - vt') + L \\ \\ t = \gamma(t' - v(x' - L)) \end{cases} . \tag{B.28}$$

Using Eq. (B.27), coherently with the description in the LAB frame, if in the COM frame the particle on the right is created in the event $C = (L; 0)$, then in the LAB frame the particle is still created in the same event $C' = (L; 0)$. Moreover, in accord with the representation in the Minkowski diagram, applying Eq. (B.28), the worldline of the particle on the right $t = -(1/v)(x - L)$ (Figure B.4) is transformed into the worldline $x' = L$.
We notice also that the event of the collision in the COM frame $S = (0; t_c)$ (where $t_c = L/v$) is transformed in the event $S' = (L, t_c/\gamma)$: the collision occurs in $x = L$ where the at rest particle is and with respect to this it happens at $t'_c = t_c/\gamma$.
This result was easily predictable as it is the proper time of the whole process for the particle on the right: indeed since in the COM frame this particle is moving, if we want to know the time interval between the two events "the particle on the right is at $(L; 0)$" and "the two particles collide at $(0; t_c)$" (that is the whole process) we have to determine the interval of proper time τ equal to $\tau = (t_c - 0)/\gamma = t_c/\gamma$.
Indeed if we consider Figure B.3, we have that the proper time is exactly:

$$\tau = \frac{1}{c}\sqrt{c^2 dt^2 - dx^2} = \sqrt{t_c^2 - \frac{L^2}{c^2}} = \sqrt{\frac{L^2}{v^2} - \frac{L^2}{c^2}} = \frac{L}{v}\sqrt{1 - \frac{v^2}{c^2}} = \frac{t_c}{\gamma} \tag{B.29}$$

From Minkowski diagram in Figure B.3, according to the external observer, the collision happens in the event $S' = (L; t_c/\gamma)$. Instead, with respect to the target particle the collision always occurs at time $t'_c = t_c/\gamma$ but in $x = 0$.
We point out that the only Lorentz transformation of event $(0; t_c)$ would yield a transformed time $t'_c \neq t_c/\gamma$.

Let now investigate the properties of the particle on the left: the event $A = (-L, 0)$ that in the COM frame is its starting point, is transformed into event $A' = (L(1 - 2\gamma), -2\gamma vL)$, set in the III quadrant.[4]
This is an important information: if in the COM frame the event A and C were simultaneous, namely the creation of the two particles happened in the same moment, in the LAB frame something different occurs. The particle at rest on the right will begin to count its

[4] $x < 0$ and $t < 0$ since the speed v is positive, having already taken into account in the equations of the transformations that the velocity of the boost is negative.

time from the instant $t'_{C'} = 0$ (with respect to the external observer) whilst the particle on the left will start moving at the instant $t' = -2\gamma v L$ (with respect to the external observer). But as far as the physical description with respect to the target is concerned, regardless of its graphical representation, instants of time less then $t'_{C'}$ do not make any sense as the particle on the right will start recording times for $t' \geq t'_{C'}$: it will see[5] the particle on the left only when $t' = t'_{C'}$ since it is created at $t'_{C'} = 0$. For smaller times, according to the target, it is as if the projectile particle does not exist.

Thus we need to determine where the particle on the left were at $t = t'_{C'}$; for this purpose we now derive its worldline. Since it crosses event S' (the collision event), then the slope m' of the line is:

$$m' = \frac{t'_{S'} - t'_{A'}}{x'_{S'} - x'_{A'}} = \frac{t'_c + 2\gamma v L}{L - L(1 - 2\gamma)} = \frac{L/(v\gamma) + 2\gamma v L}{2\gamma L} = \frac{1/\gamma^2 + 2v^2}{2v} = \frac{1 + v^2}{2v}. \tag{B.30}$$

As in a Minkowski diagram the speed v' of an uniformly moving particle is the inverse of the slope of its worldline, we have that the speed of the particle on the left in the LAB frame is

$$v' = \frac{2v}{1 + v^2}, \tag{B.31}$$

that is exactly the speed v' in Eq. (B.13): it is the speed of the particle on the left as measured by the particle on the right.

This shows once again that in the LAB frame the physical quantities are defined with respect to the particle at rest but are displayed in a Minkowski diagram not showing its point of view but the one of an external observer in $x = 0$.

Returning to the worldline of the particle on the left, it will have an expression as $t' = m'x' + q'$ where q' is the vertical intercept to be determined. As we know that the line crosses event S':

$$q' = t'_{S'} - m' x'_{S'} = \frac{t_c}{\gamma} - \frac{1 + v^2}{2v} L = \frac{L}{\gamma v} - \frac{1 + v^2}{2v} L. \tag{B.32}$$

Then the worldline of the particle on the left can be written as:

$$t' = \frac{1 + v^2}{2v}(x' - L) + \frac{L}{\gamma v}. \tag{B.33}$$

At the instant of time $t'_{C'} = 0$, this particle is in the position:

$$0 = \frac{1 + v^2}{2v}(x' - L) + \frac{L}{\gamma v} \Longrightarrow x' = L - \frac{2L}{\gamma(1 + v^2)} \tag{B.34}$$

[5]We should also take into account the time light needs to reach the particle on the right from the one on the left but it is not what we are looking for.

