A Short Course on Generalized Functions and Mathematical Modeling

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In many circumstances, the notion of function does not suffice for the needs of the model at hand, and it proves necessary to extend it. We can recall, for example, the heuristic use of symbolic methods, called *operational calculus*, introduced as long ago as 1899 by Oliver Heaviside in his book on electromagnetic theory [14]. His methods, lacking a rigorous justification, gained a bad reputation among mathematicians. In fact, a notion of generalized function did not receive wide acceptance before the introduction of the Lebesgue integral. In *Lebesgue's theory*, an integrable function is equivalent to any other one which is the same almost everywhere, implying that its value at a given point is meaningless. So, the centrality of the concept of function was replaced by that of equivalence classes of functions. Further important steps in the same direction were the introduction of *weak derivatives* and of Dirac's *delta function*. These theories were later unified by Schwartz' beautiful *theory of distributions* (see, e.g., [22, 23]), also thanks to the previous work of Leray and Sobolev. While mathematicians working in PDEs still accept Schwartz' theory as definitive (until now, at least), different notions of generalized functions have already been introduced by Colombeau [11] and Mikio Sato [20, 21].

This course deals with a new kind of generalized functions, called *ultrafunctions*, recently introduced in [3] and developed in [8, 9]. Their peculiar feature is that ultrafunctions are based on *Non-Archimedean Mathematics* (NAM), i.e., mathematics on fields containing infinite and infinitesimal numbers (non-Archimedean fields). Around the turn of the twentieth century, NAM was investigated by prominent mathematicians such as Paul du Bois-Reymond [12], Giuseppe Veronese [24], David Hilbert [15] and Tullio Levi-Civita [17]. It was later forgotten until the sixties, when Abraham Robinson presented his *Non-Standard Analysis* (NSA) [19]. We refer to Ehrlich [13] for a historical account of these developments and to Keisler [16] for a very clear exposition of NSA.

Ultrafunctions have been introduced to provide generalized solutions to equations which do not have any solutions, not even among distributions. The main features of ultrafunctions, as presented in these lectures, are the following one:

• Ultrafunctions are functions taking value in $\mathbb{R}^*$, a non-Archimedean field containing $\mathbb{R}$, which is a nonstandard extension of $\mathbb{R}$ (see, e.g., [16]).

• Any real integrable function $f:\Omega\to\mathbb{R}$ $(\Omega\subseteq\mathbb{R}^N)$ can be extended to an ultrafunction $\widetilde{f}:\Omega^*\to\mathbb{R}^*$, where $\Omega\subset\Omega^*\subset(\mathbb{R}^*)^N$.

• To any distribution $T\in\mathcal{D}'(\Omega)$ the ultrafunction $\widetilde{T}:\Omega^*\to\mathbb{R}^*$ may be associated, such that, for all $\varphi\in\mathcal{D}(\Omega)$,

$$\langle T,\varphi\rangle=\int^*\widetilde{T}(x)\widetilde{\varphi}(x)dx,$$

where $\int^*$ is a suitable extension to ultrafunctions of the definite integral.

• Any functional J defined on a function space $V(\Omega)$ can be extended to a functional $\widetilde{J}$ defined on an ultrafunction space $V_\Lambda(\Omega)\supset V(\Omega)$; moreover, if J is coercive, $\widetilde{J}$ has a minimum in $V_\Lambda(\Omega)$;

• Any (linear or nonlinear) differential operator A defined on a suitable function space $V(\Omega)$ can be extended to an operator $\widetilde{A}:V_\Lambda(\Omega)\to V_\Lambda(\Omega)$, and the equation $\widetilde{A}(u)=\widetilde{f}$ may have a solution in $V_\Lambda(\Omega)$ even if the equation $A(u)=f$ has no solution in $V(\Omega)$;

• The main strategy to prove existence of generalized solutions in the space of ultrafunctions is relatively simple—it is just a variant of the Faedo-Galerkin method.

Despite the fact that the theory of ultrafunctions makes ample use of the techniques of NSA, our approach to non-Archimedean mathematics is quite different from its spirit. There are two main differences, one in aim and one in method.

Let us examine the difference in aim first. We think that infinitesimal and infinite numbers should not be regarded as entities living in a parallel, minor universe—the non-standard universe—worth considering only as far as they are instrumental in proving statements relative to our own universe—the standard one. On the contrary, we think that they are mathematical entities which have the same status of the standar ones and can be used to build models as any other mathematical entity. *Actually, our opinion is that the advantages of a theory which includes infinitesimals lies more in making it possible to construct new models than in providing new proof techniques.* Our papers [5, 6] and these lectures are inspired by this conviction. Our point of view is not completely new: in fact, in the seventies and at the beginning of the eighties some mathematical models involving infinitesimals have been constructed in economics and physics (see, e.g., [10]).

As far as method is concerned, we introduce a non-Archimedean field via a new notion of limit, namely, the Λ-*limit*. Roughly speaking, the Λ-limit of a sequence of (standard) mathematical entities is a new object which preserves most of the properties of the approximating objects. New objects of this kind, called *internal*, in general do not exist in conventional mathematics. Infinite and infinitesimal numbers, as well as the ultrafunctions, are internal. It is noteworthy that this notion of limit allows us to make a minimal use of formal logic.

One more difference between our approach and the traditional one is that we do not assume the existence of two distinct mathematical universes. This attitude is shared by Nelson's approach to NSA called *Internal Set Theory* (see [18]). Nevertheless, our theory and IST differ in many respects. For instance, IST postulates the existence of infinitesimal elements in $\mathbb{R}$ itself, while we do not change the nature of this set.

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