Since the particle at rest is in $x' = L$, then the distance between the two particles is:

$$d' = \frac{2L}{\gamma(1+v^2)} \tag{B.35}$$

This distance is covered with speed v' (Eq. (B.31)) and thus the projectile will hit the target, according to this one, after an interval of time t^* equal to

$$t^* = \frac{d'}{v'} = \frac{2L}{\gamma(1+v^2)} \cdot \frac{(1+v^2)}{2v} = \frac{L}{\gamma v}. \tag{B.36}$$

This value is exactly the proper time of the event $\tau = t'_c$ with respect to the target particle, namely the duration of the whole process as measured by this one, since the projectile starts moving (with respect to the target) until it hits the particle.
Thus if we want to consider only the relevant Physics according to the target particle, the Minkowski diagram would be:

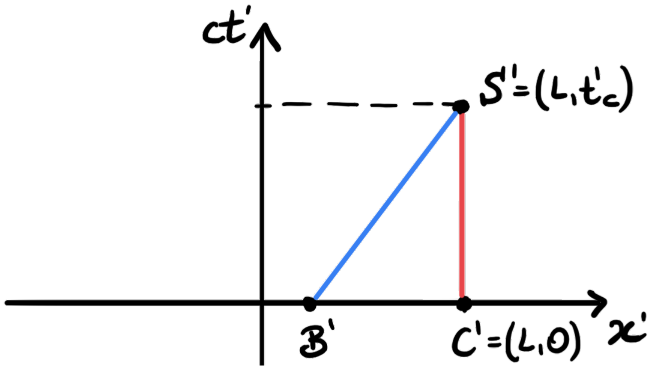


Figure B.6: Minkowski diagram of the problem in the target frame with respect to the particle at rest.

In Figure B.6 the diagram is the same as in Figure B.4 but there is no worldline for $t' < 0$ since, as we previously said, from the point of view of the target particle, it is not measurable.

B.2.6 Collision in two different reference frames

In this graphical simulation obtained using the Python codes we developed (3.4), we show a mixed use of the Minkowski diagrams as both spatiotemporal one and energy-momentum one. In particular we study the collision between two muons starting from coordinates $x = \pm 2$ according to two different reference frames: the COM one and the target one, in which one of the two particles is at rest.
At the beginning of the simulation (Figure B.7) there are two overlapping diagrams:

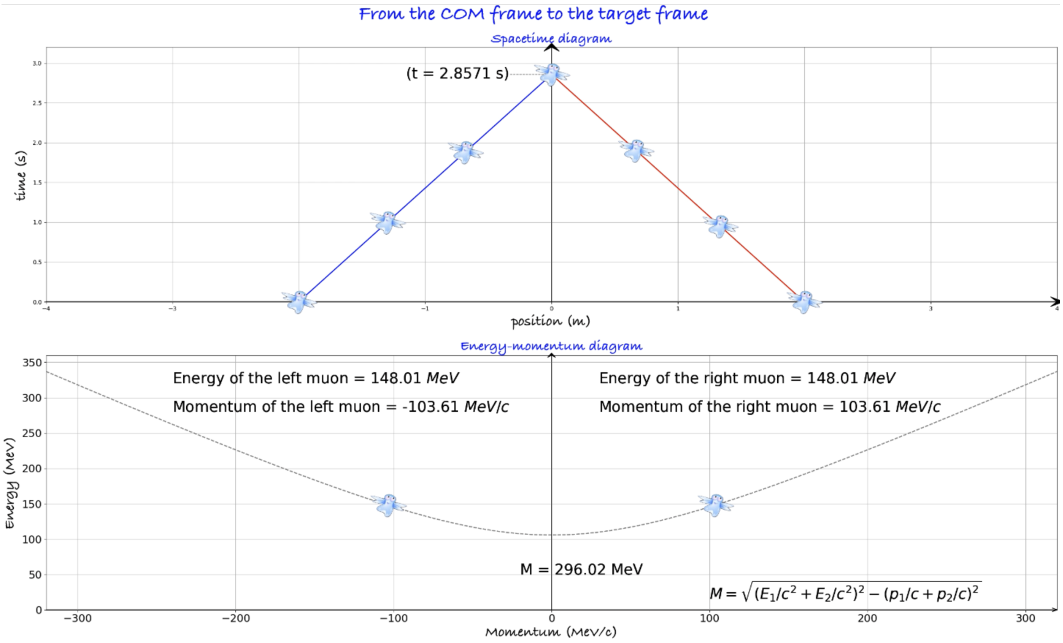


Figure B.7: Thirteenth simulation: snapshot of the beginning of the simulation

In Figure B.7 we have two graphs describing the collision in the COM frame in which it happens symmetrically in $x_c = 0$ at t_c:

- in the upper graph the worldlines of the two muons are shown within a spacetime diagram: they depart one from $x_l = -2$ m while the other one from $x_r = 2$ m, travelling with speed $v_{\mu_l} = 0.7\,c$ and $v_{\mu_r} = -0.7\,c$. In the COM frame the collision occurs at the instant of time shown in upper part of Figure B.7: $t_c = 2.85$ s.
  Indeed since the collision is symmetric, thus taking place in $x_c = 0$, this time coordinate is exactly the time one particle needs (for example the muons on the left) to travel the distance $d = |x_c - x_l|$ with speed $v = 0.7\,c$: $t_c = d/v = 2/0.7 \simeq 2.85$ s;

- in the lower graph the energy-momentum diagram of the system is shown: the muon (with mass $m_\mu = 105.7$ MeV/c) on the right has an amount of momentum $p_{\mu_r} = \gamma m v = -103.61$ MeV/c (it is negative since in the COM frame its velocity is negative) while its energy is $E_{\mu_r} = \gamma m c^2 \simeq 148.01$ MeV. Instead the muon on the left has an amount of momentum $p_{\mu_l} = -p_{\mu_r} = 103.61$ MeV/c since the speed is equal to the one of the other muon but it travels in the opposite direction. Its energy is the same energy of the other muon $E_{\mu_l} = E_{\mu_r} = E = 148.01$ MeV.
  Finally the invariant mass is also shown:

$$M = \sqrt{\left(\sum_{k=1}^{n} E_k\right)^2 - \left(\sum_{k=1}^{n} p_k\right)^2}, \tag{B.37}$$

where the sum is extended to all the n particles interacting in the collision. In this problem in the COM frame, we have that

$$M = 2E, \tag{B.38}$$

since the total momentum $\sum p_k$ is null while, as the two values of the energy of the particles are the same and it is equal to E, $\sum E_k = E_{\mu_r} + E_{\mu_l} = 2 \cdot E_\mu = 2 \cdot 148.01 = 296.02$ MeV.

Now we want to describe the collision in the target frame in which the muon on the right is at rest whilst the particle on the left is moving.
Using the transformations (B.19) for the spacetime diagram and the Lorentz transformation for the energy-momentum diagram, we have that in the LAB frame the description is:

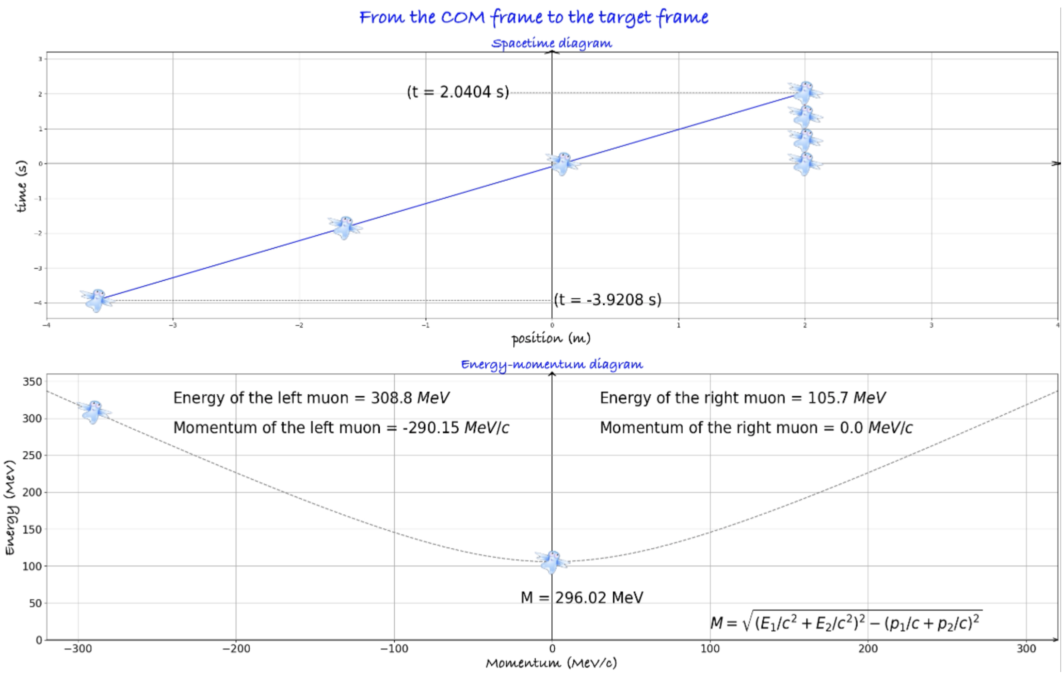


Figure B.8: Thirteenth simulation: snapshot of the end of the simulation

Let now understand the meaning of these two graphs in Figure B.8:

- in the upper graph we can see that the muon on the right is at rest (red vertical worldline) in $x = 2$ m (with respect to an external observer). It will see the muon on the left approaching with speed v', defined by the relativistic addition of velocities for a body moving with speed v_μ with respect to an observer moving with speed $-v_\mu$. We have that $v' = 2v_\mu/(1 + v_\mu^2) = 2 \cdot 0.7\,c/(1 + 0.7^2) \simeq 0.9395\,c$.
  With respect what we have said in the previous sections, adopting the same notation,

according to an external observer the particle on the left starts moving from event $A' = (L(1-2\gamma); -2\gamma v L) = (-3.60; -3.92)$, namely it is created in $x_{A'} = -3.60$ m, 3.92 seconds before that the target particle is created.
As far as this last is concerned, it will be created a time $t' = 0$ s in $x' = 2$ m and it will collide with the muon on the left in the event determined by the intersection of the two worldlines $(L; t_c/\gamma) = (2; 2.04)$. The target muon will receive the projectile muon 2.04 seconds after having being created while in the COM frame (Figure B.7) this happened after 2.85 s.
Moreover the target particle will see the projectile approaching from the coordinate $x' = L - 2L/\gamma(1+v^2) = 0.08$ m, namely the muon is $2L/\gamma(1+v^2) = 1.92$ metres away from it. Since this distance is covered with speed $v' = 0.9395\,c$ it will take exactly $t' = 1.92/0.9395 \simeq 2.04$ s.

- in the lower graph it is shown the energy-momentum diagram in the LAB frame: the target muon, being at rest, has momentum $p'_{\mu_r} = 0$ MeV/c while its energy is $E'_{\mu_r} \simeq 105.7$ MeV namely the content in energy of its mass.
  Instead, the muon on the left will have momentum[6] $p'_{\mu_l} = -2\gamma^2 m v \simeq 290.15$ MeV/c while its energy is equal to $E'_{\mu_l} = \gamma^2 m(1+v^2) \simeq 308.80$ MeV.
  Finally we can see that the invariant mass remains constant as in the LAB frame it is equal to:

$$M = \sqrt{(\gamma^2 m(1+v^2)+m)^2 - (-2\gamma^2 m v)^2} = 2m\sqrt{1+\gamma^2 v^2}, \qquad \text{(B.39)}$$

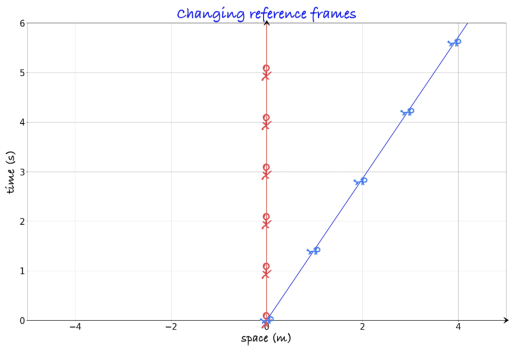
  namely $M \simeq 296.019$ MeV/c^2.

[6]Applying Lorentz transformations.

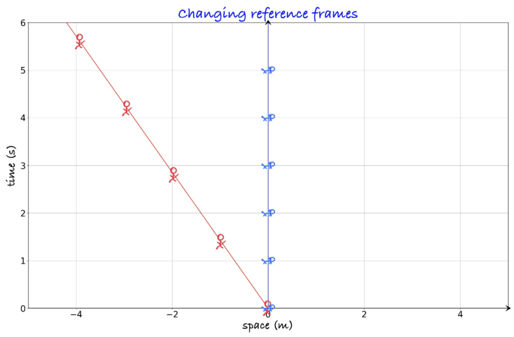
B.3 Python code example for one of the simulations

B.3.1 Main structure of the simulations

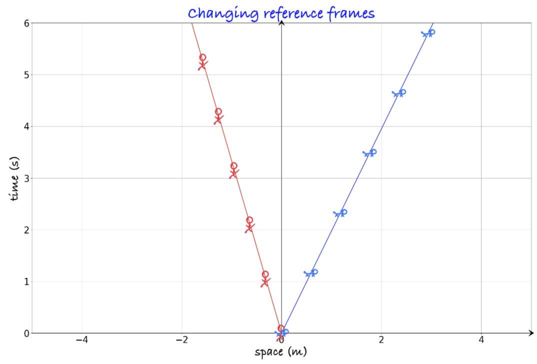
In this section we want to briefly describe the idea under our Python simulations meaning that how the codes have been developed.
The key idea is to define a parameter[7] c that changes throughout the simulation allowing the movement of the worldlines together with all the figures inside the graphs, as for instance the red man and the blue cat in Figure 3.37. This parameter will enter in each dynamical element of the simulations. Moreover all the numerical value are given in natural unit.
We conceive the change of reference frame in the following way.



(a) Beginning of the simulation: the red man's point of view.



(b) End of the simulation: the blue cat's point of view.



(c) Intermediate step of the simulation.

Figure B.9: Transition from the red man's perspective to the blue cat's perspective with Python simulation

While changing the reference frame, going from the one of the red man (Figure B.9a) to the one of the blue cat (Figure B.9b), we pass through different frames like the one in Figure

[7] It does not represent the speed of light.

B.9c in which both the red and blue worldlines change their slope from the initial configuration to the final one. From a physical point of view, each one of these steps corresponds to a particular frame of reference, namely the one of that observer (not represented in the plots of Figure B.9) whose worldlines is along the time axis. Thus for instance in Figure B.9c, the point of view of a particular observer at rest is adopted in which both the red man and the blue cat are moving with a particular speed.

Even if we do not care about these passages with the spacetime globe, it will be very useful to implement our codes. The previously defined parameter c identifies the speed[8] of this third not-represented observer which allows us to plot the intermediate graphs of the simulation. In particular it will go from 0 to v, being v the speed of the moving observer in Figure B.9a. In each frame our two primary observers have a particular speed which is given by the relativistic addition formula of speeds between their original speed in Figure B.9a and c. For instance:

- for $c = 0$ we are in Figure B.9a. Applying the relativistic addition formula we have that the red man has speed:

$$v_{\text{red man}}(c = 0) = \frac{0 - 0}{1 - 0} = 0. \tag{B.40}$$

  While the blue cat has speed:

$$v_{\text{blue cat}}(c = 0) = \frac{v - 0}{1 - v \cdot 0} = v. \tag{B.41}$$

- for $c = v$ we are in Figure B.9b. Applying the relativistic addition formula we have that the red man has speed:

$$v_{\text{red man}}(c = v) = \frac{0 - v}{1 - v \cdot 0} = -v. \tag{B.42}$$

  While the blue cat has speed:

$$v_{\text{blue cat}}(c = v) = \frac{v - v}{1 - v \cdot v} = 0. \tag{B.43}$$

For each intermediate value of c the two worldlines will have intermediate slope between their upper and lower bound.

We will describe the first of all the simulations: the one about the change between reference frames.

At the very beginning we import the necessary libraries:

[8] Always in natural unit.

```
  import matplotlib.pyplot as plt
2 import numpy as np
  import math as mt
4 import funzioni-rel-ristretta as rr
  import matplotlib.animation as animation
6 from matplotlib.offsetbox import OffsetImage, AnnotationBbox
  from mpl_toolkits.axisartist.axislines import SubplotZer
```

The library funzioni−rel− ristretta is created by ourselves and contains different functions (see for reference Appendix refappendice-coding). Three main functions are always used in each simulation:

- the first is gamma(v) which returns for a number v the value of $\beta = 1/\sqrt{1 - v^2}$;
- the second is trasforma(vector,v): one of its arguments is vector which is a vector in the form [x,t] containing the coordinates of a spacetime event. The second is a number v. This function returns a vector whose first component is the spatial coordinate of the Lorentz transformation of the previous event with speed v while the second component is its temporal coordinate;
- antitrasforma(vector,v): it is analogue to the previous function trasforma(vector,v) but it returns the inverse Lorentz transformation.

Going on with the code we find:

```
8  path_cat = "path_c"
   path_man = "path_m"
10 cat = plt.imread(path_cat)[:,:]
   man = plt.imread(path_man)[:,:]
```

Lines 8-9 contain the path of the images used in the simulation like the red man and the blue cat in Figure 3.37. Lines 10-11 transform the images in an array format that can be read easily by Python.

```
12 v=0.7
   def m(m,p):
14     return (m-p)/(1-m*p)
   def iperbole(x,k):
16     return mt.sqrt(x**2+k**2)
```

Line 12 defines the speed of the moving observer equal to $0.7\,c$ while lines 13-14 implement the function which gives the dynamical slope of a worldline. It is written as:

$$\frac{m-p}{1-m\cdot p}. \tag{B.44}$$

Indeed consider a straight line crossing the origin of the axes in the form of

$$t = mx, \tag{B.45}$$

where $m = 1/v$, being v the speed of the observer the worldline stands for.
The transformed worldline under a Lorentz transformation is written in the new coordinates as:

$$t' = m'x'. \tag{B.46}$$

Using explicitly the Lorentz transformation we have:

$$\gamma(t-px) = m'\gamma(x-pt), \tag{B.47}$$

where p is the beta factor of the boost's speed.
Hence:

$$m' = \frac{t-px}{x-pt} = \frac{x\left(\frac{t}{x}-p\right)}{x\left(1-p\frac{t}{x}\right)} \tag{B.48}$$

But from Eq. (B.45) $m = t/x$ and then:

$$m' = \frac{\left(\frac{t}{x}-p\right)}{\left(1-p\frac{t}{x}\right)} = \frac{m-p}{1-m\cdot p} \tag{B.49}$$

Thus m' is the new slope of the transformed worldline, m the original one and p the speed (in terms of c) of the boost. This relation gives us the correct dynamical slope of each worldline in a spacetime diagram, according to the previous observations given at the beginning of this section.
Lines 15-16 define a function `iperbole(x,k)` which returns for a x and k the value of the rectangular hyperbola $\sqrt{x^2+k^2}$.

```
x = np.linspace(0,15,100)
18 x2 = np.linspace(-15,0.001,100)
xt = np.linspace(-15,15,100)
20 k=[[i for i in range (1,7)],[(j**2+xt**2)**(1/2)
```

```
    for j in range (1,7)]]
22 fig = plt.figure(figsize=(10,6))
ax = fig.gca()
```

This part of the code (line 17-23) implements some graphical elements like x-intervals (lines 17-19), k which is an array containing the x, t coordinate of seven different rectangular hyperbolas $t = \sqrt{j^2 + x^2}$ (with $j = 1, \ldots, 7$) over the x interval xt. Finally (lines 22-23) a new figure is created.

In the next part of the code we implement the worldlines:

```
24 def oss_mov(c):
    y = m(1/v,c)*x
26  ax.plot(x,y,'b')

28 def oss_fisso(c):
    y = m(mt.tan(np.pi/2),c)*x2
30  ax.plot(x2,y,'r')
```

In lines 24-26 it is defined the worldline of the moving observer (the blue cat in Figure B.9): the slope is given by the value of the function m having given as the initial slope 1/v and c as the speed of the transformation. Thus given the array of points x of x-axis the corresponding t-values are obtained from y=m(1/v,c)*x. Then the straight lines is printed with blue colour as a set of point x and y.

In the same way line 28 defines the dynamical slope of the observer at rest (red man in Figure B.9). The slope initial value is actually infinite as the speed of the observer is zero: thus we choose to set is as the value of tangent at $\pi/2$. Here the range of x-values for the observer at rest is x2 (negative value as his worldline will rotate towards the fourth quadrant) and the worldline is printed on the plot with red colour.

In the next codes we describe how the figures move during the simulation.

```
def punti-oss-mobile(c):
32  for k in range(0,7):
        x=rr.trasforma([(v/mt.sqrt(1-v**2))*k,
34          (1/mt.sqrt(1-v**2))*k],c)[0]
        y=rr.trasforma([(v/mt.sqrt(1-v**2))*k,
36          (1/mt.sqrt(1-v**2))*k],c)[1]

38      im = OffsetImage(cat, zoom=75/ax.figure.dpi)
        im.image.axes = ax
40
```

```
            ab = AnnotationBbox(im,(x,y),frameon=False ,pad=0.0,)
42
            ax.add_artist(ab)
44
    def punti-oss-fisso(c):
46      for k in range(0,7):
            x=rr.trasforma([0,iperbole(0,k)],c)[0]
48          y=rr.trasforma([0,iperbole(0,k)],c)[1]

50          im = OffsetImage(man, zoom=75/ax.figure.dpi)
            im.image.axes = ax
52
            ab = AnnotationBbox(im,(x,y),frameon=False ,pad=0.0,)
54
            ax.add_artist(ab)
```

The function punti−oss−mobile(c) (lines 31-43) implements several points corresponding to the figures of the blue cat in Figures B.9. Their positions (lines 32-36) are defined by the Lorentz transformation of some coordinates, being the parameter c the speed of the boost. These coordinates are obtaining from the intersection of a line of equation $t = (1/v)x$ (the blue cat's initial worldline) with a hyperbola of equation $t = \sqrt{x^2 + k^2}$ with k a parameter running from 0 to 7 and corresponding to the different number of points we want to draw. Then in lines 38-43 the figure of the cat is anchored to the coordinate of these points. As a result, being c a changing dynamically parameter, the images will move along the hyperbolas as c varies.

In the same way in lines 45-55 the function punti−oss−fisso (c) defines the points corresponding to the figures of the red man in Figures B.9. As before there is a dynamical Lorentz transformation depending on the parameter c which defines the coordinates starting from the intersection between a line of equation $x = 0$ (the red man's initial worldline) and a hyperbola of equation $t = \sqrt{x^2 + k^2}$. In the same way, the images of the red man are anchored to these points.

Up to here, the code defines all we need to represent our problem; in the next lines we implement the animation itself together with the plot.

```
56  ax.grid()

58  plt.title("Changing reference frames",
                  fontdict={'family': 'Bradley Hand',
```

```
                          'color' : 'blue',
                          'weight': 'bold',
                          'size': 44})

plt.xticks(fontsize=18)
plt.yticks(fontsize=18)

ax.set_xlabel('space(m)',family = 'Bradley Hand',fontsize=35)
ax.set_ylabel('time(s)',family = 'Bradley Hand',fontsize=35)
```

In these lines a grid is created; then we have the title, size of the axes' markers and their labels. These lines are concerning the costume of the plot before the simulation starts.

```
def tutto(c):
    if c>0.01:
        ax.cla()

    ax.set_xlim(-6.5, 6.5)
    ax.set_ylim(0, 9)

    xmin, xmax = ax.get_xlim()
    ymin, ymax = ax.get_ylim()

    ax.grid()

##ASSE X
    ax.arrow(xmin, 0, xmax-xmin, 0., fc='k', ec='k',
    lw = 0.9, head_width=0.12, head_length=0.05,
    overhang = 2, length_includes_head= True,
    clip_on = False)

##ASSE Y
    ax.arrow(0, ymin, 0., ymax-ymin, fc='k', ec='k',
    lw = 0.9, head_width=0.12, head_length=0.05,
    overhang = 2, length_includes_head= True,
    clip_on = False)

```

```
94      plt.title("Changing reference frames",
                fontdict={'family': 'Bradley Hand',
96                        'color' : 'blue',
                        'weight': 'bold',
98                        'size': 44})

100     plt.xticks(fontsize=18)
        plt.yticks(fontsize=18)
102
        ax.set_xlabel('space (m)',family = 'Bradley Hand',
104     fontsize=35)
        ax.set_ylabel('time (s)',family = 'Bradley Hand',
106     fontsize=35)

108     punti_oss_fisso(c)
        punti_oss_mobile(c)
110     oss_fisso(c)
        oss_mov(c)
112
    N=int((v+0.01)/0.01)
114
    aniglobe = animation.FuncAnimation(fig, tutto,
116 frames=np.linspace(0, v-0.00001, N), interval=1,
    repeat=False)
118
    plt.show()
```

Line 69 opens a function tutto(c) that closes at line 111: it includes all the dynamical elements we want to update as the parameter c changes throughout the simulations. The command **if** c>0.01: allows Python to do this as the value of c starts changing. Then at each increase of the dynamical parameter, in lines 73-106 the simulation creates a new grid with all the features (arrow of the axes, maximum and minimum value,...), having before cleared the previous one (line 71). Then in lines 108-111 we add all the functions we want to update dynamically during the simulation, namely the ones defining the positions of the points of both the moving observer and the observer at rest and the functions defining their worldlines.

In line 113 N is defined, an integer value which establishes the step at which c varies. Function aniglobe in line 115-117 implements the animation requiring the figure fig of the plot, the function tutto to be updated and the number frames of frames we want. This quantity is defined by the range of values we want the parameter c to vary in. Given the physical meaning of parameter c it must be bounded between zero and v, by step of N. This animation takes 1 milliseconds as a delay between each frames and does not repeat after it is concluded (line 117). Finally in line 119 the command plt.show() shows the plot. In addition to the previous lines in order to export the simulations as a video we add the following commands:

```
Writer = animation.writers['ffmpeg']
writer = Writer(fps=12, metadata=dict(artist='Me'),
     bitrate=10000)
aniglobe.save('Name.mp4', writer = writer)
```

Appendix C

Appendix to Chapter 4

C.1 Kuder-Richardson index error

In order to evaluate the uncertainty σ above the value of the Kuder-Richardson index Eq. (4.6), we use the theory of propagation of uncertainty where

$$\sigma = \sqrt{\sum_i \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_{x_i}^2}, \tag{C.1}$$

being f the function which we want to calculate the uncertainty of, x_i the single variables which the function depends on and σ_{x_i} the uncertainty associated to the single variable. The Kuder-Richardson index

$$r_{test} = \frac{K}{K-1}\left(1 - \frac{\sum_i P_i(1-P_i)}{\sigma_x^2}\right) \tag{C.2}$$

depends only on the variable P_i with the associated uncertainty σ_P (it has the same value for all the P_i). Because of the constant multiplicative value and the constant added one, we have:

$$\sigma_{r_{test}} = \frac{K}{K-1}\sigma\left(\frac{\sum_i P_i(1-P_i)}{\sigma_x^2}\right) = \frac{K}{K-1}\frac{1}{\sigma_x^2}\sigma\left[\sum_i P_i(1-P_i)\right]. \tag{C.3}$$

Let call a_i the product $a_i = P_i(1-P_i)$; then:

$$\sigma\left(\sum_i a_i\right) = \sqrt{\sum_i a_i^2 \sigma_{a_i}^2}. \tag{C.4}$$

But using Eq. (C.1):

$$\sigma_{a_i} = \sigma[P_i(1-P_i)] = \sqrt{P_i^2\sigma_P^2 + (1-P_i)^2\sigma_P^2} = \sigma_P\sqrt{P_i^2 + (1-P_i)^2}. \tag{C.5}$$

Then:

$$\begin{aligned}\sigma\left[\textstyle\sum_i P_i(1-P_i)\right] &= \sigma_P\sqrt{\textstyle\sum_i P_i^2(1-P_i)^2(P_i^2+(1-P_i)^2)} = \\ &= \sigma_P\sqrt{\textstyle\sum_i (1-2(P_i-P_i^2))(P_i-P_i^2)^2}.\end{aligned} \tag{C.6}$$

Finally recollecting Eq. (C.3) we have:

$$\sigma_{r_{test}} = \frac{K}{K-1}\frac{\sigma_P}{\sigma_x^2}\sqrt{\textstyle\sum_i (1-2(P_i-P_i^2))(P_i-P_i^2)^2}. \tag{C.7}$$

C.2 Additional material

In Figure C.1 are shown the ranking established by teachers about the most important topics to develop as laboratorial activities.

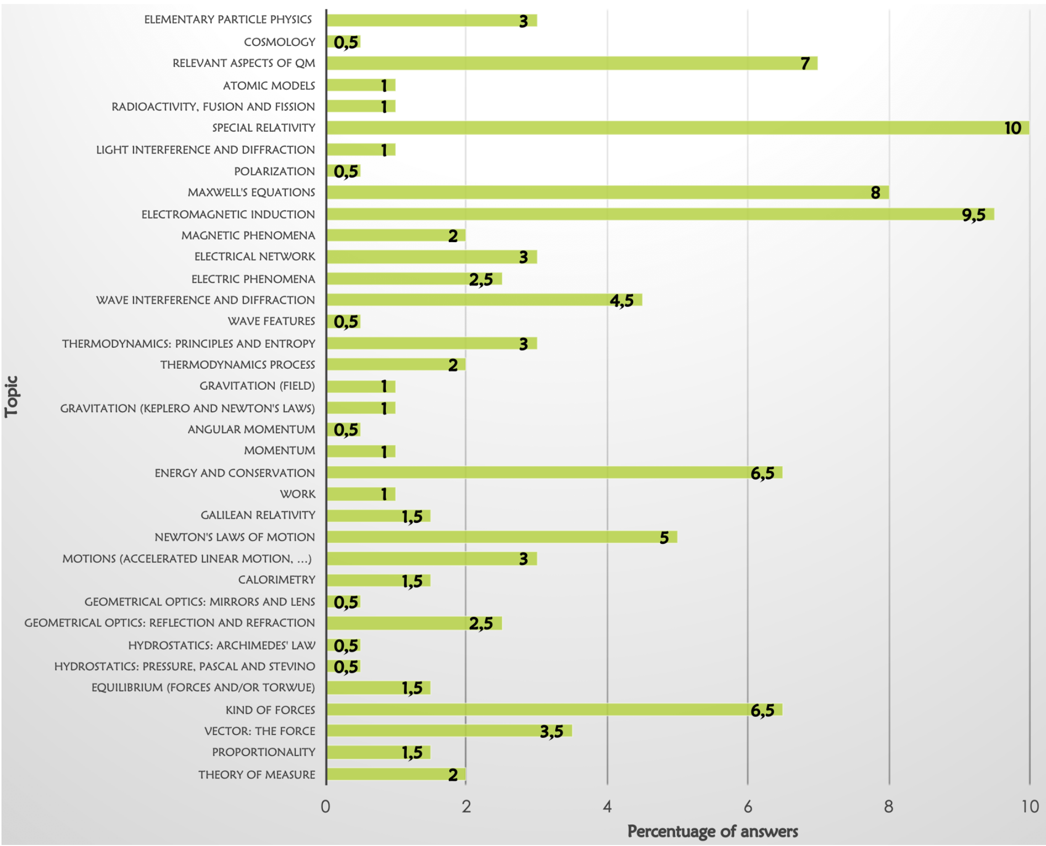


Figure C.1: Global ranking of teachers' whole choices.

In the folder at this drive link:

https://drive.google.com/drive/folders/1S6oQu28L5W-CF4ewZG-pkxutS5asWj6n?usp=sharing

they have been reported:

- the whole pdf files with the two introductory questionnaire submitted to the teachers (Section 4.1): *first-introductory-questionnaire* and *second-introductory-questionnaire*;
- the whole pdf file with the UDA of our proposal (Section 4.2): *UDA*;
- the whole pdf file with the second introductory questionnaire submitted to the teachers (Section 4.3): *Questionario-valutativo-RR*;
- the whole pdf file with the pre-test: *Pre-test*;
- the whole pdf file with student's answer to the pre-test: *Ans-pre-test*;
- the whole pdf file with the post-test: *Post-test*;
- the whole pdf file with student's answer to the post-test: *Ans-post-test*;
- a pdf file with snapshots from the beginning parts and the end parts of all the Python simulations: *Python-simulation*;

